A WELL-BALANCED NUMERICAL METHOD FOR SEVERAL MODELS OF TWO-PHASE FLOWS

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Abstract. We consider two-phase flow models which constitute from six governing equations arriving from mathematical formulation of fundamental balance laws of mass, momentum and energy, and a seventh additional closing equation. Motivated from the work of Kroener and Thanh (SIAM J. Numer. Anal, 2005), we propose a well-balanced numerical method for a general model of two-phase flows. The point is to use the addition to leave the system with only one phase having source terms. Then, restricting to the Baer-Nunziato model, where the closing equation the the compaction dynamics equation, we employ the same technique as in the Engquist-Osher scheme to deal with the compaction dynamics equation. This makes the closure of the numerical method.

Key words. numerical scheme, conservation law, two-phase flow, source term, weak solution, entropy, nozzle flow, nonconservative form.

AMS subject classifications. Primary: 35L65, 74XX. Secondary: 65Bxx, 76N10, 76L05.

1. Introduction.

Multi-phase flows occur so commonly both in nature and in technology. Here we have the fluid composites of materials such that each material may be diffused into other ones. More importantly, the fluid composites can be made to flow. Example can be seen in nature such as clouds are droplets of liquids moving in a gas. Oil, gas, and water coexist in rock. In energy industry we have gas bubbles to nucleate, grow, and coalesce. In chemical processes, we have mixing of (possibly reactive) materials.

The dynamics of two-phase flows can be described by a system of six equations arising from conservation of mass, momentum, and energy in each phase, and an additional closure equation. If we denote the volume fraction of the first phase, say solid, by α_s, and the volume fraction of the second phase, say gas, by α_g, then we obviously have

$$\alpha_s + \alpha_g = 1.$$

Precisely, the governing equations are the following: Conservation of mass, (see [9, 5]),

$$\partial_t(\alpha_s\rho_s) + \partial_x(\alpha_s\rho_s u_s) \;=\; 0, \tag{1.1}$$

$$\partial_t(\alpha_g\rho_g) + \partial_x(\alpha_g\rho_g u_g) \;=\; 0, \tag{1.2}$$

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conservation of momentum

$$\partial_t(\alpha_s\rho_s u_s) + \partial_x(\alpha_s(\rho_s u_s^2 + p_s)) \ = \ p_N\partial_x\alpha_s, \tag{1.3}$$

$$\partial_t(\alpha_g\rho_g u_g) + \partial_x(\alpha_g(\rho_g u_g^2 + p_g)) \ = - \ p_N\partial_x\alpha_s, \tag{1.4}$$

conservation of energy

$$\partial_t(\alpha_s\rho_s E_s) + \partial_x(\alpha_s u_s(\rho_s E_s + p_s)) \ = \ p_N w_N\partial_x\alpha_s, \tag{1.5}$$

$$\partial_t(\alpha_g\rho_g E_g) + \partial_x(\alpha_g u_g(\rho_g E_g + p_g)) \ = - \ p_N w_N\partial_x\alpha_s. \tag{1.6}$$

Here, as usual, $\rho_k, p_k, e_k, T_k, S_k$ are the thermodynamical variables: specific volume, pressure, internal energy, absolute temperature, and specific entropy; u_k, E_k are the velocity and total energy of the k-phase, $k = s, g$. The total energy is given by

$$E_k = e_k + \frac{u_k^2}{2}, \quad k = s, l. \tag{1.7}$$

The quantities p_N, w_N are given functions of $u_s, u_g, \rho_s, \rho_g, p_s, p_g$.

For each phase of fluid there is a constitutive relation, known as the *equation of state*, to characterize the fluid. However, the system (1.1)-(1.6) is still under-determined, as the number of equations is less than the number of variables by 1. To close the system, an obvious tendency is to find an additional equation relating the variables. In the research by Ransom and Hicks [17], the interfacial pressure p_N is given by

$$p_N = \frac{\dfrac{p_s}{a_s} + \dfrac{p_g}{a_g}}{\dfrac{1}{a_s} + \dfrac{1}{a_g}}, \tag{1.8}$$

and the transport equation for volume fraction to close the system

$$\partial_t\alpha_s \ + \ \frac{u_s + u_g}{2}\partial_x\alpha_s = \frac{p_s - p_g}{(a_s + a_g)H}, \tag{1.9}$$

