PROPERTIES TRANSFER BETWEEN TOPOLOGIES ON FUNCTION SPACES, HYPERSPACES AND UNDERLYING SPACES

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Abstract. Each collection α of families of subsets of X determines a topology $\alpha(X,Z)$ on the space of continuous maps $C(X,Z)$. Interrelations between local properties of $\alpha(X,\mathbb{R})$ and of $\alpha(X,\$)$ (on the hyperspace $C(X,\$)$), and with properties of a topological space X are studied in a general framework, which allows to treat simultaneously several classical constructions, like pointwise convergence, compact-open topology and the Isbell topology.

1. Introduction

The interrelation of properties of $C_\alpha(X,Z)$ with those of X and Z, is a fascinating theme. Here α is a collection of (openly isotone [1]) families of subsets of X, that defines a topology $\alpha(X,Z)$ on $C(X,Z)$ by a subbase

$$\{[\mathcal{A},O] : \mathcal{A}\in\alpha, O\in\mathcal{O}_Z\}\,, \tag{1.1}$$

where $[\mathcal{A},O] := \{f : f^{-}(O)\in\mathcal{A}\}$, $f^{-}(O) := \{x : f(x)\in O\}$, and $\mathcal{O}_Z$ is the set of open subsets of Z.

Its very special case, that of $C_p(X,\mathbb{R})$ (the space of real-valued functions with pointwise convergence) has attracted a lot of researchers, among whom A. V. Arhangel'skii (e.g., [1]). Its intermediate case of

$$\alpha = \alpha_{\mathcal{D}} := \{\mathcal{O}_X(D) : D\in\mathcal{D}\}\,,$$

where $\mathcal{D}$ is a family of subsets of X, is the object of a book of McCoy and Ntantu [15].

Actually the said interrelation corresponds to the upper side of a quadrilateral

$$\begin{array}{ccc} X & \leftrightarrow & C_\alpha(X,\mathbb{R}) \\ \updownarrow & & \updownarrow \\ C_\alpha(X,\$^*) & \leftrightarrow & C_\alpha(X,\$) \end{array}$$

in which, of course, one can consider also other sides, as well as diagonals. Here $\$,\$^*$ stand for the two homeomorphic variants of the Sierpiński topology on $\{0,1\}$, so that $C(X,\$)$ can be identified with the hyperspace of X, and $C(X,\$^*)$ with the set of open subsets of X.

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[1] A family $\mathcal{A}$ of open sets is *openly isotone* if $B\in\mathcal{A}$ provided that B is open and there is an element $A\in\mathcal{A}$ such that $A\subset B$.

It turns out that it is fruitful to study the three other sides in order to better grasp the interrelation of the upper side $X \leftrightarrow C_\alpha(X,\mathbb{R})$. Indeed,

(1) $C_\alpha(X,\$)$ is homeomorphic to $C_\alpha(X,\$^*)$;
(2) One can establish a dictionary of easy translations of elementary properties of $C_\alpha(X,\$^*)$ and α-properties of X;
(3) Under a separation condition (by real functions) one can evidence an intimate relationship between $C_\alpha(X,\mathbb{R})$ and $C_\alpha(X,\$)$.

More precisely, if X is completely regular and α is a compact web, then $C_\alpha(X,\mathbb{R})$ is a Hausdorff *topological vector space*, hence the topology $\alpha(X,\mathbb{R})$ is homogeneous. Roughly speaking [2], a *web* α on X is a collection of families of open subsets of X such that for each open subset Y there is $\mathcal{A} \in \alpha$ that can be reconstructed from its trace on Y. A web is *compact* if its every element $\mathcal{A}$ is a *compact family* [3].

Therefore, if X is completely regular and α is a compact web, then to characterize a local property of $C_\alpha(X,\mathbb{R})$, it is enough to study the neighborhood filter of the function 0. As we shall see, in this case, the neighborhood filter of the function 0 in $C_\alpha(X,\mathbb{R})$ and the neighborhood filter of the empty set $\varnothing$ in the hyperspace $C_\alpha(X,\$)$ belong to the same class.

Of course, in general, a hyperspace topology $\alpha(X,\$)$ is not homogeneous. As $\alpha(X,\$)$ and $\alpha(X,\$^*)$ are homeomorphic (by complementation), a property of $\mathcal{N}_{\alpha(X,\$)}(A)$ for $A \in C(X,\$)$ is also a property of $\mathcal{N}_{\alpha(X,\$^*)}(X \setminus A)$ and, as a rule, can be characterized in terms of the space $X \setminus A$ with the induced topology. Therefore a local property of $C_\alpha(X,\$)$ can be characterized by a hereditary (with respect to open subsets) property of X.

