Spectral Representation of Multiply Self-decomposable Stochastic Processes and Applications*

Nguyen Van Thu †

To Anh Dung, Duong Ton Dam‡, Nguyen Huu Thai §

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In the present paper we study multiply selfdecomposable probability measures (SDPM) and processes and prove their integral representations. Similarly, the multiple s-selfdecomposability case is treated. Our results extend some of known results due to Urbanik K.,Jurek., Rosinski J.and Rajput B.S. As an application we construct a Damped-mixed stable price process in option pricing.

1 Introduction, notation and preliminaries

The main aim of this paper is to prove that each multiply self-decomposable process (MSDP) on an Euclidean space admits a stochastic integral w.r.t.a MSD random measure (RM). Moreover, we will consider similar problems for multiply s-self-decomposable processes(MsSDP). Through the paper we shall denote by $\mathcal{X}$ a fixed d-dimensional (d=1,2,...) Euclidean space with the usual inner product $<,>$ and norm $\|.\|$.

Let $\mathcal{P}(\mathcal{X})$ denote the class of all probability measures (PM) on the σ-field $\mathcal{B}(\mathcal{X})$ of Borel subsets of $\mathcal{X}$ equipped with the weak convergence. Given a positive number

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†Department of Mathematics, International University, HCM City.

‡Department of Mathematics, University of natural sciences, HCM City.

§Department of Mathematics and Statistics, University of Economics HCM City.

c we define on $\mathcal{X}$ the following two families of mappings T_c and U_r as follows:

$$\begin{cases} T_r x = rx, \\ U_r = max(0, \|x\| - r)\frac{x}{\|x\|}, U_r(0) = 0. \end{cases} \tag{1}$$

Further, for a PM $\mu \in \mathcal{P}(\mathcal{X})$ and a mapping T on $\mathcal{X}$ let $T\mu$ denote the image of μ under T.

Recall (cf.Loéve [12] and Sato [25]) that a PM $\mu \in \mathcal{P}(\mathcal{X}$ is called SD if for each $0 < c < 1$ there exists a PM μ_c such that

$$\mu = T_c\mu * \mu_c \tag{2}$$

where $*$ denotes the ordinary convolution of PM's.

The concept of *shrinking SDPM* (shortly, s-SDPM) was introduced by Medgyessy [14]and studied by Jurek [3], [4], [6]. Namely, a PM μ is called s-SD if it is ID and for each $0 < c < 1$ there exists a PM μ_c such that

$$\mu = U_c\mu^c * \mu_c \tag{3}$$

where the power is taken in the convolution sense.

It is known [3], [4], [6], [29], [17]that if μ is SD (resp., s-SD) then μ, μ_c are both ID. Let us denote by $ID(\mathcal{X})$ the class of all IDPM's on the space. The class of all SDPM's (resp., s-SDPM's) on $\mathcal{X}$ is denoted by $\mathcal{L}(\mathcal{X})$ (resp., $\mathcal{U}(\mathcal{X})$).

Let $\mathcal{L}_n(\mathcal{X}), n = 1, 2, ...$(resp., $\mathcal{U}_n(\mathcal{X}), n = 1, 2, ...$) denote the class of all n-times SDPM's (resp., n-times s-SDPM's) which were first introduced by Urbanik[1] [29] (resp.,Jurek[4] and then studied further by many other authors (cf., for example [4], [17], [25]...).

They are defined recursively as follows: A p.m. $\mu \in \mathcal{L}_n(\mathcal{X}), n = 2, 3, ...$ if and only if $\mu \in \mathcal{L}_1(\mathcal{X})$ and for each $c \in (0, 1)$ the component μ_c in (2) belongs to $\mathcal{L}_{n-1}(\mathcal{X})$.

