# Stability of Solutions in Parametric Variational Relation Problems

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**Abstract.** The purpose of this paper is to investigate topological properties and stability of the solution sets in parametric variational relation problems. The results of the paper give a unifying way to treat these questions in the theory of variational inequalities, variational inclusions and equilibrium problems.

**Key Words:** Variational relations, variational inequalities, equilibrium, stability.

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# 1 Introduction

Throughout this paper $A, B$ and $Y$ are nonempty sets, $S_1 : A \rightrightarrows A$, $S_2 : A \rightrightarrows B$ and $T : A \times B \rightrightarrows Y$ are set-valued maps with nonempty values. Let $R(a, b, y)$ be a relation linking $a \in A$, $b \in B$ and $y \in Y$. It can be identified as a subset of the product space $A \times B \times Y$, and the relation $R(a, b, y)$ is said to hold if the triple $(a, b, y)$ belongs to that subset. It is quite frequent that a relation is expressed by equality / inequality of real functions, or by inclusion of set-valued maps of variables $a, b$ and $y$. We consider the following variational relation problem, denoted by (VR):

Find $\bar{a} \in A$ such that
(i) $\bar{a}$ is a fixed point of $S_1$, that is $\bar{a} \in S_1(\bar{a})$;
(ii) $R(\bar{a}, b, y)$ holds for every $b \in S_2(\bar{a})$ and $y \in T(\bar{a}, b)$.

The set-valued maps $S_1, S_2$ and $T$ are called constraints and $R$ is called a variational relation. This general problem has been introduced and studied in [15]. It englobes a large number of variational inequalities, variational inclusions, equilibrium problems, optimization problems and many others (see [11] for these particular problems). As already mentioned in [15], a weaker model can be formulated by requiring $R(\overline{a}, b, y)$ to hold for all $b \in S_2(\overline{a})$ and some $y \in T(\overline{a}, b)$, or for some $b \in S_2(\overline{a})$ and some $y \in T(\overline{a}, b)$. However, the methods of study of this model are similar to those developed in [15] and in the present paper, therefore, our overall concern will be sticked on (VR) only. In [15] we have given a number of sufficient conditions for existence of solutions of (VR). The purpose of the present paper is to investigate topological properties of the solution set of the variational relation problem (VR) and its stability.

The paper is structured as follows. In Section 2 we present a short study of continuities of set-valued maps. Besides the known concepts of set-limits such as outer and inner limits, we introduce two new limits: superior open limit and inferior open limit which help us to link continuities of a set-valued map with its complement. In Section 3 we establish some topological properties of the solution set such as the uniqueness, the closedness, the convexity and the boundedness under suitable conditions. In Section 4 we consider a parametric variational relation problem and prove the four continuity properties ( inner continuity, outer continuity, inner openness and outer openness) of the solution map. In Section 5 we concretize the stability criteria of Section 4 to two well-known models of variational relation problems: the equilibrium model and the variational inclusion model. These applications show that the continuity criteria for the solution set we develop in Section 4 are quite general. They extend and sometimes deepen several results of the recent literature on the topics.

Throughout the paper the following notations will be of use.

- $\Sigma$ is the solution set of the problem (VR).
- $K$ is the set of all fixed points of the map $S_1$.
- $\Gamma := \{a \in A : R(a, b, y)$ holds for all $b \in S_2(a)$ and $y \in T(a, b)\}$.
- $P_R(b) := \{a \in A : R(a, b, y)$ holds for all $y \in T(a, b)\}$ for each $b \in B$

- $P(b) := \{A \setminus S_2^{-1}(b)\} \cup \{K \cap P_R(b)\}$ for each $b \in B$.

Given a subset $D$ of $A$, its complement in $A$ is denoted by $D^c$. Thus, for instance, $K^c$ consists of all the non-fixed points of $S_1$ and $\Gamma^c$ consists of those $a$ of $A$ for which $R(a, b, y)$ does not hold for some $b \in S_2(a)$ and $y \in T(a, b)$. We mention also the following formulas, the first one is clear and the second one is already established in [15]:

$$\Sigma = K \cap \Gamma, \quad (1)$$
$$\Sigma = \cap_{b\in B} P(b). \quad (2)$$

The second formula was extensively used in [15] to establish existence conditions. We shall see in this paper that it is also useful in establishing topological properties of the solution set and the first formula is helpful in studying stability of parametric variational relation problems.

# 2 Continuities of set-valued maps

Throughout this section $\Lambda$ and $X$ denote topological spaces and $F : \Lambda \rightrightarrows X$ denotes a set-valued map. The superior and inferior limits of $F$ at $\lambda_0$ are denoted by $\liminf_{\lambda\to\lambda_0} F(\lambda)$ and $\limsup_{\lambda\to\lambda_0} F(\lambda)$ (see [17]). We define two new limits: the inferior open limit and the superior open limit:

$$\begin{aligned} \mathrm{liminfo}_{\lambda\to\lambda_0} F(\lambda) &:= \{x \in X : \text{there are open neighborhoods } U \text{ of } \lambda_0 \text{ and } V \\ &\quad \text{of } x \text{ such that } V \subseteq F(\lambda) \text{ for all } \lambda \in U, \lambda \neq \lambda_0\}; \\ \mathrm{limsupo}_{\lambda\to\lambda_0} F(\lambda) &:= \{x \in X : \text{there are an open neighborhood } V \text{ of } x \text{ and a net } \lambda_\nu \\ &\quad \text{converging to } \lambda_0 \text{ such that } V \subseteq F(\lambda_\nu) \text{ for all } \nu \text{ with } \lambda_\nu \neq \lambda_0\}. \end{aligned}$$

Here is a relationship between the above set limits. By definition, the complement map $F^c$ is given by $F^c(\lambda) = [F(\lambda)]^c := X \setminus F(\lambda)$.

**Lemma 2.1** *The following relations hold.*

(i) $\mathrm{liminfo}_{\lambda\to\lambda_0} F(\lambda) \subseteq \mathrm{liminf}_{\lambda\to\lambda_0} F(\lambda) \subseteq \mathrm{limsup}_{\lambda\to\lambda_0} F(\lambda)$.

(ii) $\mathrm{liminfo}_{\lambda\to\lambda_0} F(\lambda) \subseteq \mathrm{limsupo}_{\lambda\to\lambda_0} F(\lambda) \subseteq \mathrm{limsup}_{\lambda\to\lambda_0} F(\lambda)$.

(iii) $\mathrm{liminfo}_{\lambda\to\lambda_0} F^c(\lambda) = [\mathrm{limsup}_{\lambda\to\lambda_0} F(\lambda)]^c$.

(iv) $\mathrm{liminf}_{\lambda\to\lambda_0} F^c(\lambda) = [\mathrm{limsupo}_{\lambda\to\lambda_0} F(\lambda)]^c$.

**Proof.** The first and the second relations follow easily from the definition. Let's prove (iii). Let $x$ be an element of the set limit on the left hand side. Let $U$ and $V$ be two neighborhoods of $\lambda_0$ and $x$ as in the definition of the inferior open limit. Then $V \cap F(\lambda) = \emptyset$ for all $\lambda \in U$ which implies that $x$ does not belong to $\limsup_{\lambda\to\lambda_0} F(\lambda)$. Conversely, let $x$ be an element outside of the open inferior limit of $F^c(\lambda)$. Then for every neighborhoods $U$ of $\lambda_0$ and $V$ of $x$, there exists $\lambda_{U,V} \in U$ such that $V \not\subseteq F^c(\lambda_{U,V})$. Thus, there is some $x_{U,V}$ from the intersection $V \cap F(\lambda_{U,V})$. Choose $U$ and $V$ from a basis of neighborhoods of $\lambda_0$ and $x$ so that the nets $\lambda_{U,V}$ and $x_{U,V}$ converge respectively to $\lambda_0$ and $x$. Then $x$ belongs to $\limsup_{\lambda\to\lambda_0} F(\lambda)$.

Now we prove (iv). Let $x$ be any element of the limit in the left hand side. For every net $\lambda_\nu$ converging to $\lambda_0$, there is some $x^\nu \notin F(\lambda_\nu)$ such that the net of these $x^\nu$ converges to $x$. This implies that for every neighborhood $V$ of $x$, we have $V \not\subseteq F(\lambda_\nu)$ for all $\lambda_\nu$ sufficiently close to $\lambda_0$. Hence $x$ does not belong to $\text{limsupo}_{\lambda\to\lambda_0} F(\lambda)$. For the converse, if $x$ is outside of the set on the left hand side, then there is a net $\lambda_\nu$ converging to $\lambda_0$ and a neighborhood $V$ of $x$ such that $V \cap F^c(\lambda_\nu) = \emptyset$. Hence $V \subseteq F(\lambda_\nu)$ for all $\nu$, which shows that $x$ does not belong to the set on the right hand side. The proof is complete. □

When $F$ is a single-valued, continuous function, it is clear that the inferior open and superior open limits of $F$ are empty, while its superior and inferior limits coincide with the value of $F$ at the limit point. In this case the first inclusion of (i) and the second inclusion of (ii) are strict.

**Definition 2.2** *We say that the map $F$ is*

- *closed (respectively open) on $\Lambda$ if its graph is a closed (respectively open) set in the product space $\Lambda \times X$;*
- *outer-continuous at $\lambda_0 \in \Lambda$ if $\limsup_{\lambda\to\lambda_0} F(\lambda) \subseteq F(\lambda_0)$;*
- *inner-continuous (or lower semi-continuous) at $\lambda_0 \in \Lambda$ if $\liminf_{\lambda\to\lambda_0} F(\lambda) \supseteq F(\lambda_0)$;*
- *upper semi-continuous (usc) at $\lambda_0 \in \Lambda$ if for every open set $V$ containing $F(\lambda_0)$ there is a neighborhood $U$ of $\lambda_0$ such that $F(U) \subseteq V$;*
- *inner-open (or open) at $\lambda_0 \in \Lambda$ if $\text{liminfo}_{\lambda\to\lambda_0} F(\lambda) \supseteq F(\lambda_0)$;*
- *outer-open at $\lambda_0$ if $\text{limsupo}_{\lambda\to\lambda_0} F(\lambda) \subseteq F(\lambda_0)$.*

*When $F$ has certain continuity at every point of $\Lambda$, we say that it does so on $\Lambda$.*

In some literature (e.g. [6]) $\lambda$ is allowed to take the value $\lambda_0$ in the limsup and liminf. In this paper we require, however, that $\lambda$ converges to $\lambda_0$ with $\lambda \neq \lambda_0$. Here are some characterizations of continuous maps via their graphs and complements.