It follows from some more general facts (see [5]) that

$$f \in \lim_{\alpha(X,\mathbb{R})} \mathcal{F} \Longleftrightarrow f^-(A) \in \lim_{\alpha(X,\$)} \mathcal{F}^-(A) \tag{1.2}$$

for each closed subset A of $\mathbb{R}$, where $f^-(A) := \{x : f(x) \in A\}$ and $\mathcal{F}^-(A)$ is a filter generated by $\{\{f^-(A) : f \in F\} : F \in \mathcal{F}\}$. Consequently, each α-topology on $C(X,\mathbb{R})$ can be, in principle, characterized in terms of the corresponding α-topology on the hyperspace $C(X,\$)$, actually on its subset consisting of functionally closed subsets of X. Therefore the transfer between $C_\alpha(X,\mathbb{R})$ and $C_\alpha(X,\$)$ requires, essentially, the complete regularity of X. By the way, it is why Georgiou, Iliadis and Papadopoulos studied properties of real-valued function spaces in terms of topologies on functionally open sets [7].

The present paper restricts its scope to topologies on function spaces (almost always real-valued) and to the corresponding hyperspace topologies. This is just one aspect of a general theory of convergence function spaces and hyperspace convergences that will be discussed in [5].

2. Open-set topologies

We denote by $\mathcal{O}_X$ the set of open subsets of X, by $\mathcal{O}_X(x) := \{O \in \mathcal{O}_X : x \in O\}$, and by $\mathcal{O}_X(A) := \{O \in \mathcal{O}_X : A \subset O\}$. If now $\mathcal{A}$ is a family of subsets of X, then $\mathcal{O}_X(\mathcal{A}) := \bigcup_{A \in \mathcal{A}} \mathcal{O}_X(A)$. A family $\mathcal{A}$ of subsets of X is *openly isotone* if $\mathcal{O}_X(\mathcal{A}) = \mathcal{A}$.

[2] A precise definition is given before Lemma 3.7
[3] A precise definition is given before Lemma 3.7

If α is a non-empty collection of openly isotone families of subsets of X, then (1.1) is a subbase of a topology on $C(X,Z)$, denoted by $\alpha(X,Z)$. The corresponding topological space is denoted by $C_\alpha(X,Z)$.

In particular, for a non-empty family $\mathcal{D}$ of subsets of X, the collection $\alpha := \alpha_D$ is defined by

$$\alpha_{\mathcal{D}} := \{\mathcal{O}_X(D) : D \in \mathcal{D}\}, \tag{2.1}$$

and the symbol $C_{\alpha_{\mathcal{D}}}(X,Z)$ is abridged to $C_{\mathcal{D}}(X,Z)$. It is often required (e.g., [15]) that $\mathcal{D}$ be a (closed) *network* on X, that is, a family of closed sets such that for each $x \in X$ and $O \in \mathcal{O}_X(x)$ there is $D \in \mathcal{D}$ for which $x \in D \subset O$. However (1.1) is a topology subbase for each $\alpha = \alpha_{\mathcal{D}}$ provided that $\mathcal{D} \neq \varnothing$.

If $A \subset X$ and $B \subset Z$ then $[A,B] := \{f \in C(X,Z) : f(A) \subset B\}$. Therefore, $[\mathcal{O}_X(D), O] = [D,O]$ and thus

$$\{[\mathcal{A}, O] : \mathcal{A} \in \alpha_{\mathcal{D}}, O \in \mathcal{O}_Z\} = \{[D,O] : D \in \mathcal{D}, O \in \mathcal{O}_Z\}.$$

Example 2.1. *If $\mathcal{D} = X^{<\aleph_0}$, then*

$$\left\{[F,O] : F \in X^{<\aleph_0}, O \in \mathcal{O}_Z\right\}$$

is a base of the topological space $C_p(X,Z)$ of pointwise convergence (here p abridges $X^{<\aleph_0}$).

Example 2.2. *If $\mathcal{D} = \mathcal{K}_X$ (the family of compact subsets of X), then*

$$\{[K,O] : K \in \mathcal{K}_X, O \in \mathcal{O}_Z\}$$

is a base of the topological space $C_k(X,Z)$ of compact-open topology (here k abridges $\mathcal{K}_X$).

We consider two complementary topologies on, respectively, the hyperspace $C(X,\$)$ and the set $C(X,\$^*)$ of open subsets of X. Here $\$$ and $\$^*$ are two homeomorphic avatars of the *Sierpiński topology* on $\{0,1\}$:

$$\$:= \{\varnothing, \{1\}, \{0,1\}\} \text{ and } \$^* := \{\varnothing, \{0\}, \{0,1\}\}.$$

The *indicator function* ψ_A of a subset A of X is defined by to be 0 on A and 1 out of A. If X is a topological space, then $\psi_A \in C(X,\$)$ if and only if A is closed, and $\psi_A \in C(X,\$^*) := \mathcal{O}_X$ if and only if A is open.

