NONCLASSICAL SHOCK WAVES OF CONSERVATION LAWS: FLUX FUNCTION HAVING TWO INFLECTION POINTS

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Abstract. We consider the Riemann problem for non-genuinely nonlinear conservation laws where the flux function admit two inflection points. This is a simplification of van der Waals fluid pressure, which can be seen as a function of the specific volume for a specific entropy at which the system lacks the non-genuine nonlinearity. Two kinetic relations can be used to construct a nonclassical Riemann solution.

Key words. conservation law, non-genuine nonlinearity, nonclassical solution, kinetic relation.

AMS subject classifications. Primary: 35L65, 74XX. Secondary: 76N10, 76L05.

1. Introduction. The theory of nonclassical solutions of hyperbolic systems of conservation laws has been introduced by LeFloch and has been developed for many years. For material on this subject, see the text book [13]. See also [12, 4, 5, 13, 8, 9, 10, 11] and the references therein for the history and details of developments. Briefly, nonclassical shock waves violate the standard the Oleinik criterion [16] in the scalar case and the Lax shock inequalities [7] and the Liu entropy conditions [14] for the case of hyperbolic systems of conservation laws. To select nonclassical shock waves, by a standard way, one follows the strategy proposed by Abeyaratne-Knowles [1, 2], and by Hayes and LeFloch [4, 5, 13] to describe the whole family of *nonclassical Riemann solutions* and then to use a *kinetic relation* to determine the relevant physical solution. Related works can be found in [6, 19, 20, 17, 21, 22, 18].

Moreover, the dynamics of a fluid is governed by the following system of differential equations in Lagrange coordinates as, see [3],

$$\begin{aligned} &\partial_t v - \partial_x u = 0 && \text{(conservation of mass)},\\ &\partial_t u + \partial_x p = 0 && \text{(conservation of momentum)},\\ &\partial_t E + \partial_x (p\,u) = 0 && \text{(conservation of energy)}, \end{aligned} \tag{1.1}$$

where the unknowns are $u = u(x,t)$, $v = v(x,t)$, and $E = E(x,t)$. Here, u denotes the particle velocity, v the specific volume, $E = \varepsilon + u^2/2$ the total energy, ε the specific internal energy, and p the pressure. The system (1.1) must be supplemented by a constitutive relation (or equation of state), which characterize the properties of the material under consideration. For example, an equation of state for fluids of van der Waals may have the form

$$p(v,S) = \frac{8e^{3S/8\alpha}}{(3v-1)^{1+1/\alpha}} - \frac{3}{v^2}, \quad v > 1/3, \tag{1.2}$$

for some positive constant $\alpha = 1/(\gamma-1)$ where $\gamma \in (1,2)$ is called the adiabatic exponent. See [15] for a review. The system (1.1)-(1.2) fails to be globally

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genuinely nonlinear (in the sense of Lax [7]) or even to be globally hyperbolic, and there is a certain region in the phase space in which the system is of elliptic type. This is due to the fact that the function p (expressed in suitable variables) fails to be globally monotonous and admits *two* inflection points, see LeFloch-Thanh [8, 11].

In the work of LeFloch-Thanh [8], the presence of two inflection points in the flux function was studied. The nonclassical Riemann solver was constructed by restricting only on the first kinetic function, though we may have more than one kinetic functions on a Hugoniot curve. More clearly, following the strategy proposed by Abeyaratne-Knowles [1, 2], and by Hayes-LeFloch [4, 5, 13], the authors define the entropy dissipation to describe the whole set of nonclassical waves. It appears that the entropy dissipation may vanish three times. And this would lead to the definition of two kinetic functions. The domain as well as the range of each of these two kinetic function contains one inflection point and its values are symmetric to the variable values with respect to the inflection point. The difficulty to use the second kinetic function is that the shock speed involving the second kinetic function may be less than that of the shock speed using the first kinetic function. Consequently, the Riemann solution may not be well-defined when two kinetic functions are to be involved. In LeFloch-Thanh [11], phase transitions were observed. All nonclassical shock waves satisfying a single entropy inequality

$$\partial_t S \geq 0 \tag{1.3}$$

were also characterized.

This paper will deal with the case of two apparent kinetic functions, continuing works in [8, 11]. For simplicity, we restrict our attention to the scalar case where we have a single conservation law. The flux function will have the shape of the pressure of van der Waals fluids in the region where it admits two inflection points. Accordingly, we may have two kinetic functions, and we will consider when we can use each of them, or both. Moreover, as the entropy dissipation selects nonclassical waves like the rule of equal areas, we will define the kinetic functions relying on the rule of equal areas to set up their domains. Therefore, the construction may be more visual in some sense. We note that a similar way was constructed for classical shock waves by Oleinik [16].

Precisely, we will consider the Riemann problem for conservation laws

$$\begin{aligned} &\partial_t u + \partial_x f(u) = 0,\\ &u(x,0) = \begin{cases} u_l & \text{for } x < 0,\\ u_r & \text{for } x > 0, \end{cases} \end{aligned} \tag{1.4}$$

where u_l and u_r are constants and the flux f is a twice differentiable function of $u \in \boldsymbol{R}$. The function f is assumed to satisfy

$$\begin{aligned} &f''(u) > 0 \text{ for } u \in (-\infty, 0) \cup (1, +\infty),\\ &f''(u) < 0 \text{ for } u \in (0, 1),\\ &\lim_{u\to\pm\infty} f'(u) = +\infty, \quad \lim_{u\to\pm\infty} f(u) = +\infty. \end{aligned} \tag{1.5}$$

Thus the flux f has two inflection points at $u = 0$ and $u = 1$. The specification of these two values does not restrict the scope of consideration of this paper. By assumption, the function f is clearly convex in each interval $(-\infty, 0)$ and $(1, +\infty)$, and is concave in the interval $(0, 1)$. To specify these intervals, we denote

$$\begin{aligned} E_I &:= (-\infty, 0), \\ E_{II} &:= [0, 1], \\ E_{III} &:= (1, +\infty), \end{aligned} \tag{1.6}$$

and call each of them a *phase*.

The paper is organized as follows. In Section 2 we will investigate the properties of tangents to the graph of f, and then we review the Oleinik construction of the entropy solution. Section 3 will be devoted to geometrically select non-classical Riemann solutions relying on only one kinetic relation. Kinetic functions will be presented to give the unique admissible one between such nonclassical solutions. In Section 4 we will give a Riemann solver which permits two kinetic relations.

2. Basic Properties and Oleinik Construction. This section is aimed first to describe several essential properties of the flux function f. Tangents to the graph of f will be used to select nonclassical shocks instead of an entropy dissipation. Then we review the Oleinik construction for *classical solutions* of the Problem (1.4). A classical solution of Problem (1.4), by definition, is a weak solution satisfying the so-called Oleinik entropy condition

$$\frac{f(u) - f(u_l)}{u - u_l} \geq \frac{f(u_r) - f(u_l)}{u_r - u_l}, \quad \forall u \quad \text{between} \quad u_r \quad \text{and} \quad u_l.$$

This means that the graph of f is below (above) the line connecting u_l to u_r when $u_r < u_l$ (respectively $u_r > u_l$).