**Proposition 2.3** *The following assertions hold.*

(i) *$F$ is outer-open at $\lambda_0$ if and only if $F^c$ is inner-continuous at $\lambda_0$.*

(ii) *$F$ is outer-continuous at $\lambda_0$ if and only if $F^c$ is inner-open at $\lambda_0$.*

(iii) *$F$ is outer-continuous and closed-valued (resp. inner-open and open-valued) on $\Lambda$ if and only if its graph is closed (resp. open).*

(iv) *If $F$ is outer-continuous at $\lambda_0$, then it is outer-open there.*

(v) *$F$ is inner-open at $\lambda_0$, then it is inner-continuous there.*

**Proof.** The first and the second assertions follow from the equalities (iv) and (iii) of Lemma 2.1. The assertion (iv) is derived from the inclusion (ii) of Lemma 2.1. The assertion (v) is quite easy and already known. Let us prove (iii). If $F$ is closed, then it is clear that it is outer-continuous. Moreover, for every $\lambda$, the value $F(\lambda)$ is closed because it is the intersection of the closed set $\{\lambda\} \times X$ with the graph of $F$. For the converse, assume that $F$ is outer-continuous and closed-valued. Let $\{\lambda_\nu, x^\nu\}$ be any net of the graph of $F$ converging to $(\lambda_0, x^0)$. Then one can find a subnet, that we denote by the same symbol, such that either all $\lambda_\nu$ coincide with $\lambda_0$, which yield $x^\nu \in F(\lambda_0)$ for all $\nu$, or $\lambda_\nu \neq \lambda_0$. In the first case, $x^0 \in F(\lambda_0)$ because the latter set is closed. In the second case, $x^0$ belongs to the superior limit of $F(\lambda_\nu)$, which is included in $F(\lambda_0)$ by the outer-continuity of $F$. Thus, $(\lambda_0, x^0)$ belongs to the graph of $F$ and $F$ is closed. For the case of open maps, it suffices to notice that a map is open if and only if its complement is closed. □

It is known that inner-continuous maps are not necessarily inner-open, for instance, single-valued, continuous maps are inner-continuous, but never open. Thus, the converse of (v) is not true. The same can be said about (iv). The constant map on $\mathbb{R}$ whose values are open set $(0,1)$ is outer-open, but not outer-continuous. Now we turn to the union and intersection of set-valued maps. In the remaining part of this section we assume that $F$ and $G$ are two set-valued maps on $\Lambda$ with values in $X$ and $F \cap G$ has nonempty values at the points under consideration.

**Lemma 2.4** *The following containments and inclusions hold:*

$$\begin{aligned} \lim\sharp_{\lambda\to\lambda_0} F(\lambda) \cup G(\lambda) &\supseteq \lim\sharp_{\lambda\to\lambda_0} F(\lambda) \cup \lim\sharp_{\lambda\to\lambda_0} G(\lambda), \\ \lim\sharp_{\lambda\to\lambda_0} F(\lambda) \cap G(\lambda) &\subseteq \lim\sharp_{\lambda\to\lambda_0} F(\lambda) \cap \lim\sharp_{\lambda\to\lambda_0} G(\lambda), \end{aligned}$$

*in which the symbol $\sharp$ may be any of "sup", "supo", "inf", "info". Actually the containment for the superior limit and the inclusion for the inferior open limit are equality. Moreover, it is also true that*

$$\operatorname{limsupo}_{\lambda\to\lambda_0} F(\lambda) \cup \operatorname{limsup}_{\lambda\to\lambda_0} G(\lambda) \supseteq \operatorname{limsupo}_{\lambda\to\lambda_0} F(\lambda) \cup G(\lambda),$$

$$\operatorname{liminfo}_{\lambda\to\lambda_0} F(\lambda) \cap \operatorname{liminf}_{\lambda\to\lambda_0} G(\lambda) \subseteq \operatorname{liminf}_{\lambda\to\lambda_0} F(\lambda) \cap G(\lambda).$$

**Proof.** The containments and inclusions for the inferior and superior limits are standard (see [6]). The containment and inclusion for the superior open limit and the containment for the inferior open limit are direct from the definition. Let us prove equality for the inferior open limit of the intersection. The inclusion

$$\operatorname{liminfo}_{\lambda\to\lambda_0} F(\lambda) \cap G(\lambda) \subseteq \operatorname{liminfo}_{\lambda\to\lambda_0} F(\lambda) \cap \operatorname{liminfo}_{\lambda\to\lambda_0} G(\lambda)$$

is clear. Let $x$ be an element of the set on the right hand side. There are two neighborhoods $V_1$ and $V_2$ of $x$ and two neighborhoods $U_1$ and $U_2$ of $\lambda_0$ such that $F(\lambda)$ contains $V_1$ for all $\lambda \in U_1$ and $G(\lambda)$ contains $V_2$ for all $\lambda \in U_2$. Set $V = V_1 \cap V_2$ and $U_1 \cap U_2$. Then $V \subseteq F(\lambda) \cap G(\lambda)$ for all $\lambda \in U$. Thus, $x$ belongs to the set on the left hand side.

Let us now prove the last assertion. Let $x$ be an element of the superior open limit of $F \cup G$ at $\lambda_0$. By definition, there exist a neighborhood $V$ of $x$ and a net $\lambda_\nu$ converging to $\lambda_0$ such that

$$V \subseteq F(\lambda_\nu) \cup G(\lambda_\nu) \text{ for all } \nu. \tag{3}$$

If $x$ belongs to $\limsup_{\lambda\to\lambda_0} G(\lambda)$, then we are done. If not, in view of Lemma 2.1 (iii), it belongs to the set $\operatorname{liminfo}_{\lambda\to\lambda_0} G^c(\lambda)$ which means that there are neighborhoods $W$ of $x$ and $U$ of $\lambda_0$ such that $W \subseteq G^c(\lambda)$ for all $\lambda \in U, \lambda \neq \lambda_0$. This and (3) show that

$$V \cap W \subseteq F(\lambda_\nu) \text{ for all } \nu \text{ with } \lambda_\nu \neq \lambda_0.$$

Hence $x$ belongs to $\operatorname{limsupo}_{\lambda\to\lambda_0} F(\lambda)$.

The last inclusion is obtained from the preceding containment by means of equalities (iii) and (iv) of Lemma 2.1.

□

**Proposition 2.5** *The following assertions hold.*

(i) *If $F$ and $G$ are outer-continuous (respectively inner-open, inner-continuous) at $\lambda_0$, then their union is such at $\lambda_0$.*

(ii) *If $F$ is outer-continuous and $G$ is outer-open at $\lambda_0$, then their union is outer-open at $\lambda_0$.*

**Proof.** The first assertion follows from the first containment of Lemma 2.4. The second assertion follows from the first containment of the last assertion of Lemma 2.4. □

**Proposition 2.6** *The following assertions hold.*

(i) *If $F$ is outer-continuous (respectively outer-open) at $\lambda_0$ and if*
$$[\limsup_{\lambda\to\lambda_0} G(\lambda)] \cap F(\lambda_0) \subseteq G(\lambda_0)$$
*(respectively* $[\operatorname{limsupo}_{\lambda\to\lambda_0} G(\lambda)] \cap F(\lambda_0) \subseteq G(\lambda_0)$*),*
*then the intersection map $F \cap G$ is outer-continuous (respectively outer-open) at $\lambda_0$.*

(ii) *If $F$ is inner-open at $\lambda_0$ and if*
$$\operatorname{liminfo}_{\lambda\to\lambda_0} G(\lambda) \supseteq G(\lambda_0) \cap F(\lambda_0),$$
*then the intersection map $F \cap G$ is inner-open at $\lambda_0$.*

(iii) *If $F$ is inner-continuous (respectively inner-open) at $\lambda_0$ and if*
$$\operatorname{liminfo}_{\lambda\to\lambda_0} G(\lambda) \supseteq G(\lambda_0) \cap F(\lambda_0)$$
*(respectively* $\liminf_{\lambda\to\lambda_0} G(\lambda) \supseteq G(\lambda_0) \cap F(\lambda_0)$*),*
*then the intersection map $F \cap G$ is inner-continuous at $\lambda_0$.*

*In particular,*

(iv) *if $F$ and $G$ are outer-continuous (respectively inner-open, outer-open) at $\lambda_0$, then their intersection is such at $\lambda_0$;*

(v) *if $F$ is inner-open and $G$ is inner-continuous at $\lambda_0$, then their intersection is inner-continuous at $\lambda_0$.*

**Proof.** For the first assertion, according to the second inclusion of Lemma 2.4, we have

$$\begin{aligned}
\text{limsup}_{\lambda\to\lambda_0} F(\lambda)\cap G(\lambda) &\subseteq \text{limsup}_{\lambda\to\lambda_0} F(\lambda)\cap \text{limsup}_{\lambda\to\lambda_0} G(\lambda)\\
&\subseteq F(\lambda_0)\cap \text{limsup}_{\lambda\to\lambda_0} G(\lambda)\\
&\subseteq F(\lambda_0)\cap G(\lambda_0),
\end{aligned}$$

in which the second inclusion is due to the outer-continuity of $F$ and the last inclusion follows from the hypothesis on $G$. The proof for the outer-openness is similar.

The second assertion follows from the second equality of Lemma 2.4 as well:

$$\begin{aligned}
\text{liminfo}_{\lambda\to\lambda_0} F(\lambda)\cap G(\lambda) &= \text{liminfo}_{\lambda\to\lambda_0} F(\lambda)\cap \text{liminfo}_{\lambda\to\lambda_0} G(\lambda)\\
&\supseteq F(\lambda_0)\cap \text{liminfo}_{\lambda\to\lambda_0} G(\lambda)\\
&\supseteq F(\lambda_0)\cap G(\lambda_0),
\end{aligned}$$

in which the second containment is obtained by the inner-openness of $F$, and the second containment is by the hypothesis on $G$. As to the third assertion, by a similar reason we have

$$\begin{aligned}
\text{liminf}_{\lambda\to\lambda_0} F(\lambda)\cap G(\lambda) &\supseteq \text{liminf}_{\lambda\to\lambda_0} F(\lambda)\cap \text{liminfo}_{\lambda\to\lambda_0} G(\lambda)\\
&\supseteq F(\lambda_0)\cap \text{liminfo}_{\lambda\to\lambda_0} G(\lambda)\\
&\supseteq F(\lambda_0)\cap G(\lambda_0),
\end{aligned}$$

when $F$ is inner-continuous. The same argument works for case where $F$ is inner-open. □

We notice that the intersection of two inner continuous maps is not necessarily inner continuous. Likewise, the union of two outer-open maps is not necessarily outer-open. To see this, let us define set-valued maps $F$ and $G$ on $\mathbb{R}$ by $F(\lambda) = \mathbb{Q}$ (the set of rational numbers) and $G(\lambda) = \mathbb{R}\setminus\mathbb{Q}$ for $\lambda \neq 0$, and $F(0) = G(0) = \{0\}$. These maps are outer-open because their superior open limits are empty everywhere. Nevertheless their union is not outer-open at 0, for the superior open limit of the union at this point is the whole space $\mathbb{R}$.