The *complementation* $^c : 2^X \to 2^X$ associates $A^c := X \setminus A$ with $A \subset X$. In order to avoid ambiguity, we denote the image of $\mathcal{A} \subset 2^X$ by the complementation by

$$\mathcal{A}_c := \{A^c : A \in \mathcal{A}\}.$$

The topology $\alpha(X,\$^*)$ on the set $C(X,\$^*)$ (of all open subsets of X) has α for a subbase, because, due to our convention, the subbase consists of $\{[\mathcal{A},\{0\}] : \mathcal{A} \in \alpha\}$, and $[\mathcal{A},\{0\}] = \left\{\psi_B \in C(X,\$^*) : \psi_B^-(0) \in \mathcal{A}\right\}$ (by definition, $\psi_B^-(0) = B$).

If α is stable for finite intersections, then α is a base of $\alpha(X,\$^*)$. Hence the *neighborhood filter* $\mathcal{N}_{\alpha(X,\$^*)}(Y)$ of $Y \in C(X,\$^*)$ is generated by

$$\{\mathcal{A} \in \alpha : Y \in \mathcal{A}\}.$$

In particular, for $\alpha = \alpha_{\mathcal{D}}$ a subbase for open sets is of the form

$$\{\mathcal{O}_X(D) : D \in \mathcal{D}\},$$

and $\alpha_{\mathcal{D}}$ is stable for finite intersections provided that $\mathcal{D}$ is stable for finite unions, so that

$$\mathcal{N}_{\alpha_{\mathcal{D}}(X,\$^*)}(Y) \approx \{\mathcal{O}_X(D) : Y \supset D \in \mathcal{D}\}.$$

The homeomorphic image of $\alpha(X,\$^*)$ by the complementation is a topology on the hyperspace $C(X,\$)$ denoted by $\alpha(X,\$)$. Accordingly, $\{\mathcal{A}_c : \mathcal{A} \in \alpha\}$ is a subbase of $\alpha(X,\$)$-open sets on the hyperspace $C(X,\$)$; the neighborhood of $H \in C(X,\$)$ with respect to $\alpha(X,\$)$ is

$$\mathcal{N}_{\alpha(X,\$)}(H) \approx \{\mathcal{A}_c : H^c \in \mathcal{A} \in \alpha\}.$$

In particular, a base of $\mathcal{N}_{\alpha_{\mathcal{D}}(X,\$)}(A_0)$ consists of

$$\{\{A \in C(X,\$) : A \cap D = \varnothing\} : D \in \mathcal{D},\ A_0 \cap D = \varnothing\}$$

This form of basic neighborhoods is at the origin of the term $\mathcal{D}$*-miss topology*.

Remark 2.3. *Gruenhage introduced the so-called* γ -connection [10]*. In particular, a filter* $\Gamma(Y,X)$*, where* Y *is an open subset of* X*, is defined in a way equivalent to*

$$\Gamma(Y,X) := \left\{\mathcal{O}_X(F) : Y \supset F \in X^{<\aleph_0}\right\},$$

hence $\Gamma(Y,X)$ *is a neighborhood base of* Y *with respect to* $\alpha_{X^{<\aleph_0}} := \left\{\mathcal{O}_X(F) : F \in X^{<\aleph_0}\right\}$.

3. Compact families

An openly isotone family $\mathcal{A}$ is *compact* if each family $\mathcal{P}$ of open sets such that $\bigcup \mathcal{P} \in \mathcal{A}$ has a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ such that $\bigcup \mathcal{P}_0 \in \mathcal{A}$. We denote by $\kappa(X)$ the collection of all compact families on X. Here are fundamental examples:

$$K \text{ compact } \Rightarrow \mathcal{O}_X(K) \in \kappa(X);$$
$$x \in \lim_X \mathcal{F} \Rightarrow \mathcal{O}_X(\mathcal{F} \wedge \{x\}) \in \kappa(X),$$

where $\mathcal{F} \wedge \{x\} := \{\{F \cup \{x\}\} : F \in \mathcal{F}\}$.

The collection of (openly isotone) compact families fulfill the following properties:

$$\varnothing, \{O_X\} \in \kappa(X);$$
$$\alpha \subset \kappa(X) \Rightarrow \bigcup_{\mathcal{A} \in \alpha} \mathcal{A} \in \kappa(X);$$
$$\mathcal{A}_0, \mathcal{A}_1 \in \kappa(X) \Rightarrow \mathcal{A}_0 \cap \mathcal{A}_1 \in \kappa(X).$$

Therefore (see [9],[2])

Corollary 3.1. $\kappa(X)$ *is the collection of open sets of a topology on* $\mathcal{O}_X = C(X,\$^*)$.

Example 3.2. *If* $\kappa = \kappa(X)$ *is the collection of (openly isotone) compact families on* X*, then*

$$\{[\mathcal{A}, O] : \mathcal{A} \in \kappa(X), O \in \mathcal{O}_Z\}$$

is a subbase of the Isbell topology *on* $C(X,Z)$*. in particular,* $\kappa(X)$ *is the collection of open sets of* $C_\kappa(X,\$^*)$.

Lemma 3.3. *If* $\mathcal{A} = \mathcal{O}(\mathcal{A})$ *is a compact family of subsets of a completely regular topological space* X*, and* F *is a closed subset of* X *with* $F^c \in \mathcal{A}$*, then there is* $A \in \mathcal{A}$ *and* $h \in C(X,[0,1])$ *such that* $h(A) = \{0\}$ *and* $h(F) = \{1\}$.