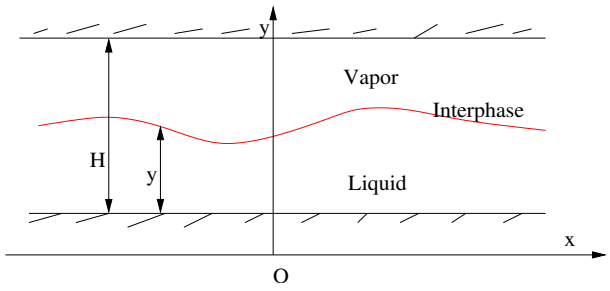

FIG. 1.1. *A two-phase flow model*

where $a_s = c_s\rho_s, c_s = \partial p_s/\partial\rho_s$, and $a_g = c_g\rho_g, c_g = \partial p_g/\partial\rho_g$ are the acoustic impedance and isentropic sound speed of the first and the second phase, relatively, and H is the channel width, see Figure 1.1.

Another form of the additional equation is known as the compaction dynamics equation

$$\partial_t\alpha_s + w_N\partial_x\alpha_s = 0. \tag{1.10}$$

The Baer-Numziato model, see [4, 5], consists of the equations (1.1)-(1.6) and (1.10) with the choice $p_N = p_g, w_N = u_s$.

Nevertheless, the additional equation as the transport volume fraction (1.8) or compaction dynamics (1.9) is still controversial. Moreover, it causes serious concerns as it is in nonconservative form. Another tendency to make the system determined is to simplify the equations so that some variable become constant and therefore the system becomes determined. In [12], the authors considered incompressible two-fluid models, and propose the closure equation to be

$$p_1 = p_2 = p. \tag{1.11}$$

The multi-phase flows models have been a major challenge for many authors. The theoretical difficulty is that the system is *nonconservative*. Roughly speaking, it conserves "wrong" quantities. A general approach to such systems was proposed by Dal Maso, LeFloch, and Murat [8]. Another way can be found in Kröner and Thanh [14]. The numerical obstacle is that for standard numerical schemes, tests often give unsatisfactory results. This is the cause of the nonconservative terms.

In this work we will propose a well-balanced numerical method to deal with the system (1.1)-1.6, as motivated from the work of Kroener-Thanh [13]. This can be described as follows: first we restrict attention to only one phase, then it looks like a model of fluid flows in a nozzle with variable

cross-sections. Second, we can sum up the corresponding balance laws to get the conservation of the total of the corresponding quantities: mass, momentum, and energy. Finally, to complete the method, we employ the idea in using backward/forward differences for the compaction dynamics equation of the Baer-Nunziato model. The last one may also be used for other models involving different closure equation.

The method we use was originally developed in [13, 11, 10], and extended to more general problems in [6, 7, 3]. We note that some proposal for the compaction dynamics equation and numerical methods for multiphase flows were proposed in the works [1, 15, 2].

2. Backgrounds: Non-strict Hyperbolicity and Non-hyperbolicity. Let us consider the following model of two-phase flows consisting of the balance laws of mass, momentum, energy, and the compaction dynamics equation

$$\begin{aligned}
&\partial_t(\alpha_s\rho_s) + \partial_x(\alpha_s\rho_s u_s) \;=\; 0,\\
&\partial_t(\alpha_s\rho_s u_s) + \partial_x(\alpha_s(\rho_s u_s^2 + p_s)) \;=\; p_N\partial_x\alpha_s,\\
&\partial_t(\alpha_s\rho_s E_s) + \partial_x(\alpha_s u_s(\rho_s E_s + p_s)) \;=\; p_N w_N\partial_x\alpha_s,\\
&\partial_t(\alpha_g\rho_g u_g) + \partial_x(\alpha_g(\rho_g u_g^2 + p_g)) \;=-\; p_N\partial_x\alpha_s,\\
&\partial_t(\alpha_g\rho_g) + \partial_x(\alpha_g\rho_g u_g) \;=\; 0,\\
&\partial_t(\alpha_g\rho_g E_g) + \partial_x(\alpha_g u_g(\rho_g E_g + p_g)) \;=-\; p_N w_N\partial_x\alpha_s.\\
&\partial_t\alpha_s + w_N\partial_x\alpha_s = 0.
\end{aligned} \tag{2.1}$$