# 3 Topological properties of the solution set

In this section we wish to know under which kind of data the solution set of (VP) has nice properties such as uniqueness, convexity, boundedness and closedness. These properties are desired because they are tightly related to convergence and efficiency of methods for computing solutions of the problem. As already said, in this paper we deal with topological properties of the solution set and its stability only, therefore, we assume throughout that the problem under consideration admits solutions. Let us begin with the uniqueness. Recall that a set-valued map $S$ on a metric space $(M, d)$ is a contraction if there is a positive number $\alpha < 1$ such that for every $u, v \in M$ one has $\max\{\sup_{x\in S(u)} d(x, S(v)); \sup_{y\in S(v)} d(y, S(u))\} \leq \alpha d(u, v)$.

**Proposition 3.1** *Problem (VR) has at most one solution if either $A$ is a metric space and $S_1$ is a contraction, or the following conditions hold*

(i) *the fixed point set $K$ of $S_1$ is contained in the image $S_2(x)$ for all $x$ being fixed point of $S_1$;*

(ii) *$R$ is anti-symmetric on $K$, that is, if $a, b \in K$ with $a \neq b$ and $R(a, b, y)$ holds for all $y \in T(a, b)$, then $R(b, a, y)$ does not hold for some $y \in T(b, a)$.*

**Proof.** It is evident that if $S_1$ is a contraction, then it has at most one fixed point, and therefore (VP) cannot have more then one solution. Assume (i) and (ii). Let $a_1$ and $a_2$ be two solutions of (VR). In particular, both of them are fixed points of $S_1$ and in view of (i), $a_1 \in S_2(a_2)$ and $a_2 \in S_2(a_1)$. This implies that $R(a_1, a_2, y)$ holds for all $y \in T(a_1, a_2)$, and $R(a_2, a_1, y)$ holds for all $y \in T(a_2, a_1)$. In view of (ii), the solutions $a_1$ and $a_2$ must coincide. □

The anti-symmetry of the relation $R$ can be guaranteed for instance when $R$ is determined by a strict inequality of a real-valued function, or by a strict inclusion of a set-valued map. Thus, let $f$ be a real-valued function on $A \times B$ and let $F$ be a set-valued map from $A \times B$ to some space $Z$. Assume that $R(a, b, y)$ holds for $a \neq b$ if and only if $f(a) < f(b)$, or it holds for $a \neq b$ if and only if $F(a) \subset F(b)$ and $F(a) \neq F(b)$. Then $R$ is anti-symmetric.

The following example of classical variational inequality problem gives a link between the anti-symmetry of a relation and the strict monotonicity of an operator.

**Example 3.2** Consider the Stampacchia variational inequality problem:

Find $\overline{a} \in A$ such that $\langle f(\overline{a}), x - \overline{a} \rangle \geq 0$ for all $x \in A$,

where $A$ is a convex subset of a real Hilbert space $H$ and $f$ is an operator from $H$ to itself. To express this problem in the form of (VR), it suffices to define $B = A, Y = H, S_1(x) = A, S_2(x) = A$ and $T(x, b) = \{b\}$ for all $x, b \in A$. The relation $R$ is defined as follows: $R(x, b, y)$ holds if and only if $\langle f(x), y - x \rangle \geq 0$. It is easy to see that the relation $R$ is anti-symmetric if the operator is strictly monotone in the sense that $\langle f(x) - f(y), x - y \rangle > 0$ when $x \neq y$. The interested reader is referred to [9], [8], [14], [16] for uniqueness of solutions in variational inequalities.

In what follows $A, B$ and $Y$ are assumed to be topological spaces. We say that $R$ is closed if the set determining it is closed in the product space $A \times B \times Y$. This is often the case when $R$ is defined by non-strict inequalities of real-valued continuous functions, or non-strict inclusions of set-valued maps. We say also that $R$ is closed in the first variable if $R(a, b, y)$ holds whenever $R(a^\nu, b, y)$ holds for all $\nu$ with $a^\nu$ converging to $a$ and with $b$ and $y$ being fixed. The next two propositions give conditions for the closedness of the solution set.

**Proposition 3.3** *The solution set $\Sigma$ is closed provided that the following conditions hold:*

(i) *$S_1$ is closed;*

(ii) *$S_2$ is inner-continuous;*

(iii) *$T$ is inner-continuous;*

(iv) *$R$ is closed.*

**Proof.** Let $\{a^\nu\}$ be a net of solutions of (VP) converging to some $a \in A$. We show that this limit is also a solution. Indeed, by (i), it belongs to the set $K$. For every $b \in S_2(a)$ and $y \in T(a,b)$, one can find $b^\nu \in S_2(a^\nu)$ converging to $b$ and $y^\nu \in T(a^\nu, b^\nu)$ converging to $y$ due to (ii) and (iii). We know that $R(a^\nu, b^\nu, y^\nu)$ holds by assumptions. In view of (iv), $R(a,b,y)$ is satisfied. Thus, $a$ belongs to $\Gamma$. By the relation (1) it is a solution of the problem. □

The closedness of the solution set can also be obtained by relaxing conditions (iii) and (iv), but strengthening condition (ii).

**Proposition 3.4** *The solution set $\Sigma$ is closed provided that the following conditions hold:*

(i) *$S_1$ is closed;*

(ii) *$S_2$ has open inverse values;*

(iii) *$T$ is inner-continuous in the first variable;*

(iv) *$R$ is closed in the first and third variables.*

**Proof.** As in the previous proof, let $\{a^\nu\}$ be a net of solutions of (VP) converging to some $a \in X$. We show that this limit is a solution by using the second relation at the end of Section 1. Let $b$ be any element of $B$. By (ii), the set $A \setminus S_2^{-1}(b)$ is closed, and by (i) the set $K$ is closed. We prove now that $P_R(b)$ is closed. Indeed, let $\{a^\nu\}$ be a net of elements of $P_R(b)$ converging to to some $a \in A$. Let $y$ be any element of $T(a,b)$. By (iii) there is $y^\nu \in T(a^\nu, b)$ converging to $y$. Since $R(a^\nu, b, y^\nu)$ are satisfied, in view of (iv), $R(a,b,y)$ is satisfied too. By this, $a$ belongs to $P_R(b)$ and hence $P_R(b)$ is closed. We conclude that $P(b)$ is closed for every $b \in B$ and hence $\Sigma$ is closed. □

In order to study the convexity of the solution set, we need some convexity concepts of set-valued maps.

**Definition 3.5** *Assume that $A$, $B$ and $Y$ are convex subsets of linear spaces.*

- *A set-valued map $S : A \rightrightarrows Y$ is said to be convex (by inclusion) if for any $x_1, x_2 \in A$ and $t \in [0,1]$ one has $tS(x_1) + (1-t)S(x_2) \subseteq S(tx_1 + (1-tx_2))$; When the inclusion "$\subseteq$" is replaced by the containment "$\supseteq$", the map $S$ is said to be concave.*

- *The relation $R$ is said to be convex if whenever $R(a_i, b_i, y_i)$ holds for $a_i \in A, b_i \in B$ and $y_i \in Y, i = 1, 2,$ the relation $R(ta_1 + (1-t)a_2, tb_1 + (1-t)b_2, ty_1 + (1-t)y_2)$ is satisfied for all $t \in [0,1]$. In other words, $R$ is convex if the set determining it is convex in the product space $A \times B \times Y$.*

In many models (see Section 5) the maps $S_1$ and $S_2$ are constant with convex values, so that they are concave and convex at the same time. Moreover, if $R$ is defined by the inequality $f(a, b, y) \leq 0$ where $f$ is a real-valued function on $A \times B \times Y$ and $A$, $B$ and $Y$ are convex sets, then $R$ is convex if and only if the function $f$ is convex.

**Proposition 3.6** *Assume that $A$, $B$ and $Y$ are convex subsets of linear spaces. Then the solution set $\Sigma$ is convex provided that the constraint map $S_1$ is convex, the maps $S_2$ and $T$ are concave and the relation $R$ is convex.*

**Proof.** It follows from the convexity of $S_1$ that $K$ is convex. We show that $\Gamma$ is convex. Indeed, let $a_1$ and $a_2$ be two elements of $\Gamma$. For each $t \in [0,1]$ and $b \in S_2(ta_1 + (1-t)a_2)$, there exist $b_1 \in S_2(a_1)$ and $b_2 \in S_2(a_2)$ such that $b = tb_1 + (1-t)b_2$. Let $y$ be any element of $T(ta_1 + (1-t)a_2, tb_1 + (1-t)b_2)$. Since $T$ is concave, there exist $y_1 \in T(a_1, b_1)$ and $y_2 \in T(a_2, b_2)$ such that $y = ty_1 + (1-t)y_2$. Then $R(a_i, b_i, y_i), i = 1, 2$ hold. The convexity of $R$ implies that $R(ta_1 + (1-t)a_2, tb_1 + (1-t)b_2, ty_1 + (1-t)y_2)$ is satisfied. Thus, $ta_1 + (1-t)a_2$ belongs to $\Gamma$. According to the relation (1) we conclude that $\Sigma$ is convex. □

When the image of every element $a \in A$ under $S_2$ is the whole set $B$, the convexity requirement on $T$ and $R$ can be weakened.

**Proposition 3.7** *Assume that $A$, $B$ and $Y$ are convex subsets of linear spaces. Then each of the following conditions is sufficient for the solution set $\Sigma$ to be convex:*

(i) *$S_1$ is convex, $S_2$ is constant of value $B$, $T$ is convex in the first variable and $R$ is convex in the first and the third variables;*

(ii) *$S_1$ is convex, $S_2$ is constant of value $B$, $T$ is constant of value $Y$ and $R$ is convex in the first variable.*

**Proof.** Apply the same argument as the proof of the preceding proposition with use of the relation (2) instead of (1). □

**Example 3.8** Let $A$ and $f$ be as in Example 3.2. Consider the Minty variational inequality problem:

Find $\overline{a} \in A$ such that $\langle f(x), x - \overline{a} \rangle \geq 0$ for all $x \in A$.

To express this problem in the form of (VR), it suffices to define $B = A, Y = H, S_1(x) = A, S_2(x) = A$ and $T(x, b) = A$ for all $x, b \in A$. The relation $R$ is defined as follows: $R(x, b, y)$ holds if and only if $\langle f(y), y - x \rangle \geq 0$. It is clear that $R$ is convex in the first variable, so the solution set of the Minty variational inequality problem is always convex if it is nonempty in view of Proposition 3.7. When $f$ is continuous and pseudo-monotone in the sense that $\langle f(y), x - y \rangle \geq 0$ implies $\langle f(x), x - y \rangle \geq 0$, the solution set of the Stampacchia problem and the Minty problem coincide, hence the solution set of the Stampacchia problem is convex too (see (Proposition 3.1,[12]) for this known result).