Proof. By complete regularity, for every $x \notin F$, there is an open neighborhood O_x of x and $f_x \in C(X,[0,1])$ such that $f_x(O_x) = \{0\}$ and $f_x(F) = \{1\}$. Therefore $F^c = \bigcup_{x \notin F} O_x \in \mathcal{A}$, so that by the compactness of $\mathcal{A}$ there is $n < \omega$ and $x_1, \ldots, x_n \notin F$ such that $A = \bigcup_{1 \le i \le n} O_{x_i} \in \mathcal{A}$. The continuous function $\min_{1 \le i \le n} f_{x_i}$ is 0 on A and 1 on F. □

Lemma 3.4. *If $\mathcal{A}$ is a compact openly isotone family on X and C is a closed subset of X, then $\mathcal{O}(\{A \cap C : A \in \mathcal{A}\})$ is compact.*

Proof. Indeed, if $\mathcal{P}$ is a family of open sets such that $\bigcup \mathcal{P} \in \mathcal{O}(\{A \cap C : A \in \mathcal{A}\})$, then $\bigcup \mathcal{P} \cup (X \setminus C) \in \mathcal{A}$, hence there exists a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ such that $\bigcup \mathcal{P}_0 \cup (X \setminus C) \in \mathcal{A}$, thus $\bigcup \mathcal{P}_0 \in \mathcal{O}(\{A \cap C : A \in \mathcal{A}\})$.[4] □

The concept of *network* is well-known. Here we introduce a notion of web that extends and weakens that of network. A collection α of openly isotone families is a *web* in X if for every $x \in X$ and each $O \in \mathcal{O}_X(x)$ there is $\mathcal{A} \in \alpha$ such that $\mathcal{A}$ is generated by a filter on O. In particular, $\alpha_{\mathcal{D}}$ (2.1) is a web if for each $x \in X$ and every $O \in \mathcal{O}_X(x)$ there is $D \in \mathcal{D}$ such that $D \subset O$. This is a weaker property than that of $\mathcal{D}$ being a network. A collection of openly isotone families is called a *compact web* if it is a web consisting of compact families.

Proposition 3.5. *If $\mathcal{D}$ is a compact network, then $\alpha_{\mathcal{D}}$ is a compact web.*

Indeed, in this case, $\alpha_{\mathcal{D}}$ is a collection of compact families. It is a web, because it includes $\{\mathcal{O}_X(\{x\}) : x \in X\}$. For instance, $\left\{\mathcal{O}_X(F) : F \in X^{<\aleph_0}\right\}$ and $\{\mathcal{O}_X(K) : K \in \mathcal{K}(X)\}$ are compact webs. Therefore,

Corollary 3.6. *$\kappa(X)$ is a compact web on X.*

In fact, $\kappa(X)$ is a web, because it includes a web, for example, $\{\mathcal{O}_X(K) : K \in \mathcal{K}(X)\}$. The following result extends [15, Theorem 1.1.5].

Lemma 3.7. *If Z is Hausdorff and α is a web, then $C_\alpha(X, Z)$ is Hausdorff.*

Proof. If $f_0 \neq f_1$ then there is $x \in X$ such that $f_0(x) \neq f_1(x)$, and because Z is Hausdorff, there exist disjoint open sets O_0 and O_1 such that $f_0(x) \in O_0$ and $f_1(x) \in O_1$. Therefore $W := f_0^-(O_0) \cap f_1^-(O_1) \in \mathcal{O}_X(x)$, and since α is a web, there exists $\mathcal{A} \in \alpha$ such that $\mathcal{A}$ is generated by a filter on W. Therefore $f_0 \in [\mathcal{A}, O_0], f_1 \in [\mathcal{A}, O_1]$ and $[\mathcal{A}, O_1] \cap [\mathcal{A}, O_0]$ is empty, for if $f \in [\mathcal{A}, O_1] \cap [\mathcal{A}, O_0]$ then there exist $W \supset A_0, A_1 \in \mathcal{A}$ such that $A_0 \subset f^-(O_0), A_1 \subset f^-(O_1)$ and $A := A_0 \cap A_1 \in \mathcal{A}$, hence $f(A) \subset O_0 \cap O_1 = \varnothing$. □

Lemma 3.8. *If X is completely regular, α is a compact web, and Z is a (real) topological vector space, then $C_\alpha(X, Z)$ is a Hausdorff topological vector space.*