For each phase of the flow, the equation of state is also needed. Frequently, one has the following kinds of equations of state:

(a) Stiffened gas equation of state

$$p_k = (\gamma_k - 1)\rho_k e_k - \gamma_k P_{k,\infty} \tag{2.2}$$

where $\gamma_k > 1, P_{k,\infty} \geq 0$ are constants, $k = s, g$. For $P_{k,\infty} = 0$ we obtain the perfect gas equation of state.

(b) Van der Waals equation of state

$$p_k = \frac{R_k T_k}{v_k - b_k} - \frac{c_k}{v_k^2}, \tag{2.3}$$

$v_k = 1/\rho_k$ is the specific volume, b_k, c_k, R_k are positive constants, $k = s, g$.

(c) Mie Gruneisen equation of state

$$p_k = (\gamma_k(\rho_k) - 1)\rho_k e_k - \gamma_k(\rho_k)P_{k,\infty}(\rho_k) \tag{2.4}$$

where $\gamma_k > 1, P_{k,\infty}$ are constants, $k = s, g$. This is a generalization of (a) and it is commonly used for high pressure flows.

Since the right-hand side of (2.1) contains nonzero terms in differential form, the system cannot be written in the conservative form. Let us consider the characteristic fields of the system. Set the unknown vector variable

$$U = (\rho_s, u_s, p_s, \rho_g, u_g, p_g, \alpha_s). \tag{2.5}$$

Then, the system of conservation laws with source terms (2.1) can be re-written for smooth flows in the nonconservative form of balance laws as

$$U_t + A(U)f(U) = 0 \tag{2.6}$$

where the Jacobian Matrix $A(U)$ is given by

$$\begin{pmatrix}
\frac{\rho_s}{\alpha_s}(u_s - w_N) & u_s & \rho_s & 0 & 0 & 0 & 0 \\
\frac{p_s - p_N}{\alpha_s \rho_s} & 0 & u_s & \frac{1}{\rho_s} & 0 & 0 & 0 \\
\frac{\rho_s c_{s,N}^2}{\alpha_s}(u_s - w_N) & 0 & \rho_s c_s^2 & u_s & 0 & 0 & 0 \\
-\frac{\rho_g}{\alpha_g}(u_g - w_N) & 0 & 0 & 0 & u_g & \rho_g & 0 \\
-\frac{p_g - p_N}{\alpha_g \rho_g} & 0 & 0 & 0 & 0 & u_g & \frac{1}{\rho_g} \\
-\frac{\rho_g c_{s,N}^2}{\alpha_g}(u_g - w_N) & 0 & 0 & 0 & 0 & \rho_g c_g^2 & u_g \\
w_N & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{2.7}$$

where

$$c_k^2 = \frac{\frac{p_k}{\rho_k^2} - \frac{\partial_k}{\partial \rho_k}}{\frac{\partial e_k}{\partial p_k}}, \qquad c_{k,N}^2 = \frac{\frac{p_N}{\rho_k^2} - \frac{\partial_k}{\partial \rho_k}}{\frac{\partial e_k}{\partial p_k}} \tag{2.8}$$

representing the sound speed and the sound speed at the interfacial pressure condition of the k phase, $k = s, g$, respectively.

Solving the characteristic equation for the system

$$|A - \lambda I| = 0$$

gives us seven eigenvalues

$$\begin{aligned} &\lambda_k^- = u_k - c_k, \\ &\lambda_k = u_k, \\ &\lambda_k^+ = u_k + c_k, \quad k = s, g \\ &\lambda_N = w_N. \end{aligned} \tag{2.9}$$

It is derived from (2.9) that the eigenvalues λ_k, λ_N may eventually coincide and interchange the order with any of the other eigenvalues, whenever defined. Consequently, the system is is *not strictly hyperbolic*. Moreover, for Van der Waals fluids for example, the eigenvalues may be complex. Thus, the system *may not be hyperbolic* in a certain region for certain real fluids such as Van der Waals fluids. The last one was also observed in incompressible two-fluid models in [12].