It has been proved by Nguyen ([18], Proposition 1.1) that *a p.m.* $\mu \in \mathcal{L}_n(\mathcal{X}), n = 1, 2, ...,$ *if and only if, for every* $c \in (0, 1)$ *there exists a p.m.* $\nu := \mu_{c,n} \in ID(\mathcal{X})$ *such that the following equality holds:*

$$\mu = *_{k=0}^{\infty}(T_{c^k}\nu)^{*r_{k,n}} \tag{4}$$

where the power is taken in the convolution sense and, for n=1,2, ... ; k=0,1,2,... we put

$$r_{k,n} = \binom{n+k-1}{k}. \tag{5}$$

[1]It should be noted, that our notation $\mathcal{L}_n(\mathcal{X})$ used here and in references [17], [18] is other than that in Urbanik and other Authors [4], [25]. In particular, in our notation, $\mathcal{L}_1(\mathcal{X})$ denotes the set of all SDPM's on $\mathcal{X}$ while in [4], [25] this class was denoted by $\mathcal{L}_0(\mathcal{X})$.

The formulas (3) and (4) lead to the following interpolation of classes $\mathcal{L}_n(\mathcal{X})$ (cf. Nguyen [17]): For each $\alpha > 0$ we put

$$\binom{\alpha}{k} = \begin{cases} 1 & k = 0, \\ \alpha(\alpha-1)...(\alpha-k+1)/k! & k = 1, 2, ... \end{cases} \tag{6}$$

and introduce the class *α-times SDPM's*, shortly, $\alpha - SDPM's$ as the following:

Definition 1.1 (cf. Nguyen [19]) *A p.m. $\mu \in \mathcal{L}_\alpha(\mathcal{X}), \alpha > 0$, if and only if, for every $c \in (0,1)$ there exists a p.m. $\nu := \mu_{c,\alpha} \in ID(\mathcal{X})$ such that the following equality holds:*

$$\mu = *_{k=0}^{\infty}(T_{c^k}\nu)^{r_{k,\alpha}} \tag{7}$$

where, the power is taken in the convolution sense and, for $\alpha > 0, k = 0, 1, 2, ...$ we put

$$r_{k,\alpha} = \binom{\alpha + k - 1}{k} \tag{8}$$

It should be noted [17] that *the infinite convolution (8) is weakly convergent if and only if*

$$\int_{\mathcal{X}} log^{\alpha}(1 + \|x\|)\nu(dx) < \infty \tag{9}$$

In the sequel, we shall denote by $ID_{log^\alpha}(\mathcal{X})$ the subclass of $ID(\mathcal{X})$ of all distributions for which the condition (9) is satisfied.

Now, let us quote the following important integral representation for SDPM's due to Vervaat-Jurek [5]:

Theorem 1.1 (Jurek-Vervaat) *A p.m. $\mu \in \mathcal{L}_1(\mathcal{X})$ if and only if there exists a $\mathcal{X}$-valued Lévy process $\{X(.)\}$ of the class ID_{log} such that*

$$\mu \stackrel{d}{=} \int_0^{\infty} exp(-t)X(dt) \tag{10}$$

The integrator $\{X(.)\}$ is called the *background driving Lévy process* (shortly, BDLP) of μ (cf.Jurek [5, 6])and the r.v. X(1) is called BD r.v.. Further, Nguyen N.H.[16], obtained the following pretty generalization of Theorem 1.1 to the case of $\alpha-$SDPM's for each $\alpha > 0$.

Theorem 1.2 (Nguyen N.H [16]) *A p.m. $\mu \in \mathcal{L}_\alpha(\mathcal{X})$ if and only if there exists a $\mathcal{X}$-valued Lévy process $\{X_\alpha(t)\}$ of the class $ID_{log^\alpha}(\mathcal{X})$ such that*

$$\mu \stackrel{d}{=} \int_0^{\infty} exp(-t^{\frac{1}{\alpha}})X_\alpha(dt) \tag{11}$$

In the sequel we shall need the following representation of ch.f.'s of ID and MSDPM's on $\mathcal{X}$:

Theorem 1.3 (cf[20], [24]) *A p.m. μ is ID if and only if its ch.f. $\hat{\mu}(y), y \in \mathcal{X}$ is of the unique form:*

$$\begin{cases} -log\hat{\mu}(y) = i < z, y > + < \Sigma y, y > \\ -\int_{\mathcal{X}} (e^{i<y,x>} - 1 - i\tau(x))M(dx) \end{cases}$$

where $z \in \mathcal{X}$ is fixed; Σ is a quadratic form on $\mathcal{X}$ and M is a Lévy measure on $\mathcal{X}$ characterized by the property that $M(0) = 0$, M is finite ouside of very neighberhood of the origin and

$$\int_{U_1} \frac{\|x\|^2}{1 + \|x\|^2} M(dx) < \infty;$$

the function $\tau(x)$ is defined by

$$\tau(x) = \begin{cases} \|x\| & x \in U_1; \\ 1 & \|x\| > 1, \end{cases}$$

U_1 being the closed unit Ball in $\mathcal{X}$.
In what follows, if μ is ID with the ch.f. given by (11) then we will identify it with the triple $[z, \Sigma, M]$. Thus,we have