We now turn to the boundedness of the solution set. To this end a concept of coerciveness of the relation $R$ is needed.

**Definition 3.9** *Assume that $A$ is a subset of a normed space. We say that $R$ is coercive in the first variable (with respect to the constraints $(S_1, S_2, T)$) if for every sequence $\{x^k\}$ in $K$ whose norm converges to $\infty$, one can find an integer $k_0$ such that whenever $k \geq k_0$ the relation $R(x^k, b^k, y^k)$ does not holds for some $b^k \in S_2(x^k)$ and $y^k \in T(x^k, b^k)$.*

It is clear that when $S_1(A)$ is bounded, then $K$ is bounded, and hence $R$ is coercive. When $R$ is defined by the inequality $f(x, b, y) \leq 0$, where $f$ is a real-valued function on $A \times B \times Y$, it is coercive in the first variable provided that $f$ is coercive in $x$ in the sense that $\lim_{\|x\| \to \infty} f(x, b, y) = \infty$ for all $b$ and $y$.

**Proposition 3.10** *Assume that $A$ is an unbounded subset of a normed space. Then the solution set $\Sigma$ is bounded if $R$ is coercive in the first variable.*

**Proof.** The conclusion follows directly from the hypothesis and from the formula (1). □

In the Stampacchia problem if the operator $f$ is coercive on $A$ in the sense that $\lim_{x \in A, \|x\| \to \infty} \langle f(x), x - a \rangle = \infty$ for some $a$ fixed, then it is clear that $R$ is coercive in the first variable. Hence, the solution set is bounded (see Theorem 3.2 of [12]). On the other hand, for the Minty problem, if all vectors $f(y), y \in A$ are strictly positive on the recession cone of $A$ without the origin, then $R$ is coercive and hence the solution set is bounded too. We refer the interested reader to [10] for more examples of coercive relations of equilibrium problems.

# 4 Stability

Throughout this section $\Lambda$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{Y}$ are topological spaces. For each $\lambda \in \Lambda$, we assume that $A^\lambda \subseteq \mathcal{A}$, $B^\lambda \subseteq \mathcal{B}$, $Y^\lambda \subseteq \mathcal{Y}$ are nonempty sets; $S_1^\lambda : A^\lambda \rightrightarrows A^\lambda$, $S_2^\lambda : A^\lambda \rightrightarrows B^\lambda$, $T^\lambda : A^\lambda \times B^\lambda \rightrightarrows Y^\lambda$ are set-valued maps with nonempty values; and $R^\lambda(a, b, y)$ is a relation linking elements $a \in A^\lambda$, $b \in B^\lambda$ and $y \in Y^\lambda$. The variational relation problem with the data $A^\lambda$, $B^\lambda$, $Y^\lambda$, $S_1^\lambda$, $S_2^\lambda$, $T^\lambda$ and $R^\lambda$ is denoted by $(VR)^\lambda$. The notations $\Sigma^\lambda$, $K^\lambda$, $\Gamma^\lambda$ etc. are defined accordingly. In this section we study the question how the solution set $\Sigma^\lambda$ changes when the data of the problem vary. More precisely, we shall establish inner and outer-continuity as well as inner and outer-openness of $\Sigma^\lambda$ as a set-valued map of the variable $\lambda$. We shall fix a value $\lambda_0 \in \Lambda$ and assume that the problem (VR) corresponds to the case of $(VR)^{\lambda_0}$. The complement of the map $\Gamma^\lambda$ is understood in the space $\mathcal{A}$.

**Theorem 4.1** *The set-valued map $\Sigma^\lambda$ is outer-open at $\lambda_0$, that is, $\mathrm{limsupo}_{\lambda \to \lambda_0} \Sigma^\lambda \subseteq \Sigma$ if the maps $K^\lambda$ is outer-open at $\lambda_0$ and $\mathrm{limsupo}_{\lambda \to \lambda_0} \Gamma^\lambda \cap K \subseteq \Gamma$. In particular this is true under the following conditions*

(i) $\limsup_{\lambda \to \lambda_0} A^\lambda \subseteq A$;

(ii) *for every $x \in A$, $x \in S_1(x)$ whenever $x \in S_1^{\lambda_\nu}(x)$ for some net $\lambda_\nu$ converging to $\lambda_0$;*

(iii) *for every $x \in K$,*

(iii$_1$) $\liminf_{\lambda\to\lambda_0} S_2^\lambda(x) \supseteq S_2(x)$;

(iii$_2$) $\liminf_{\lambda\to\lambda_0, b'\in S_2^\lambda(x)\to b} T^\lambda(x,b') \supseteq T(x,b)$;

(iii$_3$) *$R(x,b,y)$ holds whenever there are some nets $\lambda_\nu$ converging to $\lambda_0$, $b^\nu \in S_2^{\lambda_\nu}(x)$ converging to $b$ and $y^\nu \in T^{\lambda_\nu}(x,b^\nu)$ converging to $y$ such that $R^{\lambda_\nu}(x,b^\nu,y^\nu)$ holds for all $\nu$.*

**Proof.** According to (1), we have $\Sigma^\lambda = K^\lambda \cap \Gamma^\lambda$. Therefore, the first part of the theorem is derived from Proposition 2.6 (i). For the second part, we show that the set-valued map $\lambda \mapsto K^\lambda$ is outer-open at $\lambda_0$. Let $x \in \text{limsupo}_{\lambda\to\lambda_0} K^\lambda$. There is a net $\lambda_\nu$ converging to $\lambda_0$ and a neighborhood $V$ of $x$ with $V \subseteq A^{\lambda_\nu}$ and $z \in S_1^{\lambda_\nu}(z)$ for all $z \in V$. It follows from (i) that $x \in A$, and from (ii) that $x \in S_1(x)$. Thus, $x \in K$.

Now, if $x \in K \cap \text{limsupo}_{\lambda\to\lambda_0}\Gamma^\lambda$, then for some net $\lambda_\nu$ converging to $\lambda_0$ and some neighborhood $V$ of $x$ one has $V \subseteq A^{\lambda_\nu}$ and the relation $R^{\lambda_\nu}(z,b,y)$ holds for all $z \in V, b \in S_2^{\lambda_\nu}(z)$ and $y \in T^{\lambda_\nu}(z,b)$. Given $b \in S_2(x)$ and $y \in T(x,b)$, in view of (iii$_1$) and (iii$_2$) one can find $b^\nu \in S_2^{\lambda_\nu}(x)$ and $y^\nu \in T^{\lambda_\nu}(x,b^\nu)$ such that $b^\nu$ and $y^\nu$ converge respectively to $b$ and $y$. Then $R^{\lambda_\nu}(x,b^\nu,y^\nu)$ holds, and by (iii$_3$), $R(x,b,y)$ holds too. In this way, $K \cap \text{limsupo}_{\lambda\to\lambda_0}\Gamma^\lambda \subseteq \Gamma^\lambda$ and by Proposition 2.6(i) the proof is complete. □

**Theorem 4.2** *The map $\Sigma^\lambda$ is outer-continuous at $\lambda_0$, that is, $\limsup_{\lambda\to\lambda_0} \Sigma^\lambda \subseteq \Sigma$, if $K^\lambda$ is outer-continuous at $\lambda_0$ and $[\text{limsup}_{\lambda\to\lambda_0}\Gamma^\lambda] \cap K \subseteq \Gamma$. In particular this is true under the following conditions*

(i) $\limsup_{\lambda\to\lambda_0} A^\lambda \subseteq A$;

(ii) *for every $x \in A$, $x \in S_1(x)$ whenever there are a net $\lambda_\nu$ converging to $\lambda_0$ and $x^\nu \in S_1^{\lambda_\nu}(x^\nu)$ converging to $x$;*

(iii) *for every $x \in K$,*

(iii$_1$) $\liminf_{\lambda\to\lambda_0, x^\lambda\in A^\lambda\to x} S_2^\lambda(x^\lambda) \supseteq S_2(x)$;

(iii$_2$) $\liminf_{\lambda\to\lambda_0, x^\lambda\in A^\lambda\to x, b^\lambda\in S_2^\lambda(x^\lambda)\to b} T^\lambda(x^\lambda,b^\lambda) \supseteq T(x,b)$;

(iii$_3$) *$R(x,b,y)$ holds whenever $R^{\lambda_\nu}(x^\nu,b^\nu,y^\nu)$ holds for some $x^\nu \in A^{\lambda_\nu}$, $b^\nu \in S_2^{\lambda_\nu}(x^\nu)$ and $y^\nu \in T^{\lambda_\nu}(x^\nu,b^\nu)$ converging to $x, b$ and $y$ respectively, and $\lambda_\nu$ converges to $\lambda_0$.*

**Proof.** As in the proof of Theorem 4.1, the first part of the theorem is derived from Proposition 2.6 (i). For the second part, we note that due to (i) and (ii), the set-valued map $\lambda \mapsto K^\lambda$ is outer-continuous at $\lambda_0$. Now let $x \in [\text{limsup}_{\lambda\to\lambda_0}\Gamma^\lambda] \cap K$. There are $x^\nu \in \Gamma^{\lambda_\nu}$ converging to $x$ as $\lambda_\nu$ converging to $\lambda_0$. We show that $x$ belongs to $\Gamma$. Let $b$ be any element of $S_2(x)$ and let $y$ be any element of $T(x,b)$. By (iii$_1$) and (iii$_2$) there are some $b^\nu \in S_2^{\lambda_\nu}$ and $y^\nu \in T^{\lambda_\nu}(x^\nu,b^\nu)$ which converge respectively to $b$ and $y$. Since $R^{\lambda_\nu}(x^\nu,b^\nu,y^\nu)$ holds for all $\nu$, by (iii$_3$) one deduces that $R(x,b,y)$ is satisfied, and hence $x$ belongs to $\Gamma$, completing the proof. □

**Theorem 4.3** *The set-valued map $\Sigma^\lambda$ is inner-open at $\lambda_0$, that is* $\text{liminfo}_{\lambda\to\lambda_0}\Sigma^\lambda \supseteq \Sigma$, *if the maps $K^\lambda$ is inner-open at $\lambda_0$ and* $\text{liminfo}_{\lambda\to\lambda_0}\Gamma^\lambda \supseteq \Gamma\cap K$. *In particular this is true under the following conditions*

(i) $\text{liminfo}_{\lambda\to\lambda_0} A^\lambda \supseteq A$;

(ii) *for every $x \in A$,* $\text{liminfo}_{\lambda\to\lambda_0, x'\in A^\lambda\to x} S_1^\lambda(x') \supseteq S_1(x)$;