Under the hypotheses (1.5), the tangents at 1 and 0 cut the graph of the flux function f at a point a and b, respectively, with

$$a < 0 < 1 < b,$$

(see Figure 2.1). From each $u \in (a, b)$, one can draw two distinct tangents to the graph of f. Denote these tangent points by $\varphi^\natural(u)$ and $\psi^\natural(u)$ with

$$\varphi^\natural(u) < \psi^\natural(u).$$

In other words, we have

$$\begin{aligned} f'\big(\varphi^\natural(u)\big) &= \frac{f(u) - f\big(\varphi^\natural(u)\big)}{u - \varphi^\natural(u)}, \\ f'\big(\psi^\natural(u)\big) &= \frac{f(u) - f\big(\psi^\natural(u)\big)}{u - \psi^\natural(u)}. \end{aligned} \tag{2.1}$$

To the end points of the interval under consideration a, b we set

$$\varphi^\natural(a) = \psi^\natural(a) = 1 \quad \text{and} \quad \varphi^\natural(b) = \psi^\natural(b) = 0.$$

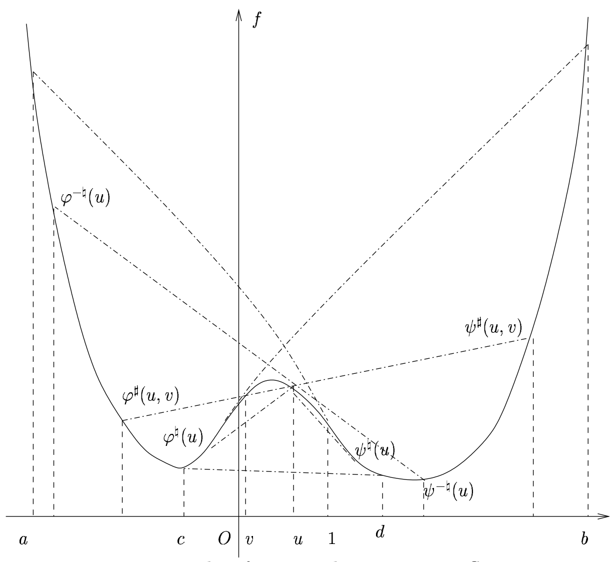

FIG. 2.1. *Flux function having two inflection points*

There are no tangents to the graph of f from any point outside the interval $[a,b]$. Besides, the values u and $\psi^\natural(u)$ always lie on different sides with respect to $u=1$, and the values u and $\varphi^\natural(u)$ always lie on different sides with respect to $u=0$, i.e.

$$\begin{aligned} \varphi^\natural(u)u &< 0 \quad \text{for } u \neq 0, \qquad \varphi^\natural(0)=0, \\ (\psi^\natural(u)-1)(u-1) &< 0 \quad \text{for } u \neq 1, \qquad \psi^\natural(1)=1. \end{aligned} \tag{2.2}$$

There are two points $c<d$ such that the epigraph of the function $\tilde{f}$ defined by

$$\tilde{f}(u) = \begin{cases} f(u), & \text{if} \quad u \in (-\infty, c] \cup [d, +\infty), \\ \text{affine} & \text{on} \quad [c,d], \end{cases} \tag{2.3}$$

coincides with the convex hull of that of the function f. Geometrically, the tangents to the graph of f from c and d coincide. Such points c and d are unique. More clearly,

$$f'(c) = \frac{f(d)-f(c)}{d-c} = f'(d).$$

It is not difficult to check that

PROPOSITION 2.1. *The function $\psi^\natural$ is increasing for $u \in [a,c]$ and decreasing for $u \in [c,b]$. The function $\varphi^\natural$ is decreasing for $u \in [a,d]$ and increasing for $u \in [d,b]$. Moreover $\varphi^\natural$ maps $\big([a,b]$ onto $[c,1]$, while $\psi^\natural$ maps $[a,b]$ onto $[0,d]$.*

Inversely, the tangent from a point $u \in (c,d)$ cuts the graph of f at exactly two distinct points, say denoted by $\varphi^{-\natural}(v)$ and $\psi^{-\natural}(v)$ with the convention

$$\varphi^{-\natural}(v) < \psi^{-\natural}(v).$$

The definition can be extended to the end values c and d as

$$\varphi^{-\natural}(c) = d, \quad \text{and} \quad \psi^{-\natural}(d) = c.$$

The functions $\varphi^{-\natural}$ and $\psi^{-\natural}$ in the interval $[c,d]$ are not monotone and therefore not one-to-one. However, they are monotone in for $u \in [0,1]$. Restricting consideration to the interval $[0,1]$, they are the inverses of the functions $\varphi^{\natural}$ and $\psi^{\natural}$ defined above, respectively:

$$\varphi^{\natural} \circ \varphi^{-\natural} = \psi^{\natural} \circ \psi^{-\natural} = id \qquad \text{on the interval } [0,1]. \tag{2.4}$$

Since we want to discuss the tangent functions in the whole interval $[a,b]$, we can assume for the global purpose

$$\begin{aligned} \varphi^{-\natural}(u) = \psi^{-\natural}(u) = +\infty, \quad u \in [a,c), \\ \varphi^{-\natural}(u) = \psi^{-\natural}(u) = -\infty, \quad u \in (d,b]. \end{aligned} \tag{2.5}$$

We were dealing with tangent points and points from which tangents can be issued. Between two these points, there is a kind of points that will be concerned to the dynamics of phase transition (to be considered in the coming sections). So, it derives the definition of these points.

The following proposition can easily be verified.

Proposition 2.2. *Given a point $u \in (a,b)$, any line between u and another point $v \in (\varphi^{\natural}(u), \psi^{\natural}(u))$ cuts the graph of f at exactly four points of which u and v are the two. Denote such the remaining two points by $\varphi^{\sharp}(u,v)$ and $\psi^{\sharp}(u,v)$, with convention*

$$\varphi^{\sharp}(u,v) < \psi^{\sharp}(u,v).$$

For the limit cases, we set

$$\begin{aligned} \varphi^{\sharp}(u, v = \varphi^{\natural}(u)) := \varphi^{\natural}(u) = v, \\ \psi^{\sharp}(u, v = \psi^{\natural}(u)) := \psi^{\natural}(u) = v. \end{aligned}$$

(See Figure 2.1.

By definition, the values $\varphi^{\sharp}(u,v)$ and $\psi^{\sharp}(u,v)$ satisfy

$$\frac{f(\varphi^{\sharp}(u,v)) - f(u)}{\varphi^{\sharp}(u,v) - u} = \frac{f(\psi^{\sharp}(u,v)) - f(u)}{\psi^{\sharp}(u,v) - u} = \frac{f(v)) - f(u)}{v - u}. \tag{2.6}$$

Next, we turn to the Oleinik construction [16] of the entropy solution of Problem (1.4). The following lemma characterize shock waves that are admissible by the Oleinik criterion.

LEMMA 2.3. (Classical shocks)

Given a left-hand state u_0, the set of right-hand states u_1 attainable by a classical shock is given by

(i) If $u_0 \in (-\infty, c) \cup (b, +\infty)$, then $u_1 \in (-\infty, u_0]$.

(ii) If $u_0 \in [c, 0]$, then $u_1 \in (-\infty, u_0] \cup [\varphi^{-\natural}(u_0), \psi^{\natural}(u_0)]$.

(iii) If $u_0 \in (0, 1)$, then $u_1 \in (-\infty, \varphi^{-\natural}(u_0)] \cup [u_0, \psi^{\natural}(u_0)]$.

(iv) If $u_0 \in [1, b]$, then $u_1 \in (-\infty, \varphi^{-\natural}(\psi^{\natural}(u_0))] \cup [\psi^{\natural}(u_0), u_0]$.

So we are at the position to construct the classical Riemann solutions. First, for $u_l \in (-\infty, c)$, Lemma 2.2 asserts that all the states $u_r \in (-\infty, u_l)$ can be reached by a single shock. States $u_r \in (u_l, 0]$ can be arrived at by a single rarefaction wave, since the characteristic speed is increasing when we move from u_l to u_r. If $u_r \in [0, d]$, we have $\varphi^{\natural}(u_r) \in [c, 0]$. So the solution is a composite of a rarefaction wave from u_l to $\varphi^{\natural}(u_r)$ followed by a shock from $\varphi^{\natural}(u_r)$ to u_r. If $u_r > d$, the solution is combined from three elementary waves: a rarefaction wave from u_l to c, followed by a shock from c to d, and then followed by a rarefaction wave from d to u_r.