Proof. Let O be open, $\mathcal{A} \in \alpha$ and $f - g \in [\mathcal{A}, O]$, that is, there is $A \in \mathcal{A}$ such that $f - g \in [A, O]$. By the assumptions on Z, for each $x \in A$, there exist open sets P_x and Q_x such that $f(x) \in P_x, g(x) \in Q_x$ and $P_x - Q_x \subset O$. Because f and g are continuous, there exist an open neighborhood V_x and a closed neighborhood W_x of x such that $W_x \subset V_x \subset A, f \in [V_x, P_x] \subset [W_x, P_x]$ and $g \in [V_x, Q_x] \subset [W_x, Q_x]$. As $\bigcup_{x \in A} V_x \in \mathcal{A}$, by the compactness of $\mathcal{A}$, there is a finite subset F of A such that $\bigcup_{x \in F} V_x \in \mathcal{A}$. On the other hand, $\mathcal{A}_x := \mathcal{O}(\mathcal{A} \vee W_x)$ is compact, $f \in [\mathcal{A} \vee W_x, P_x]$ and $g \in [\mathcal{A} \vee W_x, Q_x]$ for each $x \in A$, a fortiori for $x \in F$. Consequently,

$$f - g \in \bigcap\nolimits_{x \in F} [\mathcal{A} \vee W_x, P_x] - \bigcap\nolimits_{x \in F} [\mathcal{A} \vee W_x, Q_x] \subset [\mathcal{A}, O].$$

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[4]More generally, this holds for arbitrary compact (isotone) families in convergence spaces: denote by $\mathcal{A} \vee C$ the isotone family generated by $\{A \cap C : A \in \mathcal{A}\}$. If C is closed and $\mathcal{A}$ is compact, then $\mathcal{F} \# (\mathcal{A} \vee C)$ implies that the filter $\mathcal{F} \vee F$ meshes with $\mathcal{A}$, hence $\operatorname{adh}(\mathcal{F} \vee F) = \operatorname{adh} \mathcal{F} \cap \operatorname{adh} F = \operatorname{adh} \mathcal{F} \cap F$ meshes with $\mathcal{A}$, equivalently $\operatorname{adh} \mathcal{F}$ meshes $\mathcal{A} \vee C$.

If now O is open, $\mathcal{A} \in \alpha$ and $\lambda f \in [\mathcal{A}, O]$ for a scalar λ, then there is $A \in \mathcal{A}$ such that for each $x \in A$ there exist an open subsets P_x and I_x of Z such that $\lambda f \in I_x P_x \subset O$. By continuity, there exist an open neighborhood V_x and a closed neighborhood W_x of x such that $W_x \subset V_x \subset A, f \in [V_x, P_x] \subset [W_x, P_x]$. As $\bigcup_{x \in A} V_x \in \mathcal{A}$, by the compactness of $\mathcal{A}$, there is a finite subset F of A such that $\bigcup_{x \in F} V_x \in \mathcal{A}$. On the other hand, $\mathcal{A}_x := \mathcal{O}(\mathcal{A} \vee W_x)$ is compact and $f \in [\mathcal{A} \vee W_x, P_x]$ for each $x \in A$, a fortiori for $x \in F$. Therefore

$$\lambda f \in \bigcap\nolimits_{x \in F} I_x \cap \bigcap\nolimits_{x \in F} [\mathcal{A} \vee W_x, P_x].$$

□

It follows that if α is a compact web, then $C_\alpha(X, \mathbb{R})$ is a topological vector space.

4. Polar topologies

Recall that if $\Omega \subset V \times W$, then the Ω-*polar* $\Omega^* A$ of a subset A of V is the greatest subset B of W such that $A \times B \subset \Omega$. Dual topologies can be represented in terms of polarity.

The canonical coupling is the map that associates $\langle x, f \rangle := f(x)$ with $x \in X$ and $f \in C(X, \mathbb{R})$. For every open subset O of $\mathbb{R}$ this map defines a relation $\Omega_O := \{(x, f) : f(x) \in O\}$. Accordingly, for each $A \in C(X, \$^*)$,

$$\Omega_O^* A = \{ f : A \subset f^-(O) \} = [A, O] \tag{4.1}$$

is the polar of A by Ω_O^*. On the other hand, Ω_O^* is a relation on $C(X, \$^*) \times C(X, \mathbb{R})$, so that if $\mathcal{A}$ is a subset of $C(X, \$^*)$, then $\Omega_O^* \mathcal{A} = [\mathcal{A}, O]$. Hence for a filter (base) α on $C(X, \$^*)$,

$$\Omega_O^* \alpha \approx \{[\mathcal{A}, O] : \mathcal{A} \in \alpha\}.$$

Finally

$$\mathcal{N}_{\alpha(X,\mathbb{R})}(0) \approx \bigvee\nolimits_{O \in \mathcal{N}_{\mathbb{R}}(0)} \Omega_O^* \alpha \approx \{[\mathcal{A}, O] : \mathcal{A} \in \alpha, O \in \mathcal{N}_{\mathbb{R}}(0)\}.$$

Because of homogeneity, it is enough to establish a property of $\mathcal{N}_{\alpha(X,\mathbb{R})}(0)$ in order to prove that property for every neighborhood filter of $C_\alpha(X, \mathbb{R})$ (for a compact web α on a completely regular space X.

