A Kingman convolution approach

to Bessel processes*

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Abstract. *We study Bessel processes* $BES^{\delta}(x)$ *in terms of the Kingman convolution method. In particular, we propose a higher dimensional model of the Kingman convolution algebras in particular and Urbanik convolution algebras in general. We show that every Bessel process* $BES^{\delta}(0)$ *(starting from 0) is induced by the Kingman convolution. Moreover, a new concept of increments of stochastic processes is introduced. It permits to regard Bessel processes as "stationary and independent increments processes".*

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I. Introduction: This study is inspired by a distinguished part of Bessel processes in financial mathematics for decades. Indeed, if $\{\mathbf{W}_t\}$ is a δ-dimensional Brownian motion (BM^δ) and we put $B = \|\mathbf{W}\|$. By virtue of Revuz-Yor [10],p.439,we have

$$B_t^2 = B_0^2 + 2\int_0^t B_s d\beta_s + \delta t, \tag{1.1}$$

where W is a linear BM. Consider, for any real number $\delta \geq 0, x \geq 0$, the following SDE

$$Z_t = x + 2\int_0^t \sqrt{|Z_s|} dW_s + \delta t. \tag{1.2}$$

Note that (1.2) is a special case of the Cox-Ingersoll-Ross(**CIR**) family of diffusions ([2]) which have unique solutions. Moreover,these solutions are strong, nonnegative and adapted w.r.t. the natural filtration $\{\mathcal{F}_t\}$ of $\{W_t\}$. Consequently, in the case $\delta \geq 0, x \geq 0$, the absolute sign in (1.2) can be omitted and $\{Z_t\}$ can be modeled as short term interest rates (cf. Cox-Ingersoll-Ross [2]).

1.1 Definition(cf. Revuz-Yor [10, XI] *For every* $\delta \geq 0, x \geq 0$*, the unique strong solution of the equation (1.2) is called the square of* δ*-dimensional Bessel process started at* x *and is denoted by* $BESQ^\delta(x)$*.Further, the square root of* $BESQ^\delta(a^2)$*, is called the Bessel process of dimension* δ *started at* a *and is denoted by* $BES^\delta(a)$*.*

In the present paper we study the class of Bessel processes $BES^{\delta}(a), \delta = 2(s+1) \geq 1$ via the Kingman convolution method and will use "s" as the index of the Bessel process.

Let $\mathcal{P}$ denote the class of all p.m.'s on the positive half-line $\mathbb{R}^+$ endowed with the weak convergence and $*_{1,\delta}, \delta \geqslant 1$ denote the Kingman convolution (Hankel transforms) which was introduced by Kingman [5] in connection with the addition of independent spherically symmetric random vectors in Euclidean δ-space. Namely, for each continuous bounded function f on R^+ we write :

$$\int_0^\infty f(x)\mu *_{1,\delta} \nu(dx) = \frac{\Gamma(s+1)}{\sqrt{\pi}\Gamma(s+\frac{1}{2})}$$

$$\int_0^\infty \int_0^\infty \int_{-1}^1 f((x^2+2uxy+y^2)^{1/2}) (1-u^2)^{s-1/2}\mu(dx)\nu(dy)du, \tag{1.3}$$

where $\mu, \nu \in P$, $\delta = 2(s+1) \geqslant 1$ (cf.Kingman [5] and Urbanik [16]). In what follows, for the sake of simplicity, the Kingman convolution $*_{1,\delta}$ will be denoted shortly as $\circ$.The algebra $(\mathcal{P}, \circ)$ is the most important example of Urbanik convolution algebras (cf Urbanik [16]). In language of the Urbanik convolution algebras, the *characteristic measure*, say σ_s, of the Kingman convolution has the Rayleigh density

$$d\sigma_s(y) = 2(s+1)^{s+1}\Gamma^{-1}(s+1)y^{2s+1}exp(-(s+1)y^2) \tag{1.4}$$

with the characteristic exponent $\varkappa = 2$ and the kernel Λ_s

$$\Lambda_s(x) = \Gamma(s+1)J_s(x)/(1/2x)^s. \tag{1.5}$$

It is known (cf. Kingman [5], Theorem 1), that the kernel Λ_s itself is an ordinary ch.f. of a p.m., say F_s, defined on the interval [-1,1]. Thus if θ_s denotes a r.v. with distribution F_s then for each $t \in \mathbb{R}^+$,