(iii) *for every $x \in K$,*

(iii$_1$) *$S_2^\lambda(x')$ is upper semi-continuous and compact-valued in the variables $(\lambda, x')$ at $(\lambda_0, x)$;*

(iii$_2$) *$T^\lambda(x', b')$ is upper semi-continuous and compact-valued in the variables $(\lambda, x', b')$ at $(\lambda_0, x, b)$ with $b \in S_2(x)$;*

(iii$_3$) *$R(x, b, y)$ does not hold whenever $R^{\lambda_\nu}(x^\nu, b^\nu, y^\nu)$ does not hold for some $x^\nu \in A^{\lambda_\nu}$ converging to $x$, $b^\nu \in S_2^{\lambda_\nu}(x^\nu)$ converging to $b$ and $y^\nu \in T^{\lambda_\nu}(x^\nu)$ converging to $y$ with for all $\lambda_\nu$ converging to $\lambda_0$.*

**Proof.** The first part of the theorem follows from Proposition 2.6 (ii). For the second part, first we prove that the map $\lambda \mapsto K^\lambda$ is inner-open at $\lambda_0$. Let $x$ be an element of $K$, that is $x \in A$ and $x \in S_1(x)$. By (i), there are neighborhoods $U_1$ of $\lambda_0$ and $V_1$ of $x$ such that $V_1 \subseteq A^\lambda$ for all $\lambda \in U_1, \lambda \neq \lambda_0$. It follows from (ii) that there exist neighborhoods $U_2$ of $\lambda_0$, $W_2$ and $V_2$ of $x$ such that $V_2 \subseteq S_1^\lambda(x')$ for all $\lambda \in U_2, \lambda \neq \lambda_0$, and $x' \in W_2 \cap A^\lambda$. By taking $U = U_1 \cap U_2$ and $V = V_1 \cap V_2 \cap W_2$ we deduce that for every $\lambda \in U, \lambda \neq \lambda_0$, and $x' \in V$ one has $x' \in A^\lambda \cap S_1^\lambda(x')$. Thus, $V \subseteq K^\lambda$ for all $\lambda \in U, \lambda \neq \lambda_0$.
Now, to complete the proof, let $x \in \Gamma \cap K$. We have to show that $x \in \text{liminfo}_{\lambda\to\lambda_0}\Gamma^\lambda$. Suppose to the contrary that $x$ does not belong to the inferior open limit of $\Gamma^\lambda$ at $\lambda_0$. According to Lemma 2.1(iii), $x$ belongs to the superior limit of $(\Gamma^\lambda)^c$. Let $x^\nu \in [\Gamma^{\lambda_\nu}]^c$ converging to some $x$ as $\lambda_\nu$ tends to $\lambda_0$. For each $\nu$, either $x^\nu$ does not belongs to $A^{\lambda_\nu}$, or it does belong to $A^{\lambda_\nu}$ and $R^{\lambda_\nu}(x^\nu, b^\nu, y^\nu)$ does not holds for some $b^\nu \in S_2^{\lambda_\nu}(x^\nu)$ and $y^\nu \in T^{\lambda_\nu}(x^\nu, b^\nu)$. If the first case occurs for a subnet of $\lambda_\nu$, then in view of (i) and Proposition 2.3(ii), $x$ does not belong to $A$, and neither to $\Gamma$. In the other case, in view of (iii$_1$) and (iii$_2$) we may assume that $b^\nu$ and $y^\nu$ converge respectively to some $b \in S_2(x)$ and $y \in T(x, b)$. By (iii$_3$), $R(x, b, y)$ does not holds, which yields $x \notin \Gamma$, a contradiction. The proof is complete. $\square$

It is clear that the roles of the maps $K^\lambda$ and $\Gamma^\lambda$ in the first part of the above theorems are symmetric, so that they can be switched by each other without altering the conclusion. We now turn to the inner continuity of the solution set.

**Theorem 4.4** *The set-valued map $\Sigma^\lambda$ is inner-continuous at $\lambda_0$, that is,* $\liminf_{\lambda\to\lambda_0}\Sigma^\lambda \supseteq \Sigma$, *if either $K^\lambda$ is inner-open at $\lambda_0$ and* $\liminf_{\lambda\to\lambda_0}\Gamma^\lambda \supseteq \Gamma\cap K$, *or $K^\lambda$ is inner-continuous at $\lambda_0$ and* $\text{liminfo}_{\lambda\to\lambda_0}\Gamma^\lambda \supseteq \Gamma\cap K$. *In particular this is true under the following conditions*

(i) $\text{liminfo}_{\lambda\to\lambda_0} A^\lambda \supseteq A$;

(ii) *for every $x \in A$,* $\text{liminfo}_{\lambda\to\lambda_0, x'\in A^\lambda\to x} S_1^\lambda(x') \supseteq S_1(x)$;

(iii) *for every* $x \in K$*,*

(iii$_1$) $S_2^\lambda(x)$ *is upper semi-continuous and compact-valued at* $\lambda_0$*;*

(iii$_2$) $T^\lambda(x,b')$ *is upper semi-continuous and compact-valued at* $(\lambda_0, b)$ *with* $b \in S_2(x)$*;*

(iii$_3$) $R(x,b,y)$ *does not hold whenever there are* $\lambda_\nu$ *converging to* $\lambda_0$*,* $b^{\lambda_\nu} \in S_2^{\lambda_\nu}(x)$ *and* $y^{\lambda_\nu} \in T^{\lambda_\nu}(x, b^{\lambda_\nu})$ *converging to* $b$ *and* $y$ *respectively such that* $x \in A^{\lambda_\nu}$ *and* $R^{\lambda_\nu}(x, b^{\lambda_\nu}, y^{\lambda_\nu})$ *does not hold for all* $\nu$*.*

**Proof.** Again, the first part of the theorem is obtained from Proposition 2.6 (iii). For the second part, observe that the map $\lambda \mapsto K^\lambda$ is inner-open at $\lambda_0$ as proven in the preceding theorem. Let $x \in \Gamma \cap K$. We have to show that for every net $\lambda_\nu$ converging to $\lambda_0$ one can find some $x^\nu \in \Gamma^{\lambda_\nu}$ converging to $x$. Suppose to the contrary that this is not true, that is, there is a neighborhood $V$ of $x$ such that $V \cap \Gamma^{\lambda_\nu} = \emptyset$ for some net $\lambda_\nu$ converging to $\lambda_0$. In view of (i), we may assume that $V \subseteq A^{\lambda_\nu}$ for all $\nu$. Thus, there are some elements $b^\nu$ of $S_2^{\lambda_\nu}(x)$ and $y^\nu$ of $T^{\lambda_\nu}(x, b^\nu)$ such that $R^{\lambda_\nu}(x, b^\nu, y^\nu)$ does not hold. By (iii$_1$) and (iii$_2$) one may assume that $b^\nu$ and $y^\nu$ converge respectively to $b \in S_2(x)$ and $y \in T(x,b)$. The last hypothesis (iii$_3$) yields that $R(x,b,y)$ does not hold. This is in contradiction with $x \in \Gamma$, and the proof is complete. □

We finish this section with another kind of sufficient conditions for continuity of the map $\Gamma^\lambda$. These involve the complement of the relation $R$. We say that a set $\Lambda_0 \subseteq \Lambda$ is open (respectively closed) at $\lambda_0$ if there is an open (respectively closed) neighborhood $U_0$ of $\lambda_0$ such that $U_0 \cap \Lambda_0$ is open (respectively closed). We need the following lemma.

**Lemma 4.5** *The following assertions are true.*

(i) $\Gamma^\lambda$ *is outer-open (respectively inner-continuous) at* $\lambda_0$ *if, for every* $x \notin \Gamma$ *(respectively* $x \in \Gamma$*), the set*

$$\mathcal{U}_x = \{\lambda \in \Lambda : R^\lambda(x,b,y) \ \text{ does not hold for some } b \in S_2^\lambda(x) \ \text{ and } y \in T^\lambda(x,b)\}$$

*is open (respectively closed) at* $\lambda_0$*.*

(ii) $\Gamma^\lambda$ *is outer-continuous (respectively inner-open) at* $\lambda_0$ *if the set*

$$\mathcal{U} = \{(\lambda, x) \in \Lambda \times \mathcal{A} : \text{either } x \notin A^\lambda \text{ or } R^\lambda(x,b,y) \ \text{ does not hold for some } b \in S_2^\lambda(x) \text{ and } y \in T^\lambda(x,b)\}$$

*is open at* $(\lambda_0, x)$ *with* $x \notin \Gamma$ *(respectively closed at* $(\lambda_0, x)$ *with* $x \in \Gamma$*).*

**Proof.** To prove (i), let $x \notin \Gamma$. Then $\lambda_0 \in \mathcal{U}_x$. If $\mathcal{U}_x$ is open at $\lambda_0$, then there is a neighborhood $U$ of $\lambda_0$ which is contained in $\mathcal{U}_x$. So $x$ belongs to $[\Gamma^\lambda]^c$ for all $\lambda \in U$. Hence $x$ belongs to $\liminf_{\lambda \to \lambda_0} \Gamma^\lambda$, which, by Proposition 2.3(i), implies that $\Gamma^\lambda$ is outer-open.

Let $x \in \Gamma$. Then $\lambda_0 \notin \mathcal{U}_x$. If $\mathcal{U}_x$ is closed at $\lambda_0$, then there is a closed neighborhood $U_0$ of $\lambda_0$ such that $U_0 \cap \mathcal{U}_x$ is a closed set. It follows that there is some open neighborhood $U \subseteq U_0$ of $\lambda_0$ such that $U \cap \mathcal{U}_x$ is an empty set. Thus, $x$ belongs to $\Gamma^\lambda$ for all $\lambda \in U$.

Hence, $x$ belongs to $\text{liminf}_{\lambda\to\lambda_0}\Gamma^\lambda$, and $\Gamma^\lambda$ is inner-continuous.

For the outer-continuity in Assertion (ii), let $x_0 \notin \Gamma$. Then $(\lambda_0, x_0) \in \mathcal{U}$. Since the set $\mathcal{U}$ is open at this point, there are open neighborhoods $U$ of $\lambda_0$ and $V$ of $x_0$ such that $U \times V \subseteq \mathcal{U}$. This implies that $x \notin \Gamma^\lambda$ for all $x \in V$ and $\lambda \in U$. By definition, $x_0$ belongs to $\text{liminfo}_{\lambda\to\lambda_0}[\Gamma^\lambda]^c$ and by Proposition 2.3 (ii), $\Gamma^\lambda$ is outer-continuous.