On the other hand, it follows from (1.2) that $f \in \lim_{\alpha(X,\mathbb{R})} \mathcal{F}$ implies, in particular, $0^-(C) \in \lim_{\alpha(X,\$)} \mathcal{F}^-(C)$ for each closed subset C of $\mathbb{R}$. If $0 \in C$ then $0^-(C) = X$, hence $0^-(C) \in \lim_{\alpha(X,\$)} \mathcal{F}^-(C)$ for every $\mathcal{F}$. Hence the only case to consider is that of $0 \notin C$ that is equivalent to $0^-(C) = \varnothing$.

This observation implies that properties of $\mathcal{N}_{\alpha(X,\$)}(\varnothing)$ are intimately related to properties of $\mathcal{N}_{\alpha(X,\mathbb{R})}(0)$, hence to local properties of $C_a(X, \mathbb{R})$, thanks to homogeneity (for a compact web α on a completely regular space X). As $\alpha(X, \$)$ and $\alpha(X, \$^*)$ are homeomorphic by complementation, the properties of $\mathcal{N}_{\alpha(X,\$)}(\varnothing)$ and $\mathcal{N}_{\alpha(X,\$^*)}(X)$ are the same. On the other hand, $\mathcal{N}_{\alpha(X,\$^*)}(X)$ is generated by α.

If $\Gamma \subset X_1 \times \ldots \times X_m$ is a relation, then for $1 \leq k \leq m$, let $\Gamma_k : \Gamma \to X_k$ be the restriction to Γ of the k-th projection. Consider the *fundamental relation* $\Gamma \subset C(X, \mathbb{R}) \times C(X, \$_*) \times C(\mathbb{R}, \$_*)$ defined by

$$\Gamma := \{(f, A, O) : f \in [A, O]\}.$$

The last component of Γ is valued in (open) subsets of $\mathbb{R}$, and not in $\mathbb{R}$, because Γ is results from a polarity. Therefore, we need to define a filter on $\mathcal{O}_{\mathbb{R}}(0)$ such that its projection on $\mathbb{R}$ coincides with $\mathcal{N}_{\mathbb{R}}(0)$. A base for such filter (denoted by $\bar{\mathcal{N}}_{\mathbb{R}}(0)$) is given by $\{P \in \mathcal{O}_{\mathbb{R}}(0) : P \subset O\}$ with $O \in \mathcal{O}_{\mathbb{R}}(0)$.

Theorem 4.1. $\mathcal{N}_{\alpha(X,\mathbb{R})}(0) = \Gamma_1(\Gamma_2^- \alpha \vee \Gamma_3^- \bar{\mathcal{N}}_{\mathbb{R}}(0))$.

Proof. By definition, $\Gamma_2^- \mathcal{A} = \{(f, A, O) : f \in [A, O], A \in \mathcal{A}\}$, and $\Gamma_3^- O = \{(f, A, O) : f \in [A, O]\}$, hence $\Gamma_1(\Gamma_2^- \mathcal{A} \vee \Gamma_3^- O) = [\mathcal{A}, O]$, so that $\mathcal{N}_{\alpha(X,\mathbb{R})}(0) = \Gamma_1(\Gamma_2^- \alpha \vee \Gamma_3^- \bar{\mathcal{N}}_{\mathbb{R}}(0))$. □

Let Δ be the following subset of $C(X, \$_*) \times C(X, \mathbb{R})$:

$$\Delta := \{(A, f) : f(A) \subset B(0, 1)\} .$$

In other words, $\Delta := \Gamma_3^- B(0, 1)$. Call its projections Δ_1 and Δ_2.

Theorem 4.2. *If α is a compact web, and X is completely regular, then $\alpha = \Delta_1(\Delta_2^- \mathcal{N}_{\alpha(X,\mathbb{R})}(0))$.*

Proof. If $\mathcal{A}$ is a compact family and $A \in \mathcal{A}$, then by Lemma 3.3, there exist $H \in \mathcal{A}$ and $f_A \in C(X, [0, 1])$ such that $f_A(X \setminus A) = \{1\}$ and $f_A(H) = \{0\}$, consequently $f_A \in [A, B(0, 1)] \subset [\mathcal{A}, B(0, 1)]$. This shows that $\mathcal{A} \subset \Delta_1\Delta_2^-[\mathcal{A}, B(0, 1)]$. Conversely, if $A \in \Delta_1\Delta_2^-[\mathcal{A}, B(0, \frac{1}{n})]$, then $f \in [A, B(0, 1)] \subset [\mathcal{A}, B(0, 1)]$ for every $f \in [\mathcal{A}, B(0, \frac{1}{n})]$, and thus $\Delta_1\Delta_2^-[\mathcal{A}, B(0, \frac{1}{n})] \subset \mathcal{A}$, for each natural $n > 0$. □

5. Transfer of properties

Let $\mathbb{B}$ be a class of filters. A topology is $\mathbb{B}$-*based* if and only if each neighborhood filter is in $\mathbb{B}$. For each class $\mathbb{B}$, the $\mathbb{B}$-based topologies form a concretely coreflective subcategory. Several concretely coreflective subcategories of topologies can be represented in terms of $\mathbb{B}$-based topologies for some specific classes $\mathbb{B}$ of filters, for example, character, tightness, sequentiality, Fréchetness, strong Fréchetness, productive Fréchetness, bisequentiality, and others (see, e.g., [3]).