Theorem 1.4 (Nguyen [19],Theorem 2.4) *A p.m. $\mu \in \mathcal{L}_\alpha(\mathcal{X}), \alpha > 0$ if and only if $\mu = [z, \Sigma, M]$, where z, Σ are the same as in Theorem (1.2) and the Lévy's measure M is given by*

$$M(A) = \begin{cases} \int_{\mathcal{X}} v_\alpha(x) \\ (\int_0^\infty \chi_A(e^{-u}x)u^{\alpha-1}du)m(dx) \end{cases} \tag{12}$$

where m is a finite measure on $\mathcal{X}$ vanishing at the origin; A is a Borel subset of the real line separated from 0; the weight function $v_\alpha(x)$ is defined by

$$v_\alpha^{-1}(x) = \int_0^\infty \frac{e^{-2t}x^2}{1 + e^{-2t}x^2} t^{\alpha-1} dt \tag{13}$$

Theorem 1.5 (Nguyen[17],[19]) *A p.m. μ is mixed-stable i.e. $\mu \in \mathcal{L}_\infty(\mathcal{X})$ if and only if $\mu = [z, \Sigma, M]$, where z, Σ are the same as in Theorem (1.4) and the Lévy's measure M of μ is given by*

$$M(A) = \int_{V_1} \int_0^\infty \chi_A(tx) \frac{dt}{t^2|x| + 1} h(x)\nu(dx) \tag{14}$$

where ν is a PM on the open unit ball $V_1 := \{x \in \mathcal{X} : \|x\| < 1\}$ and $h(x)$ is a nonnegative continuous weight function on V_1.

2 Mappings $\{T_c^{(\alpha)}\}$ and classes $\{\mathcal{L}_\alpha(\mathcal{X})\}$

In this section we introduce families of mappings $\{T_c^{(\alpha)}\}$,where $0 < c < 1; \alpha > 0$ acting on the whole class $ID(\mathcal{X})$ and show that they play the same role as mappings T_c in the definition of α-SDPM's. To begin with let us consider the following particular cases:

2.1 $\alpha = n = 1, 2, ...$

Let $\mu \in \mathcal{L}_n(\mathcal{X}), n = 1, 2, ...$ By Proposition 1.1 [17], for every $0 < c < 1$ the equation (**??**) holds. Putting

$$T_c^n \mu = *_{k=1}^{\infty} (T_{c^k} \nu)^{r_{k,n}} \tag{15}$$

and taking into account (**??**) we have

$$\mu = T_c^{(n)} \mu * \mu_{c,n} \tag{16}$$

Conversely, it is also true. Namely, by induction one can prove that if a PM μ satisfies equation (16) for each $0 < c < 1$ and for a PM $\mu_{c,n}$, then it belongs to $\mathcal{L}_n(\mathcal{X})$.

2.2 $0 < \alpha < 1$.

This case was treated in [17]. Namely, for such α the mapping $T_{c,\alpha}$ is defined in [17]. Then, by Theorem 2.1 [17], it follows that a PM μ belongs to $\mathcal{L}_\alpha(\mathcal{X})$ if and only if for every $0 < c < 1$ there exists a PM $\mu_{c,\alpha}$ such that

$$\mu = T_c^{\alpha} \mu * \mu_{c,\alpha} \tag{17}$$

2.3 The general case $\alpha > 0$:

It is easy to show that

$$1 = \sum_{k=1}^{\infty} (-1)^{k-1} r_{k,\alpha} = \sum_{k=1}^{\infty} |r_{k,\alpha}| \tag{18}$$

Consequently, the mapping $T_{c,\alpha} : ID(\mathcal{X}) \to ID(\mathcal{X})$ given by

$$T_{c,\alpha} \mu = *_{k=1}^{\infty} T_{c^k} \mu^{|r_{k,\alpha}|} \tag{19}$$

for any $0 < c < 1$ and $\alpha > 0$ is well-defined.
Furthermore, the following general theorem holds:

Theorem 2.1 *A PM* $\mu \in \mathcal{L}_\alpha(\mathcal{X}), \alpha > 0$, *if and only if for each* $0 < c < 1$ *there exists a PM* $\mu_{c,\alpha}$ *such that the equation (17) holds for each* $\alpha > 0$.