Second, we deal with $u_l \in [c, 0]$. If $u_r \in (-\infty, u_l)$, the Riemann solution is a single shock. A single rarefaction wave can connect u_l with the states $u_r \in (u_l, 0]$. If $u_r \in [0, \varphi^{-\natural}(u_l)]$, then $\varphi^{\natural}(u_r) \in [u_l, 0]$ and the Riemann solution is composed by a rarefaction wave from u_l to $\varphi^{\natural}(u_r)$ followed by a shock from $\varphi^{\natural}(u_r)$ to u_r. A single shock from u_l can reach $u_r \in (\varphi^{-\natural}(u_l), \psi^{\natural}(u_l]$. Finally, if $u_r > \psi^{\natural}(u_l)$, the solution is a composite of a shock from u_l to $\psi^{\natural}(u_l)$ followed with a rarefaction wave connecting $\psi^{\natural}(u_l)$ to u_r.

Third, $u_l \in (0, 1)$. A single shock from u_l can reach $u_r \in (-\infty, \varphi^{-\natural}(u_l)] \cup [u_l, \psi^{\natural}(u_l)]$. A single rarefaction wave from u_l can connect to $u_r \in [0, u_l]$. If $u_r \in (\varphi^{-\natural}(u_l), 0)$, then there exists a unique value $u^* \in (0, u_l)$ such that $\varphi^{-\natural}(u^*) = u_r$. That is $u^* = \varphi^{\natural}(u_r)$. In that case the Riemann solution is a rarefaction wave connecting u_l to u^* followed by a shock connecting u^* to u_r. Finally, if $u_r > \psi^{\natural}(u_l)$, the Riemann solution is a shock connecting u_l to $\psi^{\natural}(u_l)$ followed with a rarefaction wave from $\psi^{\natural}(u_l)$ to u_r.

Fourth, assume that $u_l \in [1, b]$. A single shock from u_l can reach

$$u_r \in (-\infty, \varphi^{-\natural}(\psi^{\natural}(u_l))] \cup [\psi^{\natural}(u_l), u_l].$$

A single rarefaction wave from u_l can connect to $u_r \in [u_l, +\infty)$. If $u_r \in [0, \psi^{\natural}(u_l))$, the Riemann solution is combined by a shock from u_l to $\psi^{\natural}(u_l)$ followed by a rarefaction from $\psi^{\natural}(u_l)$ to u_r. If $u_r \in (\varphi^{-\natural}(\psi^{\natural}(u_l)), a)$, the solution contained three waves: a shock from u_l to $\psi^{\natural}(u_l)$, followed by a rarefaction from $\psi^{\natural}(u_l)$ to $\varphi^{\natural}(u_r)$, and followed by a shock connecting $\varphi^{\natural}(u_r)$ to u_r.

Finally, if $u_l \in (b, +\infty)$, then the Riemann solution is simply a shock if $u_r < u_l$ and a rarefaction wave otherwise.

We arrive at the following conclusion.

THEOREM 2.4. (Classical Riemann solver)

Under the assumption (1.5)*, the Riemann problem* (1.4) *admits a unique classical solution in the class of piecewise smooth self-similar functions made of rarefaction fans and shock waves satisfying the Oleinik entropy criterion. This solution depends continuously on the Riemann data.*

3. Non-classical Riemann Solvers Using One Kinetic Relation. In this section, we will present two *non-classical* Riemann solvers. The first one relying on *non-classical* jumps (see definition below) crossing the first inflection point $u = 0$. This solver can be proved to depend continuously on Riemann data. The second one using non-classical jumps crossing the second inflection point $u = 1$. The Riemann solver, however, does not depend continuouly on Riemann data.

3.1. Riemann solver relying on jumps crossing the inflection point $u = 0$. Let a function φ be given.

$$\varphi : [a, b] \to [a, b]. \tag{3.1}$$

The function φ will be called a *kinetic relation corresponding to the inflection point* $u = 0$ if there exists a neighborhood Ω_0 of $u = 0$ such that in Ω_0 the function φ satisfies the following two conditions:

(i) The function φ is monotone decreasing in $[a, b]$, $\varphi(u) \le \psi^\natural(u)$, $\varphi(u)$ lies between $\varphi^\natural(u)$ and $\varphi^{-\natural}(u), \forall u \in [a, b]$ in the sense that

$$\begin{aligned}
&\varphi^{-\natural}(u) > \varphi(u) > \varphi^\natural(u), \quad \forall u < 0,\\
&\varphi^{-\natural}(u) < \varphi(u) < \varphi^\natural(u), \quad \forall u > 0,\\
&\varphi(0) = \varphi^\natural(0) = \varphi^{-\natural}(0) = 0;
\end{aligned}$$

(ii) and the *convex-concave* condition

$$|\varphi \circ \varphi(u)| > |u|, \quad \forall u \in \Omega_0. \tag{3.2}$$

It is not difficult to check that there is a unique point $e_T \in (a, c)$ such that

$$\varphi(e_T) = \psi^\natural(e_T). \tag{3.3}$$

Moreover, in the coming construction we need an additional assumption

$$\varphi(1) = a. \tag{3.4}$$

Therefore, the largest domain Ω_0 satisfying (3.2)-(3.4) is

$$\Omega_0 = [e_T, 1], \tag{3.5}$$

and we take it as the domain of the kinetic function. (See Figures 3.1 for $u < 0$ and $u > 0$)

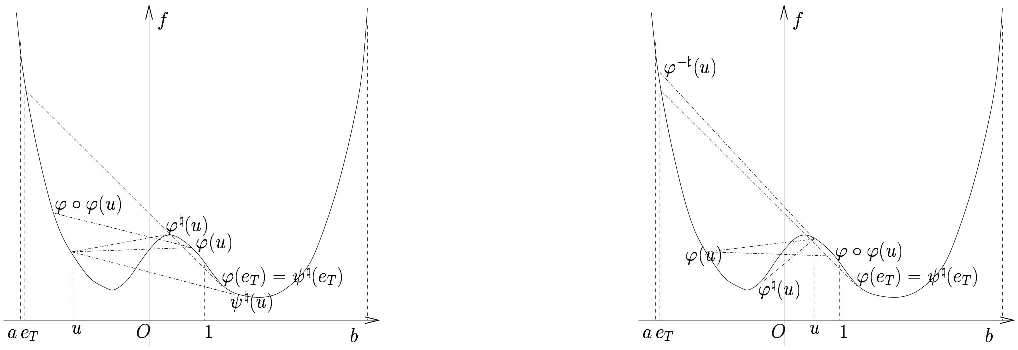

FIG. 3.1. *Kinetic Function φ*

For an arbitrary non-classical shock between a given left-hand state u_0 and a given right-hand state u_1, kinetic relation is the requirement that

$$u_1 = \varphi(u_0). \tag{3.6}$$

To select non-classical shock rather than classical ones, we postulate that

(C) Non-classical shocks are preferred whenever available.

We now solve the Riemann problem relying on the condition (C). The construction in this section is similar to the non-classical one for the 1-wave family in [8], but we want to recall it here for completeness.

Suppose first that $u_l \in (-\infty, e_T)$. Any point $u_r \in (-\infty, u_l)$ can be achieved by a single classical shock. Any point $u_r \in (u_l, 0]$ is attainable by a single rarefaction wave. If $u_r \in (0, \varphi(e_T)]$, there exists a unique point $u_* \in [e_T, 0)$ such that $u_r = \varphi(u_*)$. The solution is then the composite of a rarefaction wave from u_l to u_* followed by a nonclassical shock from u_* to u_r. If $u_r \in (\varphi(e_T), +\infty)$, the solution consists of three parts: A rarefaction wave from u_l to e_T followed by a nonclassical shock from e_T to $\varphi(e_T)$, followed by a rarefaction wave from $\varphi(e_T)$ to u_r.