Theorems 4.1 and 4.2 enable us transfer some coreflective properties from $C_\alpha(X, \mathbb{R})$ to $C_\alpha(X, \$)$ and vice versa.

If $H \subset X \times Y$, then $Hx := \{y \in Y : (x, y) \in H\}$, and if $H \subset X$ then $HA := \bigcup_{x\in A} Hx$. If now $\mathcal{F}$ and $\mathcal{H}$ are families of subsets of X and Y respectively, then

$$\mathcal{HF} := \{HF : F \in \mathcal{F}, H \in \mathcal{H}\} .$$

If $\mathcal{F}$ and $\mathcal{H}$ are filters, then, by a handy abuse of notation, $\mathcal{HF}$ stands also for the filter it generates.

Recall that $\mathbb{F}_\lambda$ denotes the class of filters admitting a filter base of cardinality $< \aleph_\lambda$. In particular, $\mathbb{F}_0$ is the class of *principal* filters, and $\mathbb{F}_1$ is the class of *countably based* filters. The class of all filters is denoted by $\mathbb{F}$.

A class $\mathbb{B}$ of filters is $\mathbb{F}_\lambda$-*composable* if $\mathcal{HF} \in \mathbb{B}$ for each $\mathcal{F} \in \mathbb{B}$ and every $\mathcal{H} \in \mathbb{F}_\lambda$ (see [6],[11],[14]). A class $\mathbb{B}$ of filters is $\mathbb{F}_\lambda$-*steady* if $\mathcal{H} \vee \mathcal{F} \in \mathbb{B}$ for each $\mathcal{F} \in \mathbb{B}$ and each $\mathcal{H} \in \mathbb{F}_\lambda$ (see [11],[14]).

If $\mathbb{H}$ is a class of filters and γ is a filter subbase, then $\gamma \in \mathbb{H}$ means that the filter generated by γ belongs to $\mathbb{H}$.

By Theorem 4.1,

Proposition 5.1. *Let $\mathbb{B}$ be $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady. If X is regular, α is a compact web, and $\alpha \in \mathbb{B}$, then $C_\alpha(X, \mathbb{R})$ is $\mathbb{B}$-based.*

Proof. If $\alpha \in \mathbb{B}$ then $\Gamma_2^- \alpha \in \mathbb{B}$, because $\mathbb{B}$ is $\mathbb{F}_0$-composable. On the other hand, $\Gamma_3^- \bar{\mathcal{N}}_{\mathbb{R}}(0)$ is a countably based filter, because $\mathcal{N}_{\mathbb{R}}(0)$ is countably based. Therefore, $\Gamma_2^- \alpha \vee \Gamma_3^- \bar{\mathcal{N}}_{\mathbb{R}}(0) \in \mathbb{B}$, because $\mathbb{B}$ is $\mathbb{F}_1$-steady. Finally, $\mathcal{N}_{\alpha(X,\mathbb{R})}(0) \in \mathbb{B}$ as the image by a map of a filter from $\mathbb{B}$. Therefore $C_\alpha(X, \mathbb{R})$ is $\mathbb{B}$-based because $C_\alpha(X, \mathbb{R})$ is homogeneous by Lemma 3.8. □

Proposition 5.2. *Let $\mathbb{B}$ be $\mathbb{F}_0$-composable. If α is a compact web, X is completely regular, and $C_\alpha(X, \mathbb{R})$ is $\mathbb{B}$-based, then $\alpha \in \mathbb{B}$.*

Proof. If $C_\alpha(X, \mathbb{R})$ is $\mathbb{B}$-based, $\mathcal{N}_{\alpha(X,\mathbb{R})}(0) \in \mathbb{B}$ hence by Theorem 4.2, $\alpha \in \mathbb{B}$, because $\mathbb{B}$ is $\mathbb{F}_0$-composable. □

Theorem 5.3. *Let $\mathbb{B}$ be $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady, and let α be a compact web on a completely regular space X. Then $C_\alpha(X, \mathbb{R})$ is $\mathbb{B}$-based if and only if $\alpha \in \mathbb{B}$.*

F. Jordan established in [11, Theorem 3] a special case of Theorem 5.3 for $\alpha = \{\mathcal{O}(D) : D \in X^{<\aleph_0}\}$, hence concerning $C_p(X, \mathbb{R})$, in terms of γ-connection (see Remark 2.3). It is enough to replace in his proofs $X^{<\aleph_0}$ by any (additively stable) family $\mathcal{D}$ of compact sets, in order that the proofs remain valid for $\alpha = \{\mathcal{O}(D) : D \in X^{<\aleph_0}\}$ and $C_{\mathcal{D}}(X, \mathbb{R})$.