Proof. The "if" part is similar to the proof of Theorem 2.1 [17]. To prove the "only if" part one may assume that $\alpha = \beta + n, \quad \text{where} \quad 0 < \beta < 1, n = 1, 2, ...$ But it is clear by virtue of the cases 2.1 and 2.2 and by noticing that the mappings $T_{c,n}$ and $T_{c,\beta}$ commute with each other.

Theorem 2.2 (α-differentiability of α-SDPM's on $\mathcal{X}$) *For every* $\alpha > 0$ *and every PM* $\mu \in \mathcal{L}_\alpha(\mathcal{X})$ *there exists a weak limit, denoted by* $\mathcal{D}^\alpha \mu$, *which belongs to* $ID_{log^\alpha}(\mathcal{X})$ *and satisfies the equation*

$$\mathcal{D}^\alpha \mu = lim_{t \to 0}\, \mu_{c,\alpha}^{t^{-\alpha}} \tag{20}$$

where t = -log c, $\mu_{c,\alpha}$ *is as in (7) and (17).*

Proof. See Nguyen (Theorem 2.4 [17]).

Definition 2.1 (cf. Nguyen [17) *The limit measure* $D^\alpha \mu$ *in Theorem (2.2) is called the* α*-derivative of* μ.

The following Theorem is obvious:

Theorem 2.3 *For each* $\alpha > 0$ *the operator* D^α *stands for an algebraic isomorphism between* $\mathcal{L}_\alpha(\mathcal{X}) \quad \text{and} \quad ID_{log^\alpha}(\mathcal{X})$.

3 Mappings $\{U_c^{(\alpha)}\}$ and classes $\{\mathcal{U}_\alpha(\mathcal{X})\}$

Following verbatim the proof of cases 2.1, 2.2 and 2.3 we have the Theorem:

Theorem 3.1 *For any* $0 < c < 1 \quad \text{and} \quad \alpha > 0$ *and for every PM* $\mu \in ID(\mathcal{X})$ *we put*

$$U_{c,\alpha}\mu = *_{k=1}^{\infty} T_{c^k} \mu^{|\binom{\alpha}{k}| c^k} = *_{k=1}^{\infty} U_{c^k} \mu^{|\binom{\alpha}{k}|} \tag{21}$$

Then we get a mapping $U_{c,\alpha}$ *which stands for a well defined continuous isomorphism of the convolution algebra* $ID(\mathcal{X})$. *Moreover, restricted to* $ID(\mathcal{X})$, *it stands for an analogue of the shrinking mapping* U_c *in (1.1).*

Definition 3.1 *A PM* $\mu \in ID(\mathcal{X})$ *is said to be of the class* $U_\alpha(\mathcal{X}), \alpha > 0$, *or equivalently,* α*-s-SD, if for each* $0 < c < 1$ *the following formula holds:*

$$\mu = U_{c,\alpha}\mu * \mu_{c,\alpha} \tag{22}$$

for some PM $\mu_{c,\alpha} \in ID(\mathcal{X})$

From the above definition we have:

Theorem 3.2 *A PM* $\mu = [z, \Sigma, M] \in \mathcal{U}_\alpha(\mathcal{X}), \alpha > 0$ *if and only if the Lévy measure* M *satisfies the following condition:*

$$\Sigma_{k=0}^{\infty} | \binom{\alpha}{k} | c^k T_{c^k} M \geq 0 \tag{23}$$

for each $0 < c < 1$*, or, equivalently,*

$$[]\Sigma_{k=0}^{\infty} | \binom{\alpha}{k} | U_{c^k} M \geq 0 \tag{24}$$

Definition 3.2 (cf. Jurek[8]) *Given* $\alpha > 0$ *let* G_α *denote a Gamma r.v. with distribution* τ_α*. Let* $\mathcal{U}^{<\alpha>}$ *denote the class of all distributions of* $\int_{(0,1)} t dY_\rho(\tau_\alpha(t))$*, where* $Y_\rho(.)$ *is a Lévy process with* $\mathcal{L}(Y_\rho(1)) = \rho$*.*