Second, suppose that $u_l \in [e_T, 0)$. A point $u_r \in (-\infty, u_l)$ can be attained by a single classical shock. A point $u_r \in (u_l, 0]$ is attainable by a single rarefaction wave. If $u_r \in (0, \varphi(u_l)]$, there exists a unique point $u_* \in [u_l, a)$ such that $u_r = \varphi(u_*)$. The solution is then the composite of the rarefaction wave from u_l to u_* followed by a nonclassical shock from u_* to u_r. If $u_r \in (\varphi(u_l), \varphi(e_T)]$, there exists a unique point $u^* \in [e_T, u_l)$ such that $u_r = \varphi(u^*)$. For this construction to make sense, one must here check whether the classical shock from u_l to u^* is slower than the nonclassical shock from u^* to u_r. So, consider the function

$$\tilde{f}(v) := \begin{cases} f(v), & \text{if } v \in (-\infty, u_l], \\ f(u_l) + f'(u_l)(v - u_l), & \text{if } v \in (u_l, +\infty). \end{cases} \tag{3.7}$$

If $u_r \in (\varphi(u_l), p)$, where

$$p := \min\{\varphi(e_T), \varphi^{-\natural}(u_l)\},$$

the function $\tilde{p}$ is convex on $(-\infty, +\infty)$ and the points u^* and u_r belong to its epigraph. Therefore, the straightline connecting u^* and u_r should lie above the graph of $\tilde{f}$ in the interval $(u^*, u_r) \ni u_l$. That is to say

$$\frac{\tilde{f}(u_l) - \tilde{f}(u^*)}{u_l - u^*} < \frac{f(u_r) - f(u^*)}{u_r - u^*},$$

i.e.,

$$s(u_l, u^*) < s(u^*, u_r). \tag{3.8}$$

The latter inequality means precisely that the classical shock from u_l to u^* can be followed by the nonclassical shock from u^* to u_r.

In the latter construction, if $u_l \in [e_T, \varphi^\natural(\varphi(e_T))$, then

$$p = \varphi(e_T),$$

and we have completed the argument when $u_r \in (\varphi(u_l), \varphi(e_T))$. For $u_r \in (\varphi(e_T),$
$+\infty)$, the Riemann solution consists of three parts: A classical shock from u_l to e_T followed by a nonclassical shock from e_T to $\varphi(e_T)$, followed by a rarefaction wave from $\varphi(e_T)$ to u_r.

Suppose next that $u_l \in [\varphi^\natural(\varphi(e_T)), 0)$, then

$$p = \varphi^{-\natural}(u_l).$$

If $u_r \in [\varphi^{-\natural}(u_l), \varphi(e_T)]$, the solution can be a classical shock connecting u_l to u^* followed by a nonclassical shock from u^* to u_r provided (3.8) holds, or else a single classical shock. For $u_r \in (\varphi(e_T), +\infty)$, if

$$s(u_l, e_T) < s(e_T, \varphi(e_T)), \tag{3.9}$$

then the solution consists of a classical shock from u_l to e_T, followed by a nonclassical shock from e_T to $\varphi(e_T)$, then followed by a rarefaction wave. If else, (3.9) fails, then the solution is either a classical shock from u_l to u_r if $u_r \le \psi^\natural(u_l)$ or a classical shock from u_l to $\psi^\natural(u_l)$ followed by a rarefaction wave from $\psi^\natural(u_l)$ to u_r if else.

Third, suppose that $u_l \in [0, 1)$. The points $u_r \in [0, +\infty)$ are reached by the classical construction described in Section 2. If $u_r \in [\varphi(u_l), 0]$, there exists a unique point $u^* \in [0, u_l]$ such that $u_r = \varphi(u^*)$. The solution then consists of a rarefaction wave connecting u_l to u^* followed by a nonclassical shock from u^* to u_r. If $u_r \in [\varphi^{-\natural}(u_l), \varphi(u_l))$, then there exists a unique point $u_* \in [u_l, 1)$ such that $u_r = \varphi(u_*)$. Since both u_l and u_* belong to $[0, 1]$ and the function f is concave in this interval, we have

$$\frac{f(u_l) - f(u_*)}{u_l - u_*} < \frac{f(\varphi(u_l)) - f(u_*)}{\varphi(u_l) - u_*} < \frac{f(u_r) - f(u_*)}{u_r - u_*}.$$

This means the shock speed $s(u_l, u_*)$ is less than the shock speed $s(u_*, u_r)$. Therefore the Riemann solution can be a classical shock from u_l to u_* followed by a nonclassical shock from u_* to u_r. If $u_r \in (a, \varphi^{-\natural}(u_l)]$, there exists a unique point $u^* \in [u_l, 1)$ such that $u_r = \varphi(u^*)$. The solution then consists of a classical shock from u_l to u^* followed by a nonclassical shock from u^* to u_r provided

$$s(u_l, u^*) < s(u^*, u_r),$$

or else a single classical shock. The states $u_r \in (-\infty, a]$ are reached by single classical shocks.

Finally, when $u_l \in [1, +\infty)$, we also use the classical construction described in Section 2.

Denote by $\varphi^{-1} : [a, \varphi(e_T)] \to [e_T, 1]$, the inverse of the kinetic function φ, which is also a monotone decreasing mapping.

The arguments presented above are summarized as follows:

THEOREM 3.1. (Construction of the Riemann solver) *Given left-hand and right-hand states u_l, u_r. Under the hypotheses (1.5) and the assumption* (3.4), *we have the following description of the Riemann solver that can be involved in a combination of rarefaction fans and shock waves, satisfying the kinetic relation* (3.6) *(for nonclassical shocks), and the condition* (C)*:*

Case 1: $u_l \in (-\infty, e_T)$.
- *If $u_r \in (-\infty, u_l)$, the solution is a single classical shock.*
- *If $u_r \in (u_l, 0]$, the solution is a single rarefaction wave.*
- *If $u_r \in (0, \varphi(e_T)]$, the solution is the composite of a rarefaction wave connecting u_l to $u_* := \varphi^{-1}(u_r)$ followed by a nonclassical shock from u_* to u_r.*
- *If $u_r \in (\varphi(e_T), +\infty)$, the solution consists of three parts: A rarefaction wave from u_l to e followed by a nonclassical shock from e to $\varphi(e_T)$, followed by a rarefaction wave from $\varphi(e_T)$ to u_r.*

Case 2: $u_l \in [e_T, 0)$.
- *If $u_r \in (-\infty, u_l)$, the solution is a single classical shock.*
- *If $u_r \in (u_l, 0]$, the solution is a single rarefaction wave.*
- *If $u_r \in (0, \varphi(u_l)]$, the solution is the composite of a rarefaction wave from u_l to $u_* := \varphi^{-1}(u_r)$ followed by a nonclassical shock from u_* to u_r.*
- *If $u_l \in [e_T, \varphi^\natural(\varphi(e_T)))$ and $u_r \in (\varphi(u_l), \varphi(e_T))$, then the solution consists of a classical shock from u_l to $u^* := \varphi^{-1}(u_r)$ followed by a nonclassical shock from u^* to u_r.*
- *If $u_l \in [e_T, \varphi^\natural(\varphi(e_T)))$ and $u_r \in (\varphi(e_T), +\infty)$, the solution consists of three waves: A classical shock from u_l to e followed by a nonclassical shock from e to $\varphi(e_T)$, followed by a rarefaction wave from $\varphi(e_T)$ to u_r.*
- *If $u_l \in [\varphi^\natural(\varphi(e_T)), 0)$ and $u_r \in (\varphi(u_l), \varphi^{-\natural}(u_l))$, the solution consists of the classical shock from u_l to $u^* := \varphi^{-1}(u_r)$ followed by a nonclassical shock from u^* to u_r.*