For the inner-openness, let $x_0 \in \Gamma$. Then $(\lambda_0, x_0) \notin \mathcal{U}$. Since the set $\mathcal{U}$ is closed, there are closed neighborhoods $U_0$ of $\lambda_0$ and $V_0$ of $x_0$ such that $U_0 \times V_0 \cap \mathcal{U}$ is closed. We can find some small open neighborhoods $U \subseteq U_0$ of $\lambda_0$ and $V \subseteq V_0$ of $x_0$ such that $U \times V \cap \mathcal{U} = \emptyset$. This implies that $V \subseteq \Gamma^\lambda$ for all $\lambda \in U$. Thus, $\Gamma^\lambda$ is inner-open at $\lambda_0$ and the proof is complete. $\square$

**Remark 4.6** In many models it is assumed that the sets $A^\lambda, B^\lambda$ and $Y^\lambda$ are constant and that the set

$$\mathcal{W} = \{(\lambda, x, b, y) \in \Lambda \times A \times B \times Y : R^\lambda(x, b, y) \text{ does not hold}\}$$

is open at $(\lambda_0, x, b, y)$ for $y \in T(x, b)$, $b \in S_2(x)$ and $x \notin \Gamma$. If in addition the map $S_2^\lambda(x)$ is inner-continuous at $(\lambda_0, x)$ and $T^\lambda(x, b)$ is inner-continuous at $(\lambda_0, x, b)$ for all $x \notin \Gamma$, $b \in S_2(x)$, then the set $\mathcal{U}$ is open at $(\lambda_0, x)$ for all $x \notin \Gamma$. Indeed, let $x_0 \notin \Gamma$. Then there are some $b_0 \in S_2(x_0)$ and $y_0 \in T(x_0, b_0)$ such that $R(x_0, b_0, y_0)$ does not hold. By hypothesis, there are open neighborhoods $U_1$ of $\lambda_0$, $V_1$ of $x_0$, $W_1$ of $b_0$ and $Z_1$ of $y_0$ such that $U_1 \times V_1 \times W_1 \times Z_1 \subseteq \mathcal{W}$. By the inner-continuity hypothesis of $T^\lambda$, there are open neighborhoods $U_2 \subseteq U_1$ of $\lambda_0$, $V_2 \subseteq V_1$ of $x_0$ and $W_2 \subseteq W_1$ of $b_0$ such that $T^\lambda(x, b) \cap Z_1 \neq \emptyset$ for all $(\lambda, x, b) \in U_1 \times V_1 \times W_1$. Similarly, by the inner-continuity of $S_2^\lambda$, one can find open neighborhoods $U \subseteq U_2$ of $\lambda_0$ and $V \subseteq V_2$ of $x_0$ such that $S_2^\lambda(x) \cap W_2 \neq \emptyset$ for all $(\lambda, x) \in U \times V$. It is clear now that for each $(\lambda, x) \in U \times V$, there are some $b \in S_2^\lambda(x)$ and $y \in T^\lambda(x, b)$ such that $(\lambda, x, b, y)$ belongs to $\mathcal{W}$, that is $R^\lambda(x, b, y)$ does not hold. In this way, $U \times V \subseteq \mathcal{U}$.

Along the same line, assume that the set $\mathcal{W}$ is closed at $(\lambda_0, x, b, y)$, the maps $S_2^\lambda$ and $T^\lambda$ are outer-continuous and compact-valued at $(\lambda_0, x)$ and $(\lambda_0, x, b)$ respectively for every $x \in \Gamma$, $b \in S_2(x)$ and $y \in T(x, b)$, and that the set $Y$ is compact. Then $\mathcal{U}$ is closed at $(\lambda_0, x)$ for all $x \in \Gamma$. In fact, let $x_0 \in \Gamma$. Then for every $b \in S_2(x_0)$ and $y \in T(x_0, b)$, $R(x_0, b, y)$ does hold. By hypothesis, there are closed neighborhoods $U_{by}$ of $\lambda_0$, $V_{by}$ of $x_0$, $W_{by}$ of $b$ and $Z_{by}$ of $y$ such that $U_{by} \times V_{by} \times W_{by} \times Z_{by} \cap \mathcal{W} = \emptyset$. By the hypothesis, the set $T(x_0, S_2(x_0))$ is compact, and hence we may choose a finite number of points, say $b_1, ..., b_k \in S_2(x_0)$ and $y_1, ..., y_k \in T(x_0, S_2(x_0))$ such that the sets $W_1 = \cup_{i=1}^k W_{b_i y_i}$ and $Z_1 = \cup_{i=1}^k Z_{b_i y_i}$ are closed neighborhoods of $S_2(x_0)$ and $T(x_0, S_2(x_0))$ respectively. Set $U_1 = \cap_{i=1}^k U_{b_i y_i}$ and $V_1 = \cap_{i=1}^k V_{b_i y_i}$. They are closed neighborhoods of $\lambda_0$ and $x_0$ respectively and the closed neighborhood $U_1 \times V_1 \times W_1 \times Z_1$ does not meet the set $\mathcal{W}$. Now, by the outer-continuity of $T^\lambda$, there are closed neighborhoods $U_2 \subseteq U_1$ of $\lambda_0$, $V_2 \subseteq V_1$ of $x_0$ and $W_2 \subseteq W_1$ of $S_2(x_0)$ such that $T^\lambda(x, S_2(x)) \subseteq Z_1$. Similarly, by the outer-continuity of $S_2^\lambda$, there are closed neighborhoods $U \subseteq U_2$ of $\lambda_0$ and $V \subseteq V_2$ of $x_0$ such that $S_2^\lambda(x) \subseteq W_2$ for all $\lambda \in U$ and $x \in V$. Consequently, the closed neighborhood $U \times V$ of $(\lambda_0, x_0)$ does not meet $\mathcal{W}$ which means that it does not intersect $\mathcal{U}$. Therefore, $\mathcal{U}$ is closed at $(\lambda_0, x_0)$.

Let us now derive sufficient conditions for continuity of the solution map $\Sigma^\lambda$ in terms of the sets $\mathcal{U}_x$, $\mathcal{U}$ and $\mathcal{W}$ above.

**Corollary 4.7** *The following assertions hold.*

(i) $\Sigma^\lambda$ *is outer-open at* $\lambda_0$ *provided that the map* $K^\lambda$ *is outer-open at* $\lambda_0$ *and the set* $\mathcal{U}_x$ *is open at* $\lambda_0$ *for all* $x \notin \Gamma$.

(ii) $\Sigma^\lambda$ *is outer-continuous at* $\lambda_0$ *provided that the map* $K^\lambda$ *is outer-continuous at* $\lambda_0$ *and that the set* $\mathcal{U}$ *is open at* $(\lambda_0, x)$ *for all* $x \notin \Gamma$*, which is true in particular when the maps* $A^\lambda, B^\lambda$ *and* $Y^\lambda$ *are constant, the maps* $S_2^\lambda$ *and* $T^\lambda$ *are inner-continuous at* $(\lambda_0, x)$ *and* $(\lambda_0, x, b)$ *respectively, and the set* $\mathcal{W}$ *is open at* $(\lambda_0, x, b, y)$ *for* $x \notin \Gamma$, $b \in S_2(x)$ *and* $y \in T(x, b)$.

(iii) $\Sigma^\lambda$ *is inner-open at* $\lambda_0$ *provided that the map* $K^\lambda$ *is inner-open at* $\lambda_0$ *and the set* $\mathcal{U}$ *is closed at* $(\lambda_0, x)$ *for all* $x \in \Gamma$*, which is true in particular when* $Y$ *is compact,* $S_2^\lambda(x)$ *and* $T^\lambda(x)$ *are outer-continuous, compact-valued respectively at* $(\lambda_0, x)$ *and* $(\lambda_0, x, b)$*, and* $\mathcal{W}$ *is closed at* $(\lambda_0, x, b, y)$ *for every* $x \in \Gamma, b \in S_2(x)$, $y \in T(x, b)$.

(iv) $\Sigma^\lambda$ *is inner-continuous at* $\lambda_0$ *provided that the map* $K^\lambda$ *is inner-open (respectively, inner-continuous) at* $\lambda_0$ *and the set* $\mathcal{U}_x$ *(respectively,* $\mathcal{U}$*) is closed at* $\lambda_0$ *(respectively,* $(\lambda_0, x)$*) for all* $x \in \Gamma$.

**Proof.** Invoke Lemma 4.5, Theorems 4.1-4.4 and the remark given after Lemma 4.5. □

Note that although the assumptions of Corollary 4.7 are difficult to be checked, they are properly weaker than the particular conditions of Theorems 4.1-4.4. Indeed, from the proof of this corollary we see that the particular assumptions of Theorems 4.1-4.4 imply the assumptions of Corollary 4.7. The following example shows that the converse is not true and Corollary 4.7 may be applicable in cases where the second parts of the above mentioned theorems are not.

**Example 4.8** Let $A^\lambda = B^\lambda = Y^\lambda = \mathbb{R}$, $\Lambda = [0, 1]$, $\lambda_0 = 0$, $S_1^\lambda(x) = [0, \lambda]$,

$$S_2^\lambda(x) = \begin{cases} (0, 1) & \text{if } \quad \lambda = 0, \\ \{-1\} & \text{otherwise,} \end{cases}$$

$$T^\lambda(x, b) = \begin{cases} (-1, 0) & \text{if } \quad \lambda = 0, \\ \{1\} & \text{otherwise,} \end{cases}$$

$$F^\lambda(b, y) = [0, 1].$$

Let the relation $R$ be defined by: $R(x, b, y)$ holds if and only if $F^\lambda(b, y) \subseteq \mathbb{R}_+$. Then $K^\lambda = [0, \lambda]$ satisfies all (i)-(v) of Corollary 4.7. All the sets $\mathcal{U}_x$, $\mathcal{U}$ and $\mathcal{W}$ are empty and hence also satisfy (i)-(v). Direct calculations give $\Sigma^\lambda = [0, \lambda]$, which have all continuity and openness properties. But $S_2^\lambda(x)$ and $T^\lambda(x, b)$ do not fulfil any assumption of Theorems 4.1-4.4.

# 5 Particular cases

In this section we apply the results of Section 4 to two models of variational relations: variational inclusion problems and quasi-equilibrium problems. To derive and improve existing results of recent literature on these models we focus on outer-continuity and inner-continuity only, since the other two kinds of continuity are newly introduced in the present paper.

## 5.1 Variational inclusion problems

Let $A, B, Y$ and $Z$ be nonempty sets. Let $S_1 : A \rightrightarrows A$, $S_2 : A \rightrightarrows B$, $T : A \times B \rightrightarrows Y$, $F : A \times B \times Y \rightrightarrows Z$ and $G : A \times A \times Y \rightrightarrows Z$ be set-valued maps. Consider the variational inclusion problem

(VIP) Find $\overline{x} \in A$ such that

(i) $\overline{x} \in S_1(\overline{x})$;

(ii) $F(\overline{x}, b, y) \subseteq G(\overline{x}, \overline{x}, y)$ for all $b \in S_2(\overline{x})$ and $y \in T(\overline{x}, b)$.