3. Component of the scheme: Stationary waves. In this section, we will investigate stationary waves of the system. These waves play a vital role in the construction of the numerical method. For simplicity, we restrict our consideration to the following under-determined system describing the dynamics of two-phase flows

$$\begin{aligned} &\partial_t(\alpha_s\rho_s) + \partial_x(\alpha_s\rho_s u_s) \;=\; 0, \\ &\partial_t(\alpha_s\rho_s u_s) + \partial_x(\alpha_s(\rho_s u_s^2 + p_s)) \;=\; p_g\partial_x\alpha_s, \\ &\partial_t(\alpha_s\rho_s E_s) + \partial_x(\alpha_s u_s(\rho_s E_s + p_s)) \;=\; p_g u_g\partial_x\alpha_s, \\ &\partial_t(\alpha_g\rho_g) + \partial_x(\alpha_g\rho_g u_g) \;=\; 0, \\ &\partial_t(\alpha_g\rho_g u_g) + \partial_x(\alpha_g(\rho_g u_g^2 + p_g)) \;=\; -\,p_g\partial_x\alpha_s, \\ &\partial_t(\alpha_g\rho_g E_g) + \partial_x(\alpha_g u_g(\rho_g E_g + p_g)) \;=\; -\,p_g u_g\partial_x\alpha_s \\ &\partial_t\alpha_s + u_s\partial_x\alpha_s = 0. \end{aligned} \tag{3.1}$$

The system (3.1) can be simplified in the following way: we keep the equations of the second phase (phase of gas), and add up each equation of the second phase to the corresponding equation of the first phase. We thus obtain an equivalent simpler system where the three new equations

have the form of conservation laws of the total mass, momentum, and energy:

$$
\begin{aligned}
&\partial_t(\alpha_g\rho_g) + \partial_x(\alpha_g\rho_g u_g) \;=\; 0,\\
&\partial_t(\alpha_g\rho_g u_g) + \partial_x(\alpha_g(\rho_g u_g^2 + p_g)) \;=\; p_g\partial_x\alpha_g,\\
&\partial_t(\alpha_g\rho_g E_g) + \partial_x(\alpha_g u_g(\rho_g E_g + p_g)) \;=\; p_g u_g\partial_x\alpha_g,\\
&\partial_t(\sum \alpha\rho) + \partial_x(\sum \alpha\rho u) \;=\; 0,\\
&\partial_t(\sum \alpha\rho u) + \partial_x(\sum \alpha(\rho u^2 + p)) \;=\; 0\\
&\partial_t(\sum \alpha\rho E) + \partial_x(\sum \alpha u(\rho E + p)) = 0\\
&\partial_t\alpha_s + u_s\partial_x\alpha_s = 0.
\end{aligned}
\tag{3.2}
$$

Let us investigate the stationary contacts of the system (3.2). As seen in [13], stationary contacts are just the limit of stationary smooth solution of (3.2). Let us investigate the later in more details. *A stationary smooth solution* U of (3.2) is a time-independent smooth solution. Consequently, the partial derivatives in t in (3.2) is equal to zero and we omit them. Furthermore, we will show next that it is sufficient to consider the equation of the gas phase and the compaction dynamics equation to determine stationary waves.

Therefore, stationary solutions of the initial value problem for (3.2) with smooth initial data satisfy the following ordinary differential equations

$$
\begin{aligned}
&\frac{d}{dx}(\alpha_g\rho_g u_g) = 0,\\
&\frac{d}{dx}(\alpha_g(\rho_g u_g^2 + p_g)) = p_g\frac{d}{dx}\alpha_g,\\
&\frac{d}{dx}(\alpha_g u_g(\rho_g E_g + p_g)) \;=\; p_g u_g\frac{d}{dx}\alpha_g,
\end{aligned}
\tag{3.3}
$$

with the smooth initial data

$$
(\rho_g, u_g, p_g, \alpha_g)(x,0) = (\rho_{g,0}(x), u_{g,0}(x), p_{g,0}(x), \alpha_{g,0}(x)). \tag{3.4}
$$

In the sequel, for simplicity we will denote

$$
\frac{d(.)}{dx} = (.)'.
$$

To determine stationary wave curve, we need the following lemma, which is just a modification of the one of [13].