$$\Lambda_s(t) = Eexp(it\theta_s) = \int_{-1}^{1} exp(itx)dF_s(x). \tag{1.6.}$$

The *radial characteristic function* (rad.ch.f.) of a p.m. $\mu \in \mathcal{P}$, denoted by $\hat{\mu}(u)$, is defined by

$$\hat{\mu}(u) = \int_0^\infty \Lambda_s(ux)\mu(dx), \tag{1.7}$$

for every $u \in \mathbb{R}^+$. In particular,the rad.ch.f. of σ_s is

$$\hat{\sigma}_s(u) = exp(-u^2/2), u \in R^+. \tag{1.8}$$

II Cartesian product of Kingman convolutions[1]

Denote by $\mathbb{R}^{+k}, k = 1, 2, ...$ the k-dimensional nonnegative cone of $\mathbb{R}^k$ and $\mathcal{P}(\mathbb{R}^{+k})$ the class of all p.m.'s on $\mathbb{R}^{+k}$ equipped with the weak convergence. Let $\mathbf{F}_1, \mathbf{F}_2 \in \mathcal{P}(\mathbb{R}^{+k})$ be of the product form

$$\mathbf{F}_i = \tau_i^1 \times ... \times \tau_i^k, \tag{2.1}$$

where $\tau_i^j \in \mathcal{P}$, j=1,2,... and i=1,2. We put

$$\mathbf{F}_1 \bigcirc_{\mathbf{k}} \mathbf{F}_2 = (F_1^1 \circ F_1^2) \times ... \times (F_1^k \circ F_2^k). \tag{2.2}$$

Since convex combinations of p.m.'s of the form (2.1) are dense in $\mathcal{P}(\mathbb{R}^{+k})$ the relation (2.2) can be extended to arbitrary p.m.'s on $\mathbb{R}^{+k}$. For every $\mathbf{F} \in \mathcal{P}(\mathbb{R}^{+k})$ the k-dimensional rad.ch.f $\hat{\mathbf{F}}(\mathbf{t}), \mathbf{t} \in \mathbb{R}^{k+}$, is defined by

$$\hat{\mathbf{F}}(\mathbf{t}) = \int_{\mathbb{R}^{+k}} \prod_{j=1}^{k} \Lambda_s(t_j x_j) \mathbf{F}(\mathbf{dx}), \tag{2.3}$$

[1] Higher dimensional Urbanik convolution algebras can be introduced in the same way as here for the Kingman convolution case but this subject will be treated systematically else where.

Let $\theta, \theta_1, ..., \theta_k$ be i.i.d. r.v's with the common distribution F_s. Further, suppose that

$$\mathbf{X} = (X_1, ..., X_k) \quad \text{and} \quad \Theta = (\theta_1, ..., \theta_k)$$

be R^{+k}-valued independent r.v.'s such that $\mathbf{X} \stackrel{\mathrm{d}}{=} \mathbf{F}$. Set

$$\Theta\mathbf{X} = \{\theta_1 X_1, ..., \theta_k X_k\}. \tag{2.4}$$

Then, the following formula is the multidimensional generalization of (1.6):

$$\widehat{\mathbf{F}}(\mathbf{t}) = E(e^{i<\mathbf{t},\Theta\mathbf{X}>}), \tag{2.5}$$

where $\mathbf{t} = (t_1, ..., t_k) \in \mathbb{R}^{+k}$ and $<,>$ denotes the inner product in $\mathbb{R}^k$. In fact, we have

$$\begin{aligned} & E(e^{i<(\theta_1 t_1,...,\theta_k t_k),\mathbf{X}>}) \\ &= \int_{R^{+k}} E(e^{i\sum_{j=1}^k (t_j x_j \theta_j} F(d\mathbf{x}) \\ &= \int_{R^{+k}} \Pi_{j=1}^k \Lambda_s(t_j x_j) F(d\mathbf{x}) \\ &= \widehat{\mathbf{F}}(\mathbf{t}). \end{aligned} \tag{2.6}$$