This model generalizes several optimization related problems, including practical problems of traffic network equilibria and has been studied in [3, 4]. To see that it is a particular case of (VR), it suffices to define a variational relation $R$ associated to it as follows: for $x \in A, b \in S_2(x)$ and $y \in T(x, b)$, $R(x, b, y)$ holds if and only if $F(x, b, y) \subseteq G(x, x, y)$. Now, assume that the data of (VIP) depend on a parameter $\lambda$ from $\Lambda$, and as in Section 4, $\mathcal{A}, \mathcal{B}, \mathcal{Y}$ and $\Lambda$ are topological spaces. In [3, 4] the authors have studied stability of (VIP) without perturbing the spaces $A$, $B$ and $Y$. Thus, the following result ([4], Corollary 4.1) is immediate from Theorem 4.2.

**Corollary 5.1** *Assume that $A^\lambda = A$, $B^\lambda = B$ and $Y^\lambda = Y$ are all constant maps and that the following conditions hold:*

(i) *$K^\lambda$ is outer-continuous at $\lambda_0$;*

(ii) *$S_2^\lambda(x)$ is inner-continuous at $(\lambda_0, x_0)$ for $x_0 \in K$;*

(iii) *$T^\lambda(x, b)$ is inner-continuous at $(\lambda_0, x_0, b_0)$ for $x_0 \in K$ and $b_0 \in S_2(x_0)$;*

(iv) *$F(x, b, y) \subseteq G(x, x, y)$ whenever there are $\lambda_\nu$ converging to $\lambda_0$, $x^\nu \in A$, $b^\nu \in S_2^{\lambda_\nu}(x^\nu)$ and $b^\nu \in T^{\lambda_\nu}(x^\nu, b^\nu)$ converging respectively to $x, b$ and $y$ such that $F^{\lambda_\nu}(x^\nu, b^\nu, y^\nu) \subseteq G^{\lambda_\nu}(x^\nu, x^\nu, y^\nu)$.*

*Then $\Sigma^\lambda$ is outer-continuous at $\lambda_0$.*

Note that in the literature on problems of type (VIP), the maps $S_1$, $T$ and $(F, G)$ are separately perturbed by independent parameters. Such a perturbation is useful when studying Hölder continuity of the solution set, since the Hölder degrees with respect to these parameters may be different. For semicontinuities, this is unnecessary because all these parameters can be considered as one by choosing it from a suitably defined product space.

Again with the spaces $A$, $B$ and $Y$ unperturbed, the following result ([3], Corollary 3.2) is immediate from Theorem 4.4.

**Corollary 5.2** *Assume that the following conditions hold*

(i) $K^\lambda$ *is inner-continuous at* $\lambda_0$*;*

(ii) $S_2^\lambda(x)$ *is upper semi-continuous and compact-valued at* $(\lambda_0, x_0)$ *for* $x_0 \in K$*;*

(iii) $T^\lambda(x, b)$ *is upper semi-continuous and compact-valued at* $(\lambda_0, x_0, b_0)$ *for* $x_0 \in K$ *and* $b_0 \in S_2(x_0)$*;*

(iv) $F(x, b, y) \nsubseteq G(x, x, y)$ *whenever there are* $\lambda_\nu$ *converging to* $\lambda_0$*,* $x^\nu \in A$*,* $b^\nu \in S_2^{\lambda_\nu}(x^\nu)$ *and* $y^\nu \in T^{\lambda_\nu}(x^\nu, b^\nu)$ *converging respectively to* $x, b$ *and* $y$ *such that* $F^{\lambda_\nu}(x^\nu, b^\nu, y^\nu) \nsubseteq G^{\lambda_\nu}(x^\nu, x^\nu, y^\nu)$.

*Then* $\Sigma^\lambda$ *is inner-continuous at* $\lambda_0$.

Together with (VIP) consider the auxiliary problem (VIP$_*$), which is obtained from (VIP) by replacing the inclusion in (ii) by

$$F(\overline{x}, b, y) \subseteq \mathrm{int}G(\overline{x}, \overline{x}, y).$$

As it is shown in the next corollary, stability of this auxiliary problem implies stability of problem (VIP).

**Corollary 5.3** *Assume that (VIP$_*$) has a nonempty solution set* $\Sigma_*^\lambda$*. Assume further that*

(i) $K^\lambda$ *is inner-continuous at* $\lambda_0$*;*

(ii) $S_2^\lambda(x)$ *is usc and compact-valued at* $(\lambda_0, x_0)$ *for* $x_0 \in K$*;*

(iii) $T^\lambda(x, b)$ *is usc and compact-valued at* $(\lambda_0, x_0, b_0)$ *for* $x_0 \in K$ *and* $b_0 \in S_2(x_0)$*;*

(iv) $\Sigma^\lambda \subseteq cl\, \Sigma_*^\lambda$ *for every* $\lambda$*;*

(v) *for all* $x \in K$*,* $F(x, b, y) \nsubseteq G(x, x, y)$*, whenever there are* $\lambda_\nu$ *converging to* $\lambda_0$*,* $b^\nu \in S_2^{\lambda_\nu}(x)$ *and* $y^\nu \in T^{\lambda_\nu}(x, b^\nu)$ *converging respectively to* $b$ *and* $y$ *with* $F^{\lambda_\nu}(x, b^\nu, y^\nu) \nsubseteq G^{\lambda_\nu}(x, x, y^\nu)$.

*Then* $\Sigma_*^\lambda$ *and* $\Sigma^\lambda$ *are inner-continuous at* $\lambda_0$.

**Proof**. By (ii), (iii) and (v), for each $x \in K$ the set $\mathcal{U}_x$ for problem (VIP$_*$) is closed at $\lambda_0$. Therefore, by Corollary 4.7(iv), the solution map $\Sigma_*^\lambda$ is inner-continuous. In view of assumption (iv) the solution map $\Sigma^\lambda$ is inner-continuous too. □

**Remark 5.4** Assumption (iv) of Corollary 5.3 can be guaranteed provided that $K^\lambda$ is convex and the following conditions hold:

(a) For every $x_1 \in \Sigma_*^\lambda$, $x_2 \in \Sigma^\lambda$, and $b \in S_2^\lambda((1-t)x_1 + tx_2)$ with $t \in [0,1)$,

$$F^\lambda(x_1, b, y) \subseteq \text{int}G^\lambda(x_1, x_1, y) \text{ for all } y \in T^\lambda(x_1, b),$$

$$F^\lambda(x_2, b, y') \subseteq G^\lambda(x_2, x_2, y') \text{ for all } y' \in T^\lambda(x_2, b);$$

(b) For every $b \in B$, the map $F^\lambda(., b, .)$ is $G^\lambda$-quasiconvex with respect to $T^\lambda(., b)$ in the sense that for all $x_1, x_2 \in K^\lambda$, $y_1 \in T^\lambda(x_1, b)$ and $y_2 \in T^\lambda(x_2, b)$, the inclusions

$$F^\lambda(x_1, b, y_1) \subseteq \text{int}G^\lambda(x_1, x_1, y_1),$$

$$F^\lambda(x_2, b, y_2) \subseteq G^\lambda(x_2, x_2, y_2),$$

imply $F^\lambda(x_t, b, y_t) \subseteq \text{int}G^\lambda(x_t, x_t, y_t)$ for all $x_t = (1-t)x + tx'$ and $y_t \in T^\lambda(x_t, b)$ with $t \in [0,1)$.

This is because if $x_1 \in \Sigma_*^\lambda$ and $x_2 \in \Sigma^\lambda$, then the interval $[x_1, x_2)$ is included in $\Sigma_*^\lambda$, and hence the set $\Sigma^\lambda$ is included in the closure of $\Sigma_*^\lambda$. The conditions of this remark have been considered in [3]. Hence, Corollary 5.3 extends Theorem 3.5 of [3]. As far as we know, [3, 4] are the only works devoted to stability of variational inclusion problems.

The reason for introducing the auxiliary problem (VIP$_*$) is that under certain continuity assumptions, the inner-openness of the maps $\Sigma_*^\lambda$ is easier to obtain than the inner-openness of the map $\Sigma^\lambda$. This can be clearly seen in the case when $G$ is a constant map, say equal to a closed convex cone $C$ with a nonempty interior, and $F^\lambda(x, b, y)$ is a single-valued continuous function. Then for the inclusion $F^{\lambda_0}(x_0, b_0, y_0) \in \text{int}C$ one can find a neighborhood $U$ of $\lambda_0$ and $V$ of $(x_0, b_0, y_0)$ such that $F^\lambda(x, b, y) \in \text{int}C$ for all $\lambda \in U$ and $(x, b, y) \in V$. This, however, is not true in general for the inclusion $F^{\lambda_0}(x_0, b_0, y_0) \in C$.

## 5.2 Quasi-equilibrium problems

Let $\Lambda$ and $X$ be topological spaces, let $C$ be a closed subset of a topological vector space $Z$ with a nonempty interior. Let $S, G : X \rightrightarrows X$ and $F : X \times X \rightrightarrows Z$ be set-valued maps. A quasi-equilibrium problem, denoted by (QEP), is the following

Find $\bar{a} \in X$ such that
(i) $\bar{a}$ is a fixed point of cl$S$, that is $\bar{a} \in \text{cl}S(\bar{a})$;
(ii) $F(b, y) \subseteq Z \setminus -\text{int}C$ for every $b \in S(\bar{a})$ and $y \in G(\bar{a})$.

This problem has been investigated in e.g. [1, 2]. It is a particular case of (VR) in which $A$, $B$ and $Y$ coincide with $X$, $S_1(x) = \text{cl}S(x)$, $S_2(x) = S(x)$, $T(x, b) = G(x)$ for all $x, b \in X$, and the relation $R(x, b, y)$ holds if and only if $F(b, y) \subseteq Z \setminus -\text{int}C$. As in the previous subsection, for simple presentation we assume that $A^\lambda = A$, $B^\lambda = B$ and $Y^\lambda = Y$ are all constant maps.

**Corollary 5.5** *Assume that the following conditions hold:*

(i) $S^\lambda(x)$ *is inner-continuous and* $clS^\lambda(x)$ *is outer-continuous, in the variables* $(\lambda, x)$ *at* $(\lambda_0, x_0)$ *for* $x_0 \in K$*;*

(ii) $G^\lambda(x)$ *is inner-continuous at* $(\lambda_0, x_0)$ *for* $x_0 \in K$*;*

(iii) $F^\lambda(x, b)$ *is inner-continuous at* $(\lambda_0, x_0, b_0)$ *for* $x_0 \in K$ *and* $b_0 \in Y$*.*

*Then* $\Sigma^\lambda$ *is outer-continuous at* $\lambda_0$*.*

**Proof.** This is immediate from Theorem 4.2. □

This corollary contains Theorem 3.4 of [1] as a special case with $G^\lambda(x) = \{x\}$.