LEMMA 3.1. *The system (3.3) for smooth solutions is equivalent to*

$$\begin{aligned} &(\alpha_g \rho_g u_g)' = 0, \\ &u_g u_g' + \frac{p_g'}{\rho_g} = 0, \\ &S_g' = 0, \end{aligned} \tag{3.5}$$

Consequently, the entropy is conserved across any stationary smooth solution of the initial value problem for (3.2).

It is implied from Lemma 3.1 that the non-isentropic system of the gas phase is reduced to the isentropic case for stationary solutions. Therefore, *stationary contacts* of (3.2) can be defined as the limit of sequences of stationary smooth solutions in a similar way of the isentropic gases (see [16]).

Proof. Given the initial data U_0. The first equation of (3.3) can be expressed as

$$\alpha_g \rho_g u_g = C,$$

for some constant C depending only on U_0. Taking into account this, we can re-write the second equation as

$$(C \cdot u_g + \alpha_g \cdot p_g)' = p_g \cdot \alpha_g',$$

or

$$C \cdot u_g' + \alpha_g \cdot p_g' = 0.$$

Substituting C, we obtain

$$\alpha_g \rho_g (u_g u_g' + \frac{p_g'}{\rho_g}) = 0.$$

Since the volume fraction is assumed to be positive, we obtain the the second equation in the lemma.

Now, the compaction dynamics equation implies that for stationary waves:

$$u_s \partial_x \alpha_s = 0.$$

Therefore, the equations of balance of energy becomes

$$\frac{d}{dx}(\alpha_g u_g(\rho_g E_g + p_g)) = 0, \tag{3.6}$$

or

$$\frac{d}{dx}(\alpha_g u_g \rho_g) E_g + (\alpha_g u_g \rho_g)\frac{dE}{dx} + \frac{d}{dx}\left(\alpha_g u_g \rho_g \frac{p_g}{\rho_g}\right) = 0.$$

The last equation is equivalent to

$$C \cdot (E_g + \frac{p_g}{\rho_g})' = 0.$$

Besides, the thermodynamics identity says that

$$TdS = de_g + p_g dv_g, \quad v_g = \frac{1}{\rho_g}.$$

Since we are considering stationary waves, i.e., solutions independent of time, the thermodynamics identity applied to this kind of waves gives

$$T_g S_g' = e_g' + p_g v_g'. \tag{3.7}$$

It is then derived from (3.6)-(3.7) that

$$e_g' + u_g u_g' + (p_g v_g)' = 0,$$

or

$$T_g S_g' + \frac{p_g'}{\rho_g} + u_g u_g' = 0.$$

Using the second equation of (3.5) which is just proved, we can reduce the last equation as

$$S_g' = \frac{dS_g}{dx} = 0$$

which is the third equation of (3.5).

The last statement of the lemma can be now proved, since

$$dS = \frac{dS}{dx}dx + \frac{dS}{dt}dt = 0, \tag{3.8}$$

The proof of Lemma 3.1 is complete.

□

The fact that the specific entropy is constant across a stationary contacts (see for example [13]), the last equation becomes trivial. And it is not difficult to check that for states connecting by stationary contacts, one has

$$\begin{aligned} &[\alpha_g \rho_g u_g] = 0 \\ &[\frac{u_g^2}{2} + h_g(\rho_g, S_g-)] = 0, \end{aligned} \tag{3.9}$$

where h_g is the enthalpy of the gas phase, given by the formula

$$\frac{\partial h(\rho_g, S_g)}{\partial \rho_g} = \frac{p_g}{\rho_g}. \tag{3.10}$$

Thus, the determination of stationary waves of the system (3.2) is similar to that in [13].