Thus, $\hat{F}(\mathbf{t})$ is also an ordinary k-dimensional ch.f., and hence it is uniformly continuous. For vectors $\mathbf{x} \in \mathbb{R}^{+k}$ the *generalized translation operators*(g.t.o.'s) $\mathbf{T}^{\mathbf{x}}$ acting on the Banach space $\mathbb{C}_b(\mathbb{R}^{+k})$ of real bounded continuous functions f on $\mathbb{R}^{+k}$ are defined, for each $\mathbf{y} \in \mathbb{R}^{+k}$, by

$$\mathbf{T}^{\mathbf{x}} f(\mathbf{y}) = \int_{\mathbb{R}^{+k}} f(\mathbf{u}) \{\delta_{\mathbf{x}} \bigcirc_{\mathbf{k}} \delta_{\mathbf{y}}\}(d\mathbf{u}). \tag{2.7}$$

In terms of these g.t.o.'s the k-dimensional rad.ch.f. of p.m.'s on $\mathbb{R}^{+k}$ can be characterized as the following:

Theorem 2.1 *A real bounded continuous function f on $\mathbb{R}^{+k}$ is a (k-dimensional) rad.ch.f. of a p.m., if and only if $f(\mathbf{0}) = 1$ and f is $\{\mathbf{T}^{\mathbf{x}}\}$-nonnegative definite in the sense that for any $\mathbf{x}_1, ..., \mathbf{x}_k \in \mathbb{R}^k$ and $\lambda_1, ..., \lambda_k \in \mathbb{C}$*

$$\sum_{i,j=1}^{k} \lambda_i \bar{\lambda}_j \mathbf{T}^{\mathbf{x}_i} f(\mathbf{x}_j) \geqslant 0.$$

(See Vólkovich [18] for the proof).

Lemma 2.2 *Every p.m. $\mathbf{F} \in \mathcal{P}(\mathbb{R}^{+k})$ is uniquely determined by its k-dimensional rad.ch.f. $\hat{\mathbf{F}}$ and the following formula holds:*

$$\widehat{\mathbf{F}_1 \bigcirc_{\mathbf{k}} \mathbf{F}_2}(\mathbf{t}) \quad = \widehat{\mathbf{F}_1}(\mathbf{t}) \widehat{\mathbf{F}_2}(\mathbf{t}), \tag{2.8}$$

$\mathbf{F}_1, \mathbf{F}_2 \in \mathcal{P}(R^{+k}) \quad and \quad \mathbf{t} \in R^{+k}.$

Proof *The formula (2.8) follows from formulas (1.3) and (2.2). Now using the formulas (2.3), (2.5) and integrating the function* $\hat{\mathbf{F}}(t_1u_1, ..., t_ku_k)$, *k-times w.r.to* σ_s , *we get*

$$(2.9) \qquad \int_{R^{+k}} \hat{\mathbf{F}}(t_1u_1, ..., t_ku_k)\sigma_s(du_1)...\sigma_s(du_k) =$$

$$\int_{R^+} ... \int_{R^{+k}} \prod_{j=1}^{k} \Lambda_s(t_jx_ju_j)\mathbf{F}(\mathbf{dx})\sigma_s(du_1)...\sigma_s(du_k)$$

$$= \int_{R^{+k}} \prod_{j=1}^{k} exp\{-t_j^2x_j^2\}\mathbf{F}(\mathbf{d}x),$$

which, by change of variables $y_j = x_j^2, j = 1, ..., k$ *and by the uniqueness of the k-dimensional Laplace transform, implies that* $\mathbf{F}$ *is uniquely determined by the left-hand side of (2.9).*

The following theorem is a simple consequence of (1.3) and (2.2).

Theorem 2.3 *The pair* $(\mathcal{P}(\mathbb{R}^{+k}, \bigcirc_{\mathbf{k}})$ *is a commutative topological semigroup with* $\delta_{\mathbf{0}}$ *as the unit element. Moreover, the operation* $\bigcirc_{\mathbf{k}}$ *is distributive w.r.t. convex combinations of p.m.'s* $\in \mathcal{P}(R^{+k})$.

In the sequel, the pair $(\mathcal{P}(R^{+k}, \bigcirc_{\mathbf{k}})$ *will be called a k-dimensional Kingman convolution algebra. It is the same as in the case k=1, the i.d. elements can be defined as the following: A p.m.* $\mu \in \mathcal{P}(R^{+k}$ *is called i.d.if for every natural* *m there exists a p.m.* μ_m *such that* $\mu = \mu_m \bigcirc_{\mathbf{k}} \mu_m ... \mu_m \bigcirc_{\mathbf{k}} \mu_m (m \quad terms)$.