By virtue of the above formulas (23) and (24) and Jurek [8],formula (29) we have the following theorem

Theorem 3.3 *The following equation hold:*

$$\mathcal{U}_\alpha(\mathcal{X}) = \mathcal{U}^{<\alpha>} \tag{25}$$

which shows that definitions 3.1 and 3.2 are equivalent.

4 Stochastic representation of classes $\mathcal{U}_\alpha(\mathcal{X})$

Definition 4.1 *Let* T *be a parameter set* Z *of all integers or* R *of all real numbers. A stochastic process* $X_t, t \in T$ *is said to be ID, stable, mixed-stable,* α*-SD,* α*-s-SD if for any* $t_1, t_2, ..., t_n \in T$ *and* $\lambda_1, \lambda_2, ..., \lambda_n, n = 1, 2, ...$ *the r.v.* $\Sigma_1^n \lambda_j X_{t_j}$ *is ID, stable, mixed-stable,* α*-SD,* α*-s-SD, respectively.*

Definition 4.2 *Let* $\Lambda = \{\Lambda(A) : A \in \mathcal{S}\}$ *be a real stochastic process defined on a probability space* $(\Omega, \mathcal{F}, \mathcal{P})$*, where* $\mathcal{S}$ *stands for a* σ*-ring of subsets of an arbitrary non-empty set* S *satisfying the following condition : There exists an increasing sequence* $S_n, n = 1, 2, ...$ *of sets in* S *with* $\bigcap_n S_n = S$*.*

We call Λ to be an *independently scattered random measure*(RM), if, for every sequence $\{A_n\}$ of disjoint sets in $\mathcal{S}$, the random variables $\Lambda(A_n), n = 1, 2, ...$ are independent, and , if $\cup_n A_n$ belongs to $\mathcal{S}$, then we also have

$$\Lambda(\cup_n A_n) = \Sigma_n \Lambda(A_n) \quad a.s.,$$

where the series is assumed to be convergent a.s. In addition, if for every $A \in \mathcal{S}$ the distribution of $\Lambda(A)$ is ID, stable, mixed-stable, MSD, respectively, then we say that it is an ID, stable, mixed-stable, MSD RM.
By virtue of Theorem 2 each r.v. $\Lambda(A), A \in S$ has the ch.f.

$$\begin{cases} -logEexp(it\Lambda(A) = it\nu_0(A) + \frac{1}{2}t^2\nu_1(A) \\ -\int_{-\infty}^{\infty}(e^{itx} - 1 - it\tau(x))F_A(dx), \end{cases} \tag{26}$$

where $t \in \mathbb{R}, A \in S$ and $-\infty < v_0(A) < \infty, 0 \leq v_1(A) < \infty$ and F_A is a Lévy measure on $\mathbb{R}$. Moreover, v_0 is a signed measure , v_1 a measure and F_A a Lévy measure.

The above representation implies the following

Theorem 4.1 (Raiput and Rosinski [24], Proposition 2.1) *The characteristic function (19) can be written in the unique form:*

$$Eexp(it\Lambda(A)) = exp(\int_A K(t,s)\lambda(ds)) \tag{27}$$

where $t \in \mathbb{R}, A \in S$ and

$$\begin{cases} K(t,s) = ita(s) - 1/2t^2\sigma^2(s) \\ +\int_A(e^{itx} - 1 - it\tau(x))\rho(s,dx), \end{cases} \tag{28}$$

with

$$a(s) = \frac{dv_0}{d\lambda}(s) \tag{29}$$

and

$$\sigma^2(s) = \frac{dv_1}{d\lambda}(s) \tag{30}$$

and ρ is given by Lemma 2.3 in [3]. Moreover, we have

$$|a(s)| + \int_{\mathbb{R}} min\{1, x^2\}\rho(s,dx) = 1 \quad a.e.[\lambda] \tag{31}$$