4. Construction of the new scheme.

4.1. Description. Observe that the conservative variable of the system (3.2) is $(U, \alpha_s)^t$ with

$$U = \begin{pmatrix} U_g \\ V = U_g + U_s \end{pmatrix}_{10}, \tag{4.1}$$

where

$$\begin{aligned} U_g &= (\alpha_g\rho_g, \alpha_g\rho_g u_g, \alpha_g\rho_g e_g)^T \\ U_s &= (\alpha_s\rho_s, \alpha_s\rho_s u_s, \alpha_s\rho_s e_s)^T, \end{aligned} \tag{4.2}$$

Given a uniform time step Δt, and a spacial mesh size Δx, setting $x_j = j\Delta x, j \in \mathbf{Z}$, and $t_n = n\Delta t, n \in \mathbf{N}$, we denote by U_j^n in the sequel the approximation of the values $U(x_j, t_n)$ of the exact solution $U = (a\rho, a\rho u, a\rho e)$ of (1.1).

Then, from the notation (4.1), the approximative vector has the form $(U_j^n, \alpha_{s,j}^n)^t$ with

$$U_j^n = \begin{pmatrix} U_{g,j}^n \\ V_j^n \end{pmatrix}. \tag{4.3}$$

We define the stationary states $(U_{j,\pm}^n, \alpha_{s,j}^n)^t$ with

$$U_{j,\pm}^n = \begin{pmatrix} U_{g,j,\pm}^n \\ V_j^n \end{pmatrix}. \tag{4.4}$$

Set

$$\lambda = \frac{\Delta t}{\Delta x}. \tag{4.5}$$

The constant λ is also required to satisfy the so-called *C.F.L. stability condition*

$$\lambda \max_U |f'(U)| \leq 1. \tag{4.6}$$

The motivation of our study in the following new scheme is to take into account the effect of stationary waves. The method can be decomposed in two stages: first we deal with the stationary waves, and then we descretize the system. The first stage constitutes from two steps:

(i) First at each grid node $x_j, j \in \mathbf{Z}$, we determine two stationary waves of (1.1) in which one stationary wave arrive at x_j with the cross-section level $\alpha^n_{g,j}$ from the given left-hand state U^n_{j-1} (with $\alpha^n_{g,j-1}$) by a right-hand state, denoted by $U^n_{g,j-1,+}$, and another stationary wave arrive at x_j with the cross-section level $\alpha^n_{g,j}$ from the given right-hand state $U^n_{g,j+1}$ (with $\alpha^n_{g,j+1}$) by a left-hand state, denoted by $U^n_{g,j+1,-}$;

(ii) Second, taking a standard numerical scheme for computing $U^{n+1}_{g,j}$ at time $t = (n+1)h$, we substitute $U^n_{g,j+1}$ by $U^n_{g,j+1,-}$, substitute $U^n_{g,j-1}$ by $U^n_{g,j-1,+}$.

Precisely, the new scheme is defined by:

$$U^{n+1}_j = U^n_j - \lambda(g^{\mathrm{N}}(U^n_j, U^n_{j+1,-}) - g^{\mathrm{N}}(U^n_{j-1,+}, U^n_j)), \tag{4.7}$$

where $g^{\mathrm{N}}(U, V)$ can be any standard numerical flux for gas dynamics equations, and $U^n_{j+1,-}, U^n_{j-1,+}$ are given shortly below.

In the scheme (4.7), the states $U^n_{j+1,-}, U^n_{j-1,+}$ defined by (4.4) where $U^n_{g,j+1,-}$ and $U^n_{g,j-1,+}$ are determined from observing that the entropy is constant across each stationary jump, and by computing $\rho^n_{g,j+1,-}, u^n_{g,j+1,-}$ from the following equations

$$\begin{aligned} &\alpha^n_{j+1}\rho^n_{g,j+1}u^n_{g,j+1} = \alpha^n_j\rho^n_{g,j+1,-}u^n_{g,j+1,-}, \\ &\frac{(u^n_{g,j+1})^2}{2} + h_g(\rho^n_{g,j+1}) = \frac{(u^n_{g,j+1,-})^2}{2} + h_g(\rho^n_{g,j+1,-}), \end{aligned} \tag{4.8}$$

and computing $\rho^n_{g,j-1,+}, u^n_{g,j-1,+}$ from the equations

$$\begin{aligned} &\alpha^n_{j-1}\rho^n_{g,j-1}u^n_{g,j-1} = \alpha^n_j\rho^n_{g,j-1,+}u^n_{g,j-1,+}, \\ &\frac{(u^n_{g,j-1})^2}{2} + h_g(\rho^n_{g,j-1}) = \frac{(u^n_{g,j-1,+})^2}{2} + h_g(\rho^n_{g,j-1,+}). \end{aligned} \tag{4.9}$$