Now observe that the function

$$\widehat{\Sigma_{s,k}}(t_1, t_2, ..., t_k) = \prod_{j=1}^{k} exp(-t_j^2), \tag{2.10}$$

where $t_j, j = 1, 2, ..., k \in \mathbb{R}^+$ *is the k-dimensional rad.ch.f. of the distribution*

$$\Sigma_{s,k} = \sigma_s \times ... \times \sigma_s \quad (k \quad terms) \tag{2.11}$$

being the k-fold Cartesian product of σ_s. *In the sequel, the* $\Sigma_{s,k}$ *will be called the k-dimensional Rayleigh distribution .*

Now, let us denote by $ID(\bigcirc_k)$ *the class of all i.d.p.m.'s in* $(\mathcal{P}(R^{+k}, \bigcirc_k)$. *The following theorem stands for a slight generalization of Theorem 7 in Kingman [5] and its proof is omitted.*

Theorem 2.5 $\mu \in ID(\bigcirc_{\mathbf{k}})$ *if and only if there exist a* σ-*finite measure* M *on* $\mathbb{R}^{+k}$ *with the property that* $M(\{\mathbf{0}\}) = 0$, M *is finite outside every neighborhood of* $\mathbf{0}$ *and*

$$\int_{\mathbb{R}^{+k}} \frac{\|\mathbf{x}\|^2}{1 + \|\mathbf{x}\|^2} M(d\mathbf{x}) < \infty$$

and for each $\mathbf{t} = (t^1, ..., t^k) \in R^k$

$$(2.12) \qquad -log\hat{\mu}(\mathbf{t}) = \int_{\mathbb{R}^{+k}} (1 - \prod_{j=1}^{k} \Lambda_s(< t_j, x_j >) \frac{\|\mathbf{x}\|^2}{1 + \|\mathbf{x}\|^2} M(d\mathbf{x}).$$

III Convolution structure of Bessel processes

Given a p.m. $\mu \in \mathcal{P}$ *and n=1,2,... we put, for any* $x \in \mathbb{R}^+, B \in \mathcal{B}(\mathbb{R}^+)$,

$$(3.1) \qquad P_n(x, E) = \delta_x \circ \mu^{\circ n}(E),$$

here the power is taken in the convolution $\circ$ *sense. Using the rad.ch.f. one can show that* $\{P_n(x, E)\}$ *satisfies the Chapman-Kolmogorov equation and therefore, there exists a homogeneous Markov sequence, say* $\{S_n^x\}$, *n=0,1,2,..., with* $\{P_n(x, E)\}$ *as its transition probability. More generally,suppose that* $\{\mu_k, k = 1, 2, ...\}$ *is a sequence of p.m's on* R^+. *Put, for any* $0 \leqslant n < m, x \in R^+, E \in \mathcal{B}(R^+)$,

$$(3.2) \qquad P_{n,m}(x, E) = \delta_x \circ \mu_n \circ \mu_{n+1} \circ ... \circ \mu_{m-1}(E).$$

Then, $P_{n,m}(x,E)$ satisfies the Chapman-Kolmogorov equation and therefore, there exists a Markov sequence $\{X_n^x\}, n = 0,1,2,...$ with the transition probability $P_{n,m}(x,E)$.

In what follows we will discuss the case of Bessel processes which stand for a continuous counter part of the above symmetric random walks.

Suppose that μ is an i.d.p.m. w.r.t. the Kingman convolution $\circ$.Putting

$$q(t,x,E) := \mu^{\circ t} \circ \delta_x(E) \tag{3.3}$$

and taking into account the fact that the family $q(t,x,.)$ of distributions satisfies the Chapman-Kolmogorov equation and therefore, it stands for a transition probability of a homogeneous strong Markov Feller process, say $\{X_t^x\}, t, x \in R^+$. and, moreover $\{X_t^x\}$ is stochastically continuous and has a CADLAG version (cf.Nguyen [13], Theorem 2.6).