- *If* $u_l \in [\varphi^\natural(\varphi(e_T)), 0)$ *and* $u_r \in [\varphi^{-\natural}(u_l), \psi^\natural(u_l)]$*, the solution is a classical shock from* u_l *to* u^* *followed by a nonclassical shock from* u^* *to* u_r *if (3.5) holds, or else a single classical shock.*
- *If* $u_l \in [\varphi^\natural(\varphi(e_T)), 0)$ *and* $u_r \in (\psi^\natural(u_l), +\infty)$*, the solution consists of a classical shock from* u_l *to* $\psi^\natural(u_l)$ *followed by a rarefaction wave from* $\psi^\natural(u_l)$ *to* u_r*.*

Case 3: $u_l \in [0, 1)$*.*

- *If* $u_r \in [0, +\infty)$*, the solution is classical (Section 2).*
- *If* $u_r \in [\varphi(u_l), 0]$*, the solution consists of the rarefaction wave from* u_l *to* $u^* := \varphi^{-1}(u_r)$ *followed by a nonclassical shock from* u^* *to* u_r*.*
- *If* $u_r \in [\varphi^{-\natural}(u_l), \varphi(u_l))$*, the solution consists of a classical shock from* u_l *to* $u_* := \varphi^{-1}(u_r)$ *followed by a nonclassical shock from* u_* *to* u_r*.*
- *If* $u_r \in [\varphi^{-\natural}(u_l), a)$*, the solution consists of the classical shock wave from* u_l *to* $u^* := \varphi(u_r)$ *followed by a nonclassical shock from* u^* *to* u_r *provided (4.3) holds, or else a single classical shock.*
- *The states* $u_r \in (-\infty, a]$ *are reached by a single classical shock.*

Case 4: $u_l \in [1, +\infty)$*.*
The construction is classical (Section 2).

3.2. Riemann solver relying on jumps crossing the inflection point $u = 1$**.** In this subsection, we will provide a Riemann solver using only nonclassical shocks crossing through the inflection point $u = 1$. As the behavior of the graph of f changes across this point from concavity to convexity, another condition will be placed instead of the convex-concave condition (3.2).

Let a function ψ be given

$$\psi : [a, b] \to [a, b]. \tag{3.10}$$

The function ψ will be called a *kinetic function corresponding to the inflection point* $u = 1$ if there exists a neighborhood Ω_1 of $u = 1$ such that in Ω_1, the function ψ fulfils the following requirements

(iii) The function ψ is monotone decreasing in $[a, b]$, $\psi(u) \geq \varphi^\natural(u)$, $\psi(u)$ lies between $\psi^\natural(u)$ and $\psi^{-\natural}(u), \forall u \in [a, b]$ in the sense that

$$\begin{aligned} &\psi^{-\natural}(u) > \psi(u) > \psi^\natural(u), \quad \forall u < 1, \\ &\psi^{-\natural}(u) < \psi(u) < \psi^\natural(u), \quad \forall u > 1, \\ &\psi(1) = \psi^\natural(1) = \psi^{-\natural}(1) = 1; \end{aligned}$$

iv) and the *concave-convex* condition

$$|\psi \circ \psi(u) - 1| < |u - 1|, \quad \forall u \in \Omega_1. \tag{3.11}$$

Similarly, there is a unique point $e^T \in (1, b)$ such that

$$\psi(e^T) = \varphi^\natural(e^T). \tag{3.12}$$

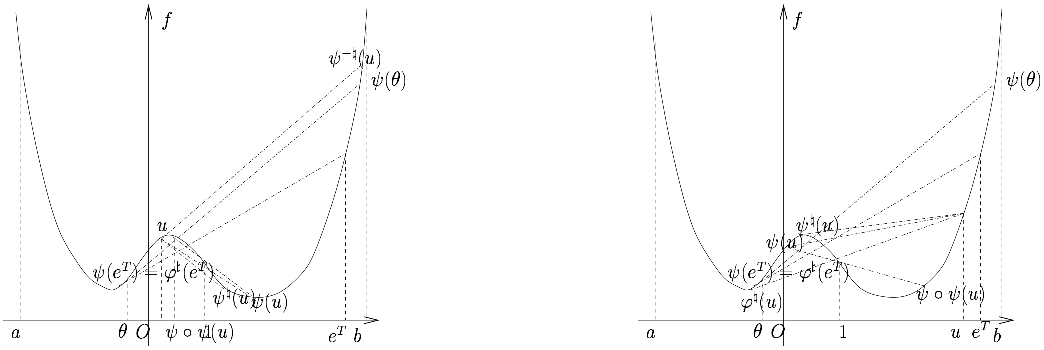

FIG. 3.2. *Kinetic Function* ψ

Assume there exists a point $\theta \in (c, 0)$ such that

$$\varphi^{\natural}(\psi(\theta)) = \theta. \tag{3.13}$$

From now on, we set definitively for the domain of the kinetic function

$$\Omega_1 = [\theta, e^T]. \tag{3.14}$$

For any non-classical shock between a given left-hand state u_0 and a given right-hand state u_1, kinetic relation for the coming construction is the requirement that

$$u_1 = \psi(u_0). \tag{3.15}$$

So, we begin to construct the Riemann solver, postulating the condition (C) in the previous subsection.

Assume first that $u_l \in (-\infty, \theta)$. A single classical shock can jump to any $u_r \in (-\infty, u_l)$. A single rarefaction wave can connect u_l from the left to any $u_r \in [u_l, 0]$ from the right. If $u_r \in (0, \varphi^{-\natural}(\theta)]$, then $\varphi^{\natural}(u_r) \in [\theta, 0)$, the solution thus is a rarefaction wave from u_l to $\varphi^{\natural}(u_r)$ followed by a classical shock from $\varphi^{\natural}(u_r$ to u_r. If now $u_r \in (\varphi^{-\natural}(\theta), \psi(\theta))$, the solution consists of a rarefaction wave from u_l to θ, followed by a non-classical shock from θ to $\psi(\theta)$, then followed by a classical shock from $\psi(\theta)$ to u_r. If $u_r \in [\psi(\theta), +\infty)$, the solution is a composite of a rarefaction wave from u_l to θ, followed by a non-classical shock from θ to $\psi(\theta)$, then followed by a rareffaction wave from $\psi(\theta)$ to u_r.

Second, let $u_l \in [\theta, 0]$. A single classical shock can jump to any $u_r \in (-\infty, u_l)$. A single rarefaction wave can connect u_l to any $u_r \in [u_l, 0]$. If $u_r \in (0, \varphi^{-\natural}(u_l)]$, then $\varphi^{\natural}(u_r) \in [u_l, 0)$, and therefore the solution is a rarefaction wave from u_l to $\varphi^{\natural}(u_r)$ followed by a classical shock from $\varphi^{\natural}(u_r)$ to u_r. If now $u_r \in (\varphi^{-\natural}(u_l), \psi^{\sharp}(u_l, \psi(u_l))]$, the solution is a single classical shock. If $u_r \in (\psi^{\sharp}(u_l, \psi(u_l)), \psi(u_l))$, then the solution is a non-classical shock from u_l to $\psi(u_l)$ followed by a classical shock from $\psi(u_l)$ to u_r. If $u_r \in [\psi(u_l), +\infty)$, then the solution is composed from a non-classical shock from u_l to $\psi(u_l)$ followed by a rarefaction wave from $\psi(u_l)$ to u_r.