Definition 4.3 (Urbanik and Woyczynski [27]) (a) *If f is a simple function on $S, f = \Sigma_j x_j\chi_{A_j}, A_j \in S$ then we put, for each $A \in \sigma(S)$*

$$\int_A fd\Lambda = \Sigma_j\lambda(A \cap A_j)$$

(b) *A measurable function* $f : (S, \sigma(S)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})$ *is said to be* $\Lambda-$*integrable if there exists a sequence* $\{f_n\}$ *of simple functions as defined in (a) such that*

(i) $f_n \to f \quad a.e.[\lambda]$,

(ii) *For every* $A \in \sigma(S)$, *the sequence* $\{\int_A f_n d\Lambda\}$ *converges in prob., as* $n \to \infty$.

If f *is* $\Lambda-$*integrable, then we put*

$$\{\int_A f d\Lambda = P - lim_{n\to\infty} \int_A f_n d\Lambda,$$

where $\{f_n\}$ *satisfies (i) and (ii).*

Now, combining Theorems 3.2, 3.3, 4.1 we get the following:

Theorem 4.2 *Given* $\alpha > 0$, *let* $\Lambda(A), A \in \mathcal{S}$ *be a* $\alpha-$*s.d.r.m. Then, the characteristic function of* $\Lambda(A)$ *is of the unique form (20) where*

$$\begin{cases} K(t,s) = ita(s) - 1/2t^2\sigma^2(s) \\ + \int_A (e^{itx} - 1 - it\tau(x))\rho(s, dx) \end{cases} \tag{32}$$

with

$$a(s) = \frac{dv_0}{d\lambda}(s) \tag{33}$$

and

$$\sigma^2(s) = \frac{dv_1}{d\lambda}(s) \tag{34}$$

and ρ *is given by Lemma 2.3 in [3]. Moreover, we have*

$$|a(s)| + \int_{\mathbb{R}} min\{1, x^2\}\rho(s, dx) = 1 \quad a.e.[\lambda].$$

Proof. By virtue of (13) it follows that for any $A \in S$ and $t \in \mathbb{R}$ $\Lambda(A)$ has the representation

$$-logEexp(it\Lambda(A)) = it\nu_0(A) + \frac{1}{2}t^2\nu_1(A) - \int_{-\infty}^{\infty} v_\alpha(x)(\int_0^\infty k(e^{-u}x, t)u^{\alpha-1}du)m(A, dx) \tag{35}$$

which, by a similar argument of Proposition 2.1 in [3], implies that there exists a unique finite measure ν on $\sigma(\mathcal{S}) \times \mathcal{B}(\mathcal{R})$ such that

$$\nu(A \times B) = m(A, B), \quad \text{for any} \quad A \in \mathcal{S}, B \in \mathcal{B}(\mathcal{R}).$$

Moreover, for every $A \in \sigma(\mathcal{S})$ we have $\nu(A, \{0\}) = 0$.

Now,we are in the position to present the following theorem whose proof is a simple combination of Theorem 6 and the Komogorov extension theorem and Threorem 5.2 in [3].

Theorem 4.3 *Given $0 < \alpha \leq \infty$ let $\{X_t : t \in T\}$ be an $\alpha -$ s.d. stochastic process defined on a probability space $(\Omega', \mathcal{P}')$. Then there exists an $\alpha-$s.d.r.m., say Λ, defined on the probability space $(\Omega, \mathcal{P})$ such that*

$$\big(\Omega = \Omega' \times [0,1], \mathcal{P} = \mathcal{P}' \times Leb\big),$$

Leb being the Lebesgue measure on [0,1] and

$$\{X_t : t \in T\} = \{\int_{\mathbb{S}} f_t(s) d\Lambda(s) : t \in T\} \quad a.s.\mathcal{P},$$

where $\{f_t(s) : t \in T, s \in S\}$ are some measurable functions on $\mathbb{S}$.