**Corollary 5.6** *Assume that the following conditions hold:*

(i) $K^\lambda$ *is inner-continuous at* $\lambda_0$*;*

(ii) $S_2^\lambda(x)$ *is outer-continuous and compact-valued at* $(\lambda_0, x_0)$ *for* $x_0 \in K$*;*

(iii) $G^\lambda(x)$ *is usc and compact-valued at* $(\lambda_0, x_0)$ *for* $x_0 \in K$*;*

(iv) $F(b, y) \subseteq Z \setminus -\text{int}C$ *whenever there are* $\lambda_\nu$ *converging to* $\lambda_0$*,* $x^\nu \in A$*,* $b^\nu \in S^{\lambda_\nu}(x^\nu)$*,* $y^\nu \in T^{\lambda_\nu}(x^\nu)$ *converging respectively to* $x, b$ *and* $y$ *such that* $F^{\lambda_\nu}(b^\nu, y^\nu) \subseteq Z \setminus -\text{int}C$*.*

*Then* $\Sigma^\lambda$ *is inner-continuous at* $\lambda_0$*.*

**Proof.** This is immediate from Theorem 4.4. □

Theorem 2.4 of [1] is a special case with $G^\lambda(x) = \{x\}$ of this corollary too. Observe that we can derive immediately Theorem 3.1 of [2] (hence improve also the corresponding result of [7]) from part (ii) of Corollary 4.6 and Theorem 2.1 of [2] (with the improvement that $\mathcal{U}_x$ replaces $\mathcal{U}$) from part (iv).

The following version of quasi-equilibrium problems is often studied in the recent literature (e.g. [13]), denoted by (QEP'):

Find $\bar{a} \in A$ such that
(i) $\bar{a} \in S(\bar{a})$
(ii) $f(\bar{a}, b) \in Z \setminus -\text{int}C(\bar{a})$ for every $b \in S(\bar{a})$.

In this model, $f$ is a single-valued function from $A \times Y$ to $Z$ with $Z$ being a topological vector space, and $C : A \rightrightarrows Z$ is a set-valued map whose values are convex cones with nonempty interior. It is clear that this problem is a particular case of the variational inclusion problem (VIP) with $S_1 = S_2 = S$, $T$ being the identity map, $F(x, b, y) = \{f(x, b)\}$ and $G(x, x, y) = Z \setminus -\text{int}C(x)$. For this problem, the following stability result ([13], Theorem 4.1) is a direct consequence of Corollary 4.7 (ii).

**Corollary 5.7** *Assume for (QEP') that*

(i) $S^\lambda(x)$ *is outer-continuous and compact-valued in* $\{\lambda_0\} \times A$*;*

(ii) *the following set* $\mathcal{W}$ *is open at* $(\lambda_0, x, b)$ *for* $x \in \Gamma$ *and* $b \in S(x)$*:*

$$\mathcal{W} = \{(\lambda, x, b) \in \Lambda \times A \times A : F(\lambda, x, b) \notin Z \setminus -intC(\lambda, x)\}.$$

*Then* $\Sigma^\lambda$ *is outer-continuous at* $\lambda_0$*.*

We denote the corresponding auxiliary problem of (QEP') by (QEP'$_*$), where $-\text{int}C(\bar{a})$ is replaced by the closure of $-C(\bar{a})$, that is:

Find $\bar{a} \in A$ such that
(i) $\bar{a} \in S(\bar{a})$
(ii) $f(\bar{a}, b) \in Z \setminus -\text{cl}C(\bar{a})$ for every $b \in S(\bar{a})$.

Similar to the case of variational inclusions, stability of this auxiliary problem implies stability of the quasi-equilibrium problem (QEP') under a suitable condition.

**Corollary 5.8** *Assume that (QEP'$_*$) has solutions and that the following conditions hold:*

(i) $K^\lambda$ *is inner-continuous at* $\lambda_0$*;*

(ii) $S^\lambda(x)$ *is upper semi-continuous and compact-valued at* $(\lambda_0, x_0)$ *for every* $x_0 \in A$*;*

(iii) *the set* $\{(\lambda, x, b) \in \Lambda \times A \times A : F^\lambda(x, b) \in Z \setminus -\text{cl}C^\lambda(x)\}$ *is open at* $(\lambda_0, x, b)$ *for every* $x, b \in A$*;*

(iv) $\Sigma^\lambda \subseteq cl\ \Sigma^\lambda_*$ *for every* $\lambda$*.*

*Then the set-valued maps* $\Sigma^\lambda_*$ *and* $\Sigma^\lambda$ *are inner-continuous at* $\lambda_0$*.*

**Proof.** It follows from conditions (ii) and (iv) that the set $\mathcal{U}_x$ of Lemma 4.5 is closed. Hence, by Corollary 4.7 (iv), the solution map $\lambda \to \Sigma^\lambda_*$ is inner-continuous at $\lambda_0$. By condition (iv), the map $\lambda \to \Sigma^\lambda$ is inner-continuous there too. □

We notice that this corollary improves Theorem 5.1 of [13]. In fact, the above-said theorem requires the following conditions in addition to the assumptions of Corollary 5.8 (except assumption (iv)): 1) $S^\lambda(x)$ is convex-valued and concave with respect to $x$; 2) $K^\lambda$ is convex; 3) a generalized quasiconvexity stronger than the condition that for each $\lambda$, $F^\lambda(.,b)$ is C-quasiconvex (with respect to the identity map), see the definition in Remark 5.4(b). It follows from these conditions (the assumption on convex-values of $S^\lambda(x)$ is not used) that for every $x \in \Sigma^\lambda_*$ and $x' \in \Sigma^\lambda$, the interval $[x, x')$ is included in $\Sigma^\lambda_* \subseteq \Sigma^\lambda$. By this, $\Sigma^\lambda$ is included in the closure of $\Sigma^\lambda_*$. It is evident that the latter inclusion can be true without the three conditions above.

It is not hard to see that the results on upper and lower semi-continuities for symmetric quasi-equilibrium problems of [5] are also consequences of the results of Section 4.

## 6 Final remark

Let $X$ and $Z$ be nonempty sets, $M$ a nonempty subset of $Z$ and $F$ a set-valued map from $X$ to $Z$. Consider the following general inclusion problem:

(GI) Find $\overline{x} \in X$ such that $F(\overline{x}) \subseteq M$.

There are several ways to interpret (GI) as a variational relation problem. One of them is as follows. Set $A = \{x \in X : F(x) \neq \emptyset\}$, $B = A$, $Y = Z$, $S_1(x) = A$, $S_2(x) = A$ and $T(x, b) = F(x)$ for all $x$ and $b$ of $A$. Given $(x, b, z) \in A \times A \times Z$, we say that $R(x, b, z)$ holds if $x \in M$. Then with these data, the problem (VR) is exactly the problem (GI).
It turns out that (VR) can also be expressed as an inclusion problem. For this purpose, set $X = A$ and $Z = A \times B \times Y$. Define a subset $M$ of the space $Z$ as the set determining the relation $R$, that is, $(x, b, y) \in M$ if and only if $R(x, b, y)$ holds. The map $F : X \rightrightarrows Z$ is given by

$$F(x) = \{(x, b, y) \in Z : b \in S_2(x), y \in T(x, b)\} \cap S_1(x) \times B \times Y.$$

Then the obtained problem (GI) is exactly the problem (VR). The specific form of (VR) which involves the constraint maps $S_1, S_2$ and $T$ makes it mathematically attractive and applicable to several models of practice as it is shown in [15] and the present study.

## References


[1] ANH, L.Q. and KHANH, P.Q., *Semicontinuity of the solution set of parametric multivalued vector quasiequilibrium problems*, J. Math. Anal. Appl. 294, 699-711, 2004.

[2] ANH, L.Q. and KHANH, P.Q., *On the stability of the solution set of general multivalued vector quasiequilibrium problems*, J. Optim. Theory Appl. 135, 271-284, 2007.

[3] ANH, L.Q. and KHANH, P.Q., *Semicontinuity of the solution set to parametric quasivariational inclusion problems with applications to traffic network problems (I)*, Set-Valued Anal., to appear.

[4] ANH, L.Q. and KHANH, P.Q., *Semicontinuity of the solution set to parametric quasivariational inclusion problems with applications to traffic network problems (II)*, submitted for publication.

[5] ANH, L.Q. and KHANH, P.Q., *Various kinds of semicontinuity and the solution sets of parametric multivalued symmetric vector quasiequilibrium problems,* J. Global Optim., to appear.

[6] AUBIN, J.P. and FRANKOWSKA, H., *Set-Valued Analysis*, Springer, New York, 1990.

[7] BIANCHI, M. and PINI, R., *A note on stability for parametric equilibrium problems*, Oper. Res. Lett. 31, 445-450, 2003.

[8] COTTLE, R. W., *Nonlinear Programs with Positively Bounded Jacobians*, SIAM Journal on Applied Mathematics, Vol. 14, pp.147-158, 1966.

[9] FACCHINEI, F. and PANG, J. S., *Finite-Dimensional Variational Inequalities and Complementary Problems, I and II,* Springer, New York Inc., 2003.

[10] FLORES-BAZAN F., *Existence theorems for generalized noncoercive equilibrium problems: the quasi-convex case*, SIAM J. Optimization, Vol. 11, pp. 675-690, 2000.

[11] GIANNESSI, F., *Vector Variational Inequalities and Vector Equilibria*, Kluwer Academic Publishers, London, England, 2000.

[12] HARKER, P.T. and PANG, J.S., *Finite-dimentional variational inequality and nonlinear complementary problems: a survey of theory, algorithms and applications*, Mathematical Programming, Vol. 48, pp. 169-220, 1990.

[13] KIMURA, K. and YAO, J.-C., *Sensitivity analysis of solution mappings of parametric vector quasiequilibrium problems*, J. Global Optim., in press, available online 2007.

[14] LUC, D. T., *Frechet Approximate Jacobian and Local Uniqueness of Solutions in Variational Inequalities*, Journal of Mathematical Analysis and Applications, Vol.266, pp. 629-646, 2002.

[15] LUC, D.T., *An abstract problem of variational analysis*, J. Optim. Theory Appl., to appear.

[16] LUC, D. T. and NOOR, M. A., *Local uniqueness of Solutions of general variational inequalities*, J. Optim. Theory Appl., Vol. 117, pp. 103-119, 2003.

[17] ROCKAFELLAR, R.T. and WETS, R. J-B., *Variational Analysis* , Springer, Berlin Heidelberg, 1998.