Third, let $u_l \in (0, 1)$. A single classical shock can arrive at any $u_r \in (-\infty, \varphi^{-\natural}(u_l)]$. If $u_r \in (\varphi^{-\natural}(u_l), 0]$, then $\varphi^{\natural}(u_r) \in [0, u_l)$. The solution is thus a rarefaction wave from u_l to $\varphi^{\natural}(u_r)$ attached by a classical shock from $\varphi^{\natural}(u_r)$ to u_r. If $u_r \in (0, u_l]$, the solution is a single rarefaction wave. A single classical shock can arrive at any $u_r \in (u_l, \psi^{\sharp}(u_l, \psi(u_l))]$. If $u_r \in (\psi^{\sharp}(u_l, \psi(u_l)), \psi(u_l))$, then the solution is a composite of a non-classical shock from u_l to $\psi(u_l)$ followed by a classical shock from $\psi(u_l)$ to u_r. If $u_r \in [\psi(u_l), +\infty)$, then the solution is combined from a non-classical shock from u_l to $\psi(u_l)$ followed by a rarefaction wave from $\psi(u_l)$ to u_r.

Fourth, assume $u_l \in (1, \psi^{-1}(0))$. By the monotony, we have

$$\psi(u_l) > 0.$$

If $u_r \in [u_l, +\infty)$, then the solution is a rarefaction wave. A single classical shock can jump from u_l to any $u_r \in [\psi^{\sharp}(u_l, \psi(u_l)), u_l)$. If $u_r \in (\psi(u_l), \psi^{\sharp}(u_l, \psi(u_l)))$, then the solution is combined from two shocks: a non-classical shock from u_l to $\psi(u_l)$ followed by a classical one from $\psi(u_l)$ to u_r. If $u_r \in [0, \psi(u_l)]$, then the solution is a non-classical shock from u_l to $\psi(u_l)$ followed by a rarefaction wave from $\psi(u_l)$ to u_r. If now $u_r \in (\varphi^{-\natural}(\psi(u_l)), 0)$, then $\varphi^{\natural}(u_r) \in (0, \psi(u_l))$. The solution is thus a non-classical shock from u_l to $\psi(u_l)$ followed by a rarefaction wave from $\psi(u_l)$ to $\varphi^{\natural}(u_r)$ attached by a classical shock from $\varphi^{\natural}(u_r)$ to u_r. If $u_r \in [\psi^{\sharp}(u_l, \psi(u_l)), \varphi^{-\natural}(\psi(u_l))]$, then the solution is a non-classical shock from u_l to $\psi(u_l)$ followed by a classical shock from $\psi(u_l)$ to u_r. If $u_r \in (-\infty, \psi^{\sharp}(u_l, \psi(u_l)))$, then no non-classical shocks can be involved in the construction. We thus use the classical construction in Section 2 in this interval. The discontinuity in this regime is

$$u_l \in (1, \psi^{-1}(0)), \quad u_r = \psi^{\sharp}(u_l, \psi(u_l)). \tag{3.16}$$

Fifth, let $u_l \in [\psi^{-1}(0), e^T]$. The monotony of ψ yields

$$\psi(u_l) \leq 0.$$

A single rarefaction wave can connect u_l to any $u_r \in [u_l, +\infty)$. A single classical shock can jump from u_l to any $u_r \in [\psi^{\sharp}(u_l, \psi(u_l)), u_l)$. If $u_r \in [\varphi^{-\natural}(\psi(u_l)), \psi^{\sharp}(u_l, \psi(u_l)))$, then the solution is combined from two shocks: a non-classical shock from u_l to $\psi(u_l)$ followed by a classical one from $\psi(u_l)$ to u_r. If $u_r \in (0, \varphi^{-\natural}(\psi(u_l)))$, then $\varphi^{\natural}(u_r) \in (\psi(u_l), 0)$. The solution is therefore a composite of a non-classical shock from u_l to $\psi(u_l)$ followed by a rarefaction wave from $\psi(u_l)$ to $\varphi^{\natural}(u_r)$, then attached by a classical shock from $\varphi^{\natural}(u_r)$ to u_r. If $u_r \in [\psi(u_l), 0]$, then the solution is a non-classical shock from u_l to $\psi(u_l)$ followed by a rarefaction wave from $\psi(u_l)$ to u_r. If $u_r \in [\psi^{\sharp}(u_l, \psi(u_l)), \psi(u_l))$, then the solution is a non-classical shock from u_l to $\psi(u_l)$ followed by a classical one from $\psi(u_l)$ to u_r. If $u_r \in (-\infty, \psi^{\sharp}(u_l, \psi(u_l)))$, then there are no non-classical shocks and we have a situation similar to the previous one. In this construction, we have a discontinuity sharinf the same formula for u_r but $u_l \in [\psi^{-1}(0), e^T]$ instead. Combining this argument and (3.16), we obtain the curve of discontinuity of the construction

$$\{u_l \in (1, e^T], \quad u_r = \psi^\sharp(u_l, \psi(u_l))\}. \tag{3.17}$$

Finally, let $u_l \in (e^T, +\infty)$. In this case we have no non-classical shocks and we use the classical construction as well.

The above arguments can be summarized in the following theorem

THEOREM 3.2. *Given the initial Riemann data (u_l, u_r). Under the hypotheses (1.5) and the condition (C), the Riemann problem (1.4) admits a unique self-similar soltion made of rarefaction waves, classical shocks and non-classical shocks satisfying the kinetic relation (3.15). The Riemann solver is described by*

Case 1: $u_l \in (-\infty, \theta)$.

- *If $u_r \in (-\infty, u_l)$, the solution is a single classical shock.*
- *If $u_r \in [u_l, 0]$, the solution is a single rarefaction wave.*
- *If $u_r \in (0, \varphi^{-\natural}(\theta)]$, the solution is a rarefaction wave from u_l to $\varphi^\natural(u_r)$ followed by a classical shock from $\varphi^\natural(u_r$ to u_r.*
- *If $u_r \in (\varphi^{-\natural}(\theta), \psi(\theta))$, the solution is a composite of a rarefaction wave from u_l to θ, followed by a non-classical shock from θ to $\psi(\theta)$, followed by a classical shock from $\psi(\theta)$ to u_r. If $u_r \in [\psi(\theta), +\infty)$, the solution is a rarefaction wave from u_l to θ, followed by a non-classical shock from θ to $\psi(\theta)$, then followed by a rarefaction wave from $\psi(\theta)$ to u_r.*

Case 2: $u_l \in [\theta, 0]$.

- *If $u_r \in (-\infty, u_l)$, then the solution is a classical shock.*
- *If $u_r \in [u_l, 0]$, the solution is a single rarefaction wave.*
- *If $u_r \in (0, \varphi^{-\natural}(u_l)]$, the solution is a rarefaction wave from u_l to $\varphi^\natural(u_r)$ followed by a classical shock from $\varphi^\natural(u_r)$ to u_r.*
- *If $u_r \in (\varphi^{-\natural}(u_l), \psi^\sharp(u_l, \psi(u_l))]$, the solution is a single classical shock.*
- *If $u_r \in (\psi^\sharp(u_l, \psi(u_l)), \psi(u_l))$, the solution is a non-classical shock from u_l to $\psi(u_l)$ followed by a classical shock from $\psi(u_l)$ to u_r.*
- *If $u_r \in [\psi(u_l), +\infty)$, the solution is a composite of a non-classical shock from u_l to $\psi(u_l)$ followed by a rarefaction wave from $\psi(u_l)$ to u_r.*

Case 3: $u_l \in (0, 1)$.

If $u_r \in (-\infty, \varphi^{-\natural}(u_l)]$, the solution is a classical shock.

- *If $u_r \in (\varphi^{-\natural}(u_l), 0]$, the solution is a rarefaction wave from u_l to $\varphi^\natural(u_r)$ attached by a classical shock from $\varphi^\natural(u_r)$ to u_r.*
- *If $u_r \in (0, u_l]$, the solution is a single rarefaction wave.*
- *If $u_r \in (u_l, \psi^\sharp(u_l, \psi(u_l))]$, the solution is a single classical shock.*
- *If $u_r \in (\psi^\sharp(u_l, \psi(u_l)), \psi(u_l))$, the solution is a composite of a non-classical shock from u_l to $\psi(u_l)$ followed by a classical shock from $\psi(u_l)$ to u_r.*

- *If* $u_r \in [\psi(u_l), +\infty)$, *then the solution is a composite of a non-classical shock from* u_l *to* $\psi(u_l)$ *followed by a rarefaction wave from* $\psi(u_l)$ *to* u_r.