4.2. Closure of method: discretization of the compaction dynamics equation. Let us consider the compaction dynamics equation of the Baer-Nonziato model

$$\partial_t\alpha_s + u_s\partial_x\alpha_s = 0. \tag{4.10}$$

Since this is a nonconservative balance law, it is expected to have certain difficulty dealing with it. Here, we propose to employ the technique in the Engquist-Osher scheme. We first write

$$u = \max\{u, 0\} + \min\{u, 0\} = u^+ + u^-. \tag{4.11}$$

and then we apply the backward difference scheme for u^+ and forward difference scheme for u^-. This can be done as arrive at

$$\alpha_j^{n+1} = \alpha_j^n - \lambda\Big(u_j^{+,n}(\alpha_j^n - \alpha_{j-1}^n) + u_j^{-,n}(\alpha_{j+1}^n - \alpha_j^n)\Big). \tag{4.12}$$

Thus, the scheme is well-defined from (4.7) and (4.12).

4.3. Efficiency of the New Scheme. The scheme constructed in the last subsection was shown by experiments to be balanced and fast when we consider the following one-dimensional space gas dynamics equations that describe the evolution of a gas flow in a nozzle with cross-sectional area $a = a(x) > 0, x \in \boldsymbol{R}$:

$$\begin{aligned}
&\partial_t(a\rho) + \partial_x(a\rho u) \;=\; 0,\\
&\partial_t(a\rho u) + \partial_x(a(\rho u^2 + p)) \;=\; p\partial_x a,\\
&\partial_t(a\rho E) + \partial_x(au(\rho E + p)) = 0, \quad x \in \boldsymbol{R},\, t > 0.
\end{aligned} \tag{4.13}$$

The model (4.13) can be seen as the restriction on one phase of the model of two-phase flow (1.1)-(1.6).

In this case, we have the exact solutions (see [16]) to compare between the new scheme and other known schemes. For example, we can see from the Figure 4.1 that the stationary contact is well-approximated with only a short time by the new scheme, while there is significant oscillation if we use known schemes, see Figure 4.2. The reader is referred to see [13] where many tests are available to justify the efficiency of the method.

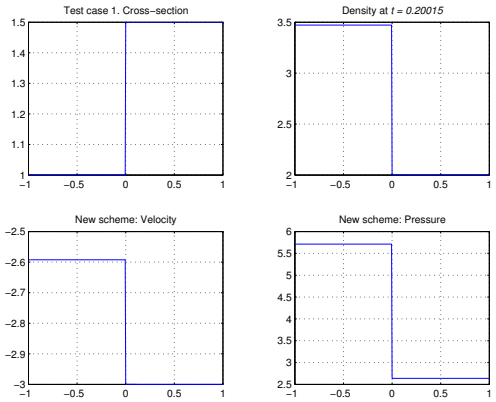

FIG. 4.1. *A stationary contact approximated by the new scheme*

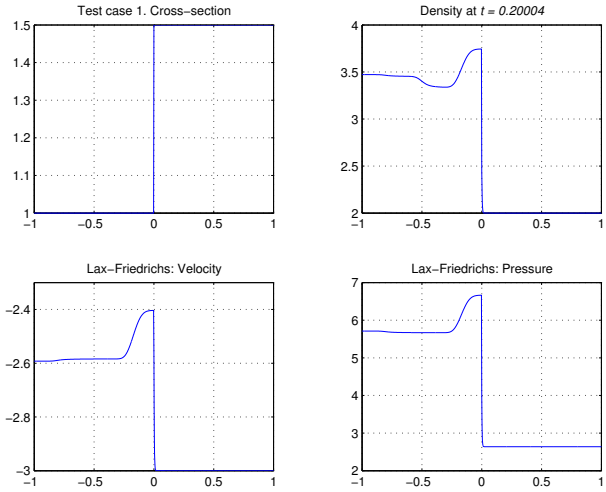

FIG. 4.2. *A stationary contact approximated by Lax-Friedrichs scheme*

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