3.1 Definition *A stochastic process $\{X_t^x\}$ is called a Lévy-type (or, $\circ$-Lévy) process if (i) $X_0^x = x$ (P.1);(ii) $\{X_t^x\}$ is strong Markov Feller process with transition probability of the form (3.3); (iii)$\{X_t^x\}$ is a stochastically continuous process with CADLAG realizations with (P.1).*

It is evident that all Lévy processes are $*$-Lévy ones. The simplest example of Lévy-type but non-Lévy processes is absolute value of the linear BM. Similarly,the following theorem *shows that Bessel processes starting from 0 stands for Lévy type processes induced by Kingman convolutions.*

3.2 Theorem *Let $\{B_t^\delta\}$ denote a Lévy-type process which has transition probability (3.3) with x=0 and $\mu = \sigma_s$. Then, up to a scale change, $\{B_t^\delta\}$ and $BES^\delta(0)$ have the same distribution. Consequently, they are induced by the Kingman convolution.*

Proof. *Let p_x^δ denote the law of $BES^\delta(x), \delta \geq 0, x \geq 0$ on $C(\mathbb{R}^+, \mathbb{R})$ (cf. Revuz-Yor [10],XII P.445) which entails that the density $p_t^\delta(0,y)$ of the Bessel semigroup is found (cf.Revuz-Yor [10],XII P.446), for $\delta \geq 0, x = 0$, to*

$$(3.4) \quad P_t^\delta(0,y) = 2^{-s} t^{-(s+1)} \Gamma(s+1)^{-1} y^{2s+1} exp(-y^2/2t).$$

It should be noted that functions (3.4) are Rayleigh functions of y. In addition, if t=2 we get $P^\delta(0) = \sigma_s$. Next, by(1.8), we have

$$\widehat{\sigma_s^{\circ t}}(u) = exp(-tu^2/4(s+1)), u \geq 0.$$

Our further aim is to prove that, up to a scale change, the rad.ch.f. of $\sigma_s^{\circ t}$ is equal to the rad.ch.f.of $P_t^\delta(0,y)$. Accordingly, integrating the kernel $\Lambda_s(uz)$ w.r.t. $P_t^\delta(0,z)$ it follows, by (1.3), (1.5), (3.4) that the rad.ch.f. of $P_t^\delta(0,y)$ is given, for each $u \geq 0$, by

$$(3.5) \qquad \widehat{P_t^\delta}(0,y)(u) = \int_0^\infty \Lambda_s(uz) P_t^\delta(0,z) dz$$

$$= 2^{-s} t^{-(s+1)} \Gamma(s+1)^{-1} \int_0^\infty z^{2s+1} \Lambda_s(uz) exp(-z^2/2t) dz$$

Hence and by virtue of the Weber integral[1] *we have*

$$\widehat{q_t^\delta}(0,y)(u)$$

$$= \{2^{-s} t^{-(s+1)} \Gamma(s+1)^{-1}\}\{2^{-1} 2^{s+1} t^{s+1} \Gamma(s+1) e^{-\frac{tu^2}{2}}$$

$$= \widehat{\sigma_s^{\circ t}}(u), u \geq 0,$$

which shows that

$$q_t^\delta(0) = \sigma_s^{\circ t}.$$

IV Bessel processes as stationary independent "increments" processes

Suppose that $X_j, j = 1, 2, ...$ *are nonnegative independent r.v.'s with the corresponding distributions* $F_{X_j}, j = 1, 2, ...$ *and* $\theta, \theta_1, \theta_2 ...$ *are i.i.d. r.v's with the common distribution*

[1] From Watson [19] , p.394 we have, for $s \geq -1/2, a \geq 0, p > 0$,

$$\int_0^\infty t^{s+1} J_s(at) e^{-p^2 t^2} dt = a^s (2p^2)^{-s-1} e^{-a^2/4p^2}$$

which may be written as

$$\int_0^\infty t^{2s+1} \Lambda_s(at) e^{-p^2 t^2} dt = \frac{1}{2} \Gamma(s+1) p^{-2(s+1)} e^{-a^2/4p^2}.$$

F_s and the r.v.'s $X_j, j = 1, 2, ...\theta, \theta_1, \theta_2, ...$ are independent. Following Kingman [5] we say that for a fixed $s \geqslant -1/2$ any one of the equivalent r.v.'s

$$X_1 \oplus X_2 := \sqrt{X_1^2 + X_2^2 + 2X_1 Y_2 \theta_1} \tag{4.1}$$

is a radial sum of the two independent nonnegative r.v.'s X_1, X_2. By induction, the radial sum $X_1 \oplus X_2 \oplus ... \oplus X_k$ is defined for any finite k=2,3,.... It should be noted [5] that the operation $\oplus$ is associative.