Case 4: $u_l \in (1, \psi^{-1}(0))$.

- *If* $u_r \in [u_l, +\infty)$, *the solution is a rarefaction wave.*
- *If* $u_r \in [\psi^\sharp(u_l, \psi(u_l)), u_l)$, *the solution is a single classical shock.*
- *If* $u_r \in (\psi(u_l), \psi^\sharp(u_l, \psi(u_l)))$, *the solution is a composite of two shocks: a non-classical shock from* u_l *to* $\psi(u_l)$ *followed by a classical one from* $\psi(u_l)$ *to* u_r.
- *If* $u_r \in [0, \psi(u_l)]$, *the solution is a non-classical shock from* u_l *to* $\psi(u_l)$ *followed by a rarefaction wave from* $\psi(u_l)$ *to* u_r.
- *If* $u_r \in (\varphi^{-\natural}(\psi(u_l)), 0)$, *the solution is a composite of a non-classical shock from* u_l *to* $\psi(u_l)$ *followed by a rarefaction wave from* $\psi(u_l)$ *to* $\varphi^\natural(u_r)$ *attached by a classical shock from* $\varphi^\natural(u_r)$ *to* u_r.
- *If* $u_r \in [\psi^\sharp(u_l, \psi(u_l)), \varphi^{-\natural}(\psi(u_l))]$, *the solution is a non-classical shock from* u_l *to* $\psi(u_l)$ *followed by a classical shock from* $\psi(u_l)$ *to* u_r.
- *If* $u_r \in (-\infty, \psi^\sharp(u_l, \psi(u_l)))$, *then the construction is classical (Section 2).*

Case 5: $u_l \in [\psi^{-1}(0), e^T]$.

- *If* $u_r \in [u_l, +\infty)$, *the solution is a single rarefaction wave.*
- *If* $u_r \in [\psi^\sharp(u_l, \psi(u_l)), u_l)$, *the solution is a single classical shock.*
- *If* $u_r \in [\varphi^{-\natural}(\psi(u_l)), \psi^\sharp(u_l, \psi(u_l)))$, *the solution is a composite of two shocks: a non-classical shock from* u_l *to* $\psi(u_l)$ *followed by a classical one from* $\psi(u_l)$ *to* u_r.
- *If* $u_r \in (0, \varphi^{-\natural}(\psi(u_l)))$, *the solution is a composite of three elementary waves: a non-classical shock from* u_l *to* $\psi(u_l)$ *followed by a rarefaction wave from* $\psi(u_l)$ *to* $\varphi^\natural(u_r)$, *then attached by a classical shock from* $\varphi^\natural(u_r)$ *to* u_r.
- *If* $u_r \in [\psi(u_l), 0]$, *the solution is a non-classical shock from* u_l *to* $\psi(u_l)$ *followed by a rarefaction wave from* $\psi(u_l)$ *to* u_r.
- *If* $u_r \in [\psi^\sharp(u_l, \psi(u_l)), \psi(u_l))$, *then the solution is a non-classical shock from* u_l *to* $\psi(u_l)$ *followed by a classical one from* $\psi(u_l)$ *to* u_r.
- *If* $u_r \in (-\infty, \psi^\sharp(u_l, \psi(u_l)))$, *then the construction is classical.*

Case 6: $u_l \in (e^T, +\infty)$, *the construction is classical.*

The curve of discontinuity is

$$\{u_l \in (1, e^T], \quad u_r = \psi^\sharp(u_l, \psi(u_l))\} \subset \boldsymbol{R}^2.$$

4. Non-Classical Riemann Solver Using two Kinetic Relations. In this section, we discuss the Riemann solver to the problem (1.4) using two kinetic relations for non-classical shock-waves between two phases. It turns out that even under the condition (C), non-uniqueness appears. A stronger condition is imposed to guarantee there is a unique choice of non-classical

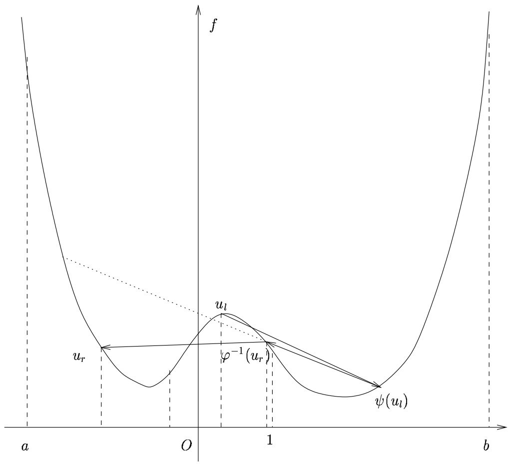

FIG. 4.1. *A 3-shock solution:* $N(u_l, \psi(u_l)) - C(\psi(u_l), \varphi^{-1}(u_r)) - N(\varphi^{-1}(u_r), u_r)$.

shocks. As expected, the unique Riemann solution does not depend continuously globally on the Riemann data.

Let us first point out several circumstances in which there are distinct choices of non-classical solutions adaptable to the condition (C). Firtly, assume that $u_l \in (1, \psi^{-1}(0))$, then $\psi(u_l) \in (0, 1)$. Therefore,

$$\varphi(\psi(u_l)) \in (\varphi^{-\natural}(\psi(u_l)), \varphi^{\natural}(\psi(u_l))) \subset (\varphi^{-\natural}(\psi(u_l)), 0).$$

If we take

$$u_r = \varphi(\psi(u_l)),$$

then we obtain a solution contains two nonclassical shocks: one nonclassical shock corresponding to the kinetic function ϕ from u_l to $\psi(u_l)$, followed by one nonclassical shock corresponding to the kinetic function ψ from $\psi(u_l)$ to $\varphi(\psi(u_l))$. However, as derived from the construction in the subsection 3.2 that in this case, we obtained a non-classical solution containing a non-classical shock corresponding to the kinetic function ψ: one nonclassical shock from u_l to $\psi(u_l)$ followed by a rarefaction wave from $\psi(u_l)$ to $\varphi^{-\natural}(u_r)$, attached by a classical shock from $\varphi^{-\natural}(u_r)$ to u_r. This illustrates the co-existence of two nonclassical solutions, one contains more nonclassical shocks than the other. (See Figure 4.1).

Secondly, assume now

$$u_l \in [\theta, 0], \quad \text{and} \quad u_r \in (\psi^{\sharp}(u_l, \psi(u_l)), 1).$$

According to the description in Section 3, we could have two nonclassical Riemann solutions, each of them contain one nonclassical shock. Precisely, the solution would be

- either a classical shock corresponding to the kinetic function φ from u_l to $\varphi^{-1}(u_r)$, followed by a nonclassical shock from $\varphi^{-1}(u_r)$ to u_r;
- or a nonclassical shock corresponding to the kinetic function ψ from u_l to $\psi(u_l)$ followed by a classical shock from $\psi(u_l)$ to u_r.

This is an example of the co-existence of nonclassical solutions including the same number of nonclassical shocks.

For short, in the sequel we will name a $0-$ *shock* (or a $1-$ *shock*) is a nonclasscial shock corresponding to the kinetic function φ (the kinetic function ψ, resp.).

In order to select a unique solution, at least we must avoid the above circumstances. In the following, we need a more restrictive procedure than (C). That is the procedure

(P) – A classical solution is understood to contain zero nonclassical shock.

– Nonclassical shocks are preferred whenever available in the extended sense that: if a solver $R_1(u_l, u_r)$ contains m nonclassical shocks, and a solver $R_2(u_l, u_r)$ contains n nonclassical shocks with $m > n$, then R_1 excludes R_2.