5 An Application in Option Pricing

If X is Lévy - stable random variable with index $0 < \alpha < 1$, then it does not have any integer moment, and for the case $1 < \alpha < 2$ only the first integer moment exists. Therefore, to overcome this dificulties, following Cartea and Howinson [1], we introduce the following Damped - Lévy - mixed - stable process which will lead to a mathematical model for our purpose of option pricing. Suppose that $X_i(t)$, $i = 1, 2$ are independent Lévy -stable processes with indexes $0 < \alpha_1 < \alpha_2 < 2$, respectively such that the logarithm of the characteristic function of $X_i(1)$ is given by

$$\psi_j(u) = \int_{-\infty}^{+\infty} (e^{iux} - 1 - iu\tau_{\alpha_j}(x))W_j(x)dx, \quad j = 1, 2. \tag{36}$$

where

$$W_j(x) = \begin{cases} C_q|x|^{-1-\alpha_j} & \text{for} \quad x < 0 \\ C_p x^{-1-\alpha_j} & \text{for} \quad x > 0 \end{cases}$$

and

$$\tau_{\alpha_j}(x) = \begin{cases} x & \text{for} \quad \alpha_j > 1 \\ sinx & \text{for} \quad \alpha_j = 1 \\ 0 & \text{for} \quad \alpha_j < 1. \end{cases}$$

Here $C_p, C_q > 0$ are scale constants, $p, q \geq 0$ and $p + q = 1$.
Following Cartea and Howinson [1] the exponential cut-off $e^{-\lambda|x|}$ is introduced to obtain the Damped Lévy measures

$$W_j^\lambda(x) = \begin{cases} C_q|x|^{-1-\alpha}e^{-\lambda|x|}, & \text{for} \quad x < 0 \\ C_p x^{-1-\alpha}e^{-\lambda|x|}, & \text{for} \quad x > 0 \end{cases} . \tag{37}$$

Let $W_j^\lambda, j = 1, 2$, denote the Damped Lévy measures corresponding to Lévy processes $X_j^\lambda(t), j = 1, 2$ with

$$\phi_j(u) = \int_{-\infty}^{+\infty} (e^{iux} - 1 - iu\tau_{\alpha_j}(x))e^{-\lambda|x|}W_j(dx) \tag{38}$$

Putting, for $t \geq 0, X(t) = X_1(t) + X_2(t)$ we get a Lévy process $X(t)$ which is also a mixed - stable - Lévy
process with $\Phi(u) = \Phi_1(u) + \Phi_2(u)$ where $\Phi_j(u), j = 1, 2$ are given by (38). Putting

$$W_j^\lambda(x) = \begin{cases} C_q|x|^{-1-\alpha_j}e^{-\lambda|x|}, & \text{for} \quad x < 0 \\ C_p x^{-1-\alpha_j}e^{-\lambda|x|}, & \text{for} \quad x > 0 \end{cases} , j = 1, 2 \tag{39}$$

and taking into account (38) we infer that the logarithm of the characteristic function, denoted by $\phi^\lambda(u)$, for a Damped-Levy's process $\{X^\lambda(t)\}$ is of the form

$$\phi^\lambda(u) = \phi_1^\lambda(u) + \phi_2^\lambda(u)$$

where

$$\phi_j^(\lambda)(u) = \int_{-\infty}^{+\infty} (e^{iux} - 1 - iu\tau_{\alpha_j}(x))e^{-\lambda|x|}W_j^(\lambda)(dx), \; j = 1, 2 \tag{40}$$

which implies that the Damped Lévy process $X^\lambda(t) := X_1^\lambda(t) + X_2^\lambda(t)$ has the following property:
(i) $\{X^\lambda(t)\}$ is a Lévy process.
(ii) It is not a stable process.
(iii) $\lim_{\lambda \to 0} X^\lambda(t) = X(t)$ (in distribution and in probability).
(iv) The process $\{X^\lambda(t)\}$ has finite moments of all orders. Moreover, its exponential moments exist.

Definition 5.1 (Power - Jump Process) *For $k = 2, 3, \ldots$ we define, following M. Corunera, D. Nualart and W.Schouten, the following Power - Jump process.*

$$Z_t^{(k)} = \sum_{0 < s \leq t} [\Delta X^\lambda(t)]^k, \; k \geq 2 \tag{41}$$

where

$$\Delta X\lambda(t) = X^{\lambda}(t) - X^{\lambda}(t^{-})$$

and, for convenience, we put

$$Z_t^{(k)} = X^{\lambda}(t).$$

Each process $Z_t^{(k)},\ k \geq 2$ is called the k-th power - jump process.