4.1 Definition *Let $\mathcal{B}_b$ be the ring of subsets of a non-empty bounded Borel subsets of $\mathbb{R}^+$. A function*

$$M : \mathcal{B}_b \to L^+,$$

where $L^+ = K^+(\Omega, \mathcal{F}, P)$ denotes the class of all nonnegative r.v.'s on the probability space $(\Omega, \mathcal{F}, P)$, is said to be an $\circ$-scattered random measure, if (i) $M(\emptyset) = 0$ (P.1), (ii) For any $A, B \in \mathcal{B}_b, A \cap B = \emptyset$, then $M(A)$ and $M(B)$ are independent and

$$M(A \cup B) \stackrel{d}{=} M(A) \oplus M(B)$$

(iii) For any pairwise disjoint sets $A_1, A_2, ... \in \mathcal{B}_b$, with the union in Cal_b the r.v.'s $M(A_j), j = 1, 2, ...$ are independent and

$$M(\cup_{j=1}^{\infty} A_j) \stackrel{d}{=} \bigoplus_{j=1}^{\infty} M(A_j).$$

It is well known that if $\{W(t)\}, t \in R^+$ is a Wiener process, then there exists a Gaussian stochastic measure $M(A), A \in \mathcal{B}_0$, where $\mathcal{B}_b$ is the ring of bounded Borel subsets of R^+ with the property that, for every $t \geqslant 0$, we have $W(t) = M((0,t])$. The same it is also true for Bessel processes. Namely, we get

4.2 Theorem *Let $\{B_t^\delta\}$ denote a Bessel process starting from 0. Then there exists a unique (up to finite dimensional distributions) $\circ$-scattered r.m.$B(A), A \in \mathcal{B}_b$ with the Lebesgue measure as its control measure such that for each $t \geqslant s \geqslant 0$ we have*

$$B([0,t]) = B_s^\delta \oplus B((s,t]) \stackrel{d}{=} \sigma_s^{t-s}. \tag{4.2}$$

We proceed the proof of the Theorem by proving the following Lemma.

4.3 Lemma *Let $\pi := \{0 = t_0 < t_1 < t_2 < ...\}$ be a subdivision of R^+. Then there exist independent r.v.'s $X_1, X_2, ...$ such that*

$$\sigma_s^{t_k - t_{k-1}} \stackrel{d}{=} X_k, k = 0, 1, 2, ...$$

.Moreover,we have

$$B_{t_n}^\delta \stackrel{d}{=} X_1 \oplus X_2 \oplus ... \oplus X_n \quad (n = 2, 3, ... \tag{4.3}$$

and

$$B((t_n, t_{(n+r]}) \stackrel{d}{=} \sigma_s^{t_{n+r} - t_n} \tag{4.4}$$

Proof. *Following the ideal of Kingman([5], pp.20) let us take as sample space* Ω *the Cartesian product of countably many intervals* R^+ *with countably many intervals [-1,1]. The probability measure is defined on* Ω *as the product of the distributions* $\sigma_s^{t_k - t_{k-1}}, k = 1, 2, ...$ *on each of the first set of* R^+ *together with the distribution* F_s *(see(1.6))on each of the second set. If the typical point* $\omega \in \Omega$ *has components*

$$X_1(\omega), X_2(\omega), ...; \eta_1(\omega), \eta_2(\omega), ...,$$

then $S_m(\omega)$ *is defined inductively by*

$$S_0 = 0,$$

$$S_{m+1}(\omega) = \{S_m^2(\omega) + X_{m+1}^2(\omega) + 2\eta_m(\omega)S_m(\omega)X_{m+1}(\omega)\}^{\frac{1}{2}}. \tag{4.4}$$

Thus, we have

$$S_{m+1} = S_m \oplus X_{m+1}$$

which, by virtue of the associativity of $\oplus$*, implies that for each m =2,3,...*

$$S_m = X_1 \oplus X_2 \oplus ... \oplus X_m. \tag{4.5}$$

Moreover, since $X_k, k = 2, 3, ...$ *are independent it follows that*

$$S_m \stackrel{d}{=} \sigma^{t_m} \stackrel{d}{=} B(t_m). \tag{4.6}$$

Now, since the operation $\oplus$ is associative (cf. Kingman [5],Theorem 1), we can show that

$$S_{m+r} = S_m \oplus S_r^m, \tag{4.7}$$

where S_r^m is defined inductively by

$$S_0^m = 0, S_{r+1}^m = S_r^m \oplus X_{m+r+1}. \tag{4.8}$$

Note, by (4.6,7,8), that

$$\sigma^{t_{m+r}-t_m} \overset{d}{=} S_r^m \overset{d}{=} \big(X_m \oplus ... \oplus X_{m+r}\big) \tag{4.9}$$

which entails (4.3,4).