– If the left-hand state belongs to the phase E_I, then the 0-shocks are preferred than the 1-shocks in the sense that: if $R_1(u_l, u_r)$ and $R_2(u_l, u_r)$ contain the same total number of nonclassical shocks, and $R_1(u_l, u_r)$ contain m 0-shocks and $R_2(u_l, u_r)$ contain n 0-shocks with $m > n$, then R_1 excludes R_2. Similarly, if the left-hand state belongs to the phase E_{III}, then 1-shocks are preferred than the 0-shocks.

For the construction, we first make it clear that a $1-$ *shock* can not follow a $0-$ *shock*.

Proposition 4.1. *In any Riemann solution, a* $1-$ *shock can not follow a* $0-$ *shock.*

Proof. Let the states u_0, u_1, u_2 be given. Denote $N_0(u_0, u_1)$ is the 0-shock from u_0 to u_1 and $N_1(u_1, u_2)$ is the 1-shock from u_1 to u_2. That is to say

$$u_1 = \varphi(u_0), \quad \text{and} \quad u_2 = \psi(u_1).$$

In order to for N_1 to follow N_0 we must have the condition on shock speeds:

$$s(u_1, u_2) > s(u_0, u_1). \tag{4.1}$$

By the definition of kinetic functions, the shock speed $s(u_1, u_2)$ has to be smaller than the slope of the tangent at u_1, which is greater than the shock speed $s(u_0, u_1)$. This contradicts with the condition (4.1). The proposition is proved. □

Based on the procedure (P), we proceed now to construct the Riemann solution. First, assume that $u_l \in (-\infty, \min\{\theta, \varphi^\natural(\varphi(e_T))\}]$. Since u_l is out of the domain of the kinetic function ψ and, as described in the subsection 3.1, any $u_r \in (0, +\infty)$ can be arrived at by a solution contain one 0-shock. By virtue of the procedure (P), we thus use the construction in the subsection 3.1 for this interval.

Second, let $u_l \in (\min\{\theta, \varphi^\natural(\varphi(e_T)), 0)$. The construction of the subsection 3.1 is valid for $u_r < \varphi^{-\natural}(u_l)$. If $u_r \in [\varphi^{-\natural}(u_l), \varphi(e_T)]$, the solution can be a classical shock connecting u_l to u^* followed by a nonclassical shock from u^* to u_r provided (3.8) holds. If (3.8) fails, then the construction in the subsection 3.2 can be applied here: if $u_l \le \theta$ we have a rarefaction wave from u_l to θ followed by a 1-shock from θ to $\psi(\theta)$, then followed by a classical shock from $\psi(\theta)$ to u_r, if $u_l > \theta$, then we have a 1-shock from u_l to $\psi(u_l)$ followed by a classical one from $\psi(u_l)$ to u_r. For $u_r \in (\varphi(e_T), +\infty)$, if (3.9) holds then we use the construction in the subsection 3.1 to cover 0-shocks, else we use the one in the subsection 3.2 to cover 1-shocks or classical construction.

Third, let $u_l \in [0, 1]$. We know from Proposition 4.1 that 1-shocks can not follow 0-shocks, so we need only find the possibility of a 0-shock following a 1-shock. The interval $[0, 1]$ can be separated by two regions

$$\begin{aligned} &\mathcal{A} := \{u \in [0,1] : \psi^\sharp(u, \psi(u)) < 1\} \quad \text{relatively open in} \quad [0,1], \\ &\mathcal{A}^C = [0,1] \setminus \mathcal{A}. \end{aligned} \tag{4.2}$$

The relatively open set $\mathcal{A}$ is thus a union of certain relatively open subintervals of the interval $[0, 1]$. For any $u \in \mathcal{A}$, there corresponds a set defined by

$$\mathcal{B} := \{v \in (a, 0) : v < \varphi(\psi^\sharp(u, \psi(u))), \quad \text{and} \quad v > \varphi^\sharp(\psi(u), \varphi^{-1}(v))\}. \tag{4.3}$$

The set $\mathcal{B}$ is an open subset of $\boldsymbol{R}$. By definition, given any left-hand state $u_r \in \mathcal{B}$, the Riemann solution for the initial datum (u_l, u_r) is a three-jump wave: first a 1-shock from u_l to $\psi(u_l)$, followed by a classical jump from $\psi(u_l)$ to $\varphi^{-1}(u_r)$, then followed by a 0-shock from $\varphi^{-1}(u_r)$ to u_r. For $u_r \in (-\infty, u_l] \setminus \mathcal{B}$, no 1-shocks to be followed by a 0-shock, so we use the construction in the subsection 3.1. The states $u_r \in (u_l, +\infty)$ can be reached by the construction in the subsection 3.2, as no 0-shocks are available.

Fourth, assume $u_l \in (1, \psi^{-1}(0)]$. By the monotony, we have

$$\psi(u_l) > 0. \tag{4.4}$$

Due to (4.4) the right-hand states $u_r \in [0, +\infty)$ should be involved with 1-shocks and the construction is the one of the subsection 3.2. If $u_r \in (\varphi(\psi(u_l)), 0)$, then the solution is a 1-shock from u_l to $\psi(u_l)$ followed by a rarefaction wave from $\psi(u_l)$ to $\varphi^{-1}(u_r)$, then followed by a 0-shock from $\varphi^{-1}(u_r)$ to u_r. If $u_r \in (\varphi^\sharp(u_l, \psi(u_l)),$
$\varphi(\psi(u_l))]$, then $\varphi^{-1}(u_r) \in (\psi(u_l), 1)$. The solution is a 1-shock from u_l to $\psi(u_l)$ followed by a classical shock from $\psi(u_l)$ to $\varphi^{-1}(u_r)$, then followed by a 0-shock from $\varphi^{-1}(u_r)$ to u_r iff

$$s(\psi(u_l), \varphi^{-1}(u_r)) < s(\varphi^{-1}(u_r), u_r). \tag{4.5}$$

If (4.5) fails, then no 0-shocks are involved in the construction and we use the one in the subsection 3.2. If now $u_r \in (-\infty, \varphi^\sharp(u_l, \psi(u_l))]$, then the classical construction is invoked.

Fifth, let $u_l \in (\psi^{-1}(0), e^T]$, then

$$\psi(u_l) < 0. \tag{4.6}$$

Right-hand states $u_r \in [\psi^\sharp(u_l, \psi(u_l)), +\infty) \cup (-\infty, 0]$ can be arrived at as in the construction of the subsection 3.2. If $u_r \in (0, \varphi(\psi(u_l)))$, then the solution is a 1-shock from u_l to $\psi(u_l)$ followed by a rarefaction wave from $\psi(u_l)$ to $\varphi^{-1}(u_r)$ by virtue of (4.6), then followed by a 0-shock from $\varphi^{-1}(u_r)$ to u_r. If $u_r \in [\varphi(\psi(u_l)), \psi^\sharp(u_l, \psi(u_l)))$, then $\varphi^{-1}(u_r) \in (e_T, \psi(u_l))$. The solution is a 1-shock from u_l to $\psi(u_l)$ followed by a classical shock from $\psi(u_l)$ to $\varphi^{-1}(u_r)$, then followed by a 0-shock from $\varphi^{-1}(u_r)$ to u_r iff

$$s(\psi(u_l), \varphi^{-1}(u_r)) < s(\varphi^{-1}(u_r), u_r). \tag{4.7}$$

If (4.7) fails, then we use the one in the subsection 3.2.

Finally, if $u_l \in (e^T, +\infty)$, then the classical construction is valid.

Summarizing the above arguments, we arrive at the following theorem

THEOREM 4.2. *Given the initial Riemann data (u_l, u_r). Under the hypotheses (1.5), There exists a unique Riemann solution made of rarefaction waves, classical shocks and non-classical shocks satisfying the kinetic relations (3.6) and (3.15), and the selective procedure (P).*

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