Since α is a filter subbase of $\mathcal{N}_{\alpha(X,\$^*)}(X)$, and $\alpha(X, \$^*)$ is homeomorphic to $\alpha(X, \$)$ by complementation, we have

Corollary 5.4. *Let $\mathbb{B}$ be $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady, and let α be a compact web on a completely regular space X. Then $C_\alpha(X, \mathbb{R})$ is $\mathbb{B}$-based if and only if $\mathcal{N}_{\alpha(X,\$)}(\varnothing) \in \mathbb{B}$.*

6. Transferable properties

We shall discuss several $\mathbb{F}_0$-composable $\mathbb{F}_1$-steady classes of filters, in other words, of trasferable local properties. Several results on composability and steadiness can be found in [11],[14].

We say that a property of topological spaces is *local* if there is a class $\mathbb{P}$ of filters [5] such that a topology has the property whenever each neighborhood filter belongs to $\mathbb{P}$. Character and tightness are local properties.

The *character* $\chi(\mathcal{F})$ of a filter $\mathcal{F}$ is the least cardinal τ such that $\mathcal{F}$ has a base of cardinality $\leq \tau$. The *tightness* $t(\mathcal{F})$ of a filter $\mathcal{F}$ is the least cardinal τ for which if $A \in \mathcal{F}^{\#}$ then there is $B \subset A$ with $\operatorname{card} B \leq \tau$ such that $B \in \mathcal{F}^{\#}$.

Proposition 6.1. *(Infinite) character and tightness are $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady.*

Proof. If $\mathcal{B}$ is a filter base of a filter $\mathcal{F}$ on X and $A \subset X \times Y$, then $\{AB : B \in \mathcal{B}\}$ is a base of $A\mathcal{F}$. Indeed, if $H \in A\mathcal{F}$ then there exists $F \in \mathcal{F}$ such that $AF \subset H$, hence there is $B \in \mathcal{B}$ with $B \subset F$, so that $AB \subset AF$. Therefore $\chi(A\mathcal{F}) \leq \chi(\mathcal{F})$, because $\operatorname{card}\{AB : B \in \mathcal{B}\} \leq \operatorname{card} \mathcal{B}$.

If $\mathcal{B}$ is a base of $\mathcal{F}$, and $\mathcal{D}$ is a base of $\mathcal{E}$, then $\{B \cap D : B \in \mathcal{B}, D \in \mathcal{D}\}$ is a base of $\mathcal{F} \vee \mathcal{E}$. As $\operatorname{card}\{B \cap D : B \in \mathcal{B}, D \in \mathcal{D}\} \leq \operatorname{card} \mathcal{B} \times \operatorname{card} \mathcal{D}$. Therefore if $\chi(\mathcal{F})$ is infinite and $\chi(\mathcal{E}) \leq \aleph_0$ then $\chi(\mathcal{F} \vee \mathcal{E}) = \chi(\mathcal{F})$.

The tightness is $\mathbb{F}_0$-composable, because $H \# A\mathcal{F}$ if and only if $A^- H \in \mathcal{F}$. The (infinite) tightness is $\mathbb{F}_1$-steady, for if $\mathcal{E} \approx \{E_n : n < \omega\}$ and $H \in (\mathcal{F} \vee \mathcal{E})^{\#}$ then

[5] possibly depending on topology.

$H \in (\mathcal{F} \vee E_n)^{\#}$ for each $n < \omega$. Hence there exists $B_n \subset H$ with $\operatorname{card} B_n \leq \chi(\mathcal{F})$ and such that $B_n \in (\mathcal{F} \vee E_n)^{\#}$. Consequently $\bigcup_{n<\omega} B_n \in (\mathcal{F} \vee \mathcal{E})^{\#}$ and $\operatorname{card}\left(\bigcup_{n<\omega} B_n\right) \leq \chi(\mathcal{F})$ if $\chi(\mathcal{F}) \geq \aleph_0$. □

A filter $\mathcal{F}$ is $\mathbb{G}$ *to* $\mathbb{E}$ *refinable* [12] ($\mathcal{F} \in (\mathbb{G}/\!\!/\mathbb{E})_{\geq}$) if for each filter $\mathcal{G} \in \mathbb{G}$ with $\mathcal{G}\#\mathcal{F}$ there exists a filter $\mathcal{E} \in \mathbb{E}$ such that $\mathcal{E} \geq \mathcal{F} \vee \mathcal{G}$; a filter $\mathcal{F}$ is $\mathbb{G}$ *to* $\mathbb{E}$ *me-refinable* [12] ($\mathcal{F} \in (\mathbb{G}/\!\!/\mathbb{E})\#_{\geq}$) if for each filter $\mathcal{G} \in \mathbb{G}$ with $\mathcal{G}\#\mathcal{F}$ there exists a filter $\mathcal{E} \in \mathbb{E}$ such that $\mathcal{E} \geq \mathcal{F}$ and $\mathcal{E}\#\mathcal{G}$.

Lemma 6.2. *The property $(\mathbb{F