Proof of Theorem 4.2. *Let* $\mathbb{B}_0)$ *denote the class of finite unions of disjoint finite intervals* $(a, b]$ *i.e.*

$$\cup_{j=1}^k I_j, I_j = (t_{2j}, t_{2j+1}], j = 0, 1, ..., k = 1, 2, ...$$

We put

$$B(\cup_{j=1}^k I_j) = \bigoplus_{j=1}^k B((I_j)).$$

Finally, using the transfinite induction and by Lemma 4.4 and the usual extension method of random interval functions one can gets an $\circ$-random measure B(.) on $\mathcal{B}_b$ with the required properties.

4.2 Definition *For every $0 \leqslant a \leqslant b$ the quantity $M((a,b])$ is called the increment-type of the Bessel processes BES_t^δ. Mor3eover, from Theorems 3.2 and 4.2 we have*

Theorem *Every Bessel process which starts from 0 has a modification as a process with stationary and increments-type process.*

The above theorem permits us to construct a new stochastic integration with respect Bessel processes with convergence in distribution which will be discuss in a subsequent paper.

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References

1 *Bingham, N.H.(1973). Random walks on spheres, Z. Wahrscheinlichkeitstheorie Verw. Geb.,* **22***,169-172.*

2 *Cox, J.C., Ingersoll, J.E.Jr., and Ross, S.A. (1985). A theory of the term structure of interest rates. Econometrica,* **53***(2).*

3 *Ito, K., Mackean H.P. Diffusion processes and their sample paths. Berlin-Heidelberg-New York. Springer 1965.*

4 Kalenberg O.(1983). Random measures, 3rd ed. New York: Academic Press.

5 Kingman, J.F.C.(1963). Random walks with spherical symmetry, Acta Math., **109***, 11-53.*

6 Levitan B.M.(1962). Generalized translation operators and some of their applications, Israel program for Scientific Translations, Jerusalem.

7 Nualart, D.and Zakai, N.(1989). On the relation between the Stratonovich and Ogawa integrals, The Annals of Probability **17***, No. 2, 1536-1540.*

*8 Ogawa, S.(1984). Quelques proprietés de intégrale stochastique du type noncausal.Japan J. Appl. math.***1** *405-416.*

9 Raiput B.S., Rosinski J., Spectral representation of infinitely divisible processes, Probab. Th. Rel. Fields **82** *(1989),451-487.*

1 0 Revuz, D. and Yor, M.(1991). Continuous martingals and Brownian motion. Springer-verlag Berlin Heidelberg.

1 1 Sato K (1999). Lévy processes and infinitely divisible distributions, Cambridge University of Press.

1 2 Shiga T.,Watanabe S.(1973). Bessel diffusions as a one-parameter family of diffusion processes, Z.Warscheinlichkeitsth Verw. geb. 27, 34-46.

*1 3 Nguyen V.T (1994). Generalized independent increments processes, Nagoya Math. J.***133***155-175.*

1 4 Nguyen V.T Generalized translation operators and Markov processes, Demonstratio Mathematica, **34** *No 2, 295-304.*

1 5 Nguyen V.T.(2004) Double-indexes Bessel diffusions, Abstract and applied analysis, 563-567, World Sci. Publishing, River Edge, NJ.

1 6 Urbanik K., Generalized convolutions, Studia math., **23** *(1964), 217-245.*

*1 7 Urbanik K., Cramér property of generalized convolutions,Bull. Polish Acad. Sci. Math.***37** *No 16 (1989), 213-218.*

1 8 Vólkovich, On symmetric stochastic convolutions, J. Theor. Prob. **5**, *No. 3(19929, 117-430.*

1 9 Watson G.N., A Treatise on the Theory of Bessel Functions. Cambridge, 1944.

2 0 Yor, M.(1992).Some Aspects of Brownian Motion, Part I: Some special functionals. Lecture notes in mathematics, ETH Zurich. Birkhauser Verlag Basel.