We have

$$E[Z_t^{(1)}] = E[X^{\lambda}(t)] = ta^{\lambda}$$

$$E[Z_t^{(k)}] = E[\sum_{0<s\leq t} [\Delta X^{\lambda}(t)]^k] = t\int_{-\infty}^{+\infty} x^k \gamma(dx) = m_k t$$

Definition 5.2 *Put*

$$Y_t^{(k)} = Z_t^{(k)} - m_k t, \quad k = 1, 2, 3, \ldots.$$

Then, $\{Y_t^{(k)}\}$ is a normal martingale and is called Teugels martingale of order k

By taking a suitable linear combination of the $\{Y_t^{(k)}\}$, one get pairwise strongly orthonormal martingale, say $\{T^{(k)}\}$, which correspond to the procedure of the orthonormal lization of the polynomial $1,\ x,\ x^2$ with respect to to measure

$$\mu(dx) = x^2\gamma(dx) + c^2\delta(dx)$$

The resulting process

$$\{T_t^{(k)},\ t \geq 0\}$$

are called the orthonormalization k -th jump process.

Theorem 5.1 (Nualart - Schouten) *Let the σ-fields $(\mathcal{F}_t)$ be generated by $\{K_t\}$ and $L_T^2(T > 0)$ be the space of all square integrable and $(\mathcal{F}_t)$ - measurable $\Phi = \Phi_t, \quad t \in [0, T]$ such that*

$$\|\Phi\|^2 = E[\int_0^T |\Phi_t^2|dt] < \infty$$

Then each square integrable random variable $F \in L_T^2$ has the representation

$$F = E[F] + \sum_{k=1}^{\infty} V_s^{(k)}$$

where$\{V_s^{(k)}\}$ is a predictable process in $M_T^2(l^2)$, the space of predictable l^2- valued processes

Suppose that we work under the framework of the market with the k-th power - jump assets. Note that the value of the contingent claim at time t is given by

$$F(S_t) = exp(-r(T-t))E_Q[X|\mathcal{F}_t].$$

We call $F(t, S_t)$ the price function. Note that we have choosen an equivalent martingale measure Q under which all discounted assets price processes are martingales.

The Black - Scholes Formula under Lévy Mixed Stable Shocks
Let $0 < \alpha_1 < \alpha_2 < 2$. Suppose that our stock price $X(t) = X_1(t) + X_2(t)$ satisfies the condition that $X_j(t), j = 1, 2$ are α_k - stable Lévy processes under measure Q, respectively.
Let

$$X_{t+\Delta t} = X_t e^{\mu \Delta t + \sigma \Phi}$$

where Φ is a parameter for Damped -(α_1, α_2) mixed - stable - Lévy process $X^\lambda(t)$. Then as $\Delta t \to 0$ the "Damped Black - Scholes" partial differential equation (PDE) satisfies $\frac{\delta \hat{V}}{\delta t} = \psi(-\bar{\sigma}\xi) + i\xi[\psi(-i\bar{\sigma}) + \bar{D}_t - r(1-i\xi)\hat{V}$ when ψ is the logarithm of the characteristic function of the Damped - Lévy - mixed - stable process and

$$V(S,t) \stackrel{\text{def}}{=} E_Q[e^{-r(T-t)}\Pi(S,T)]$$

where $\Pi(S,T)$ stands for payoff at time t=T.
Notice that we can write

$$V(S,t) = e^{-r\delta t}E_Q[e^{-r(T-t-\Delta t)}\Pi(S,T)] = e^{-r\delta t}E_Q[V(St + \Delta t, t) + \delta t]$$

which is equivalent to the equation

$$rV(S,t)\Delta t = E_Q[\Delta V(S,t)]$$

where

$$\Delta(S,t) \stackrel{\text{def}}{:=} V(V + \Delta S, t + \Delta t) - V(S,t).$$

Finally, one can solve the Black -Scholes equation by a similar method as for the case of Lévy - stable process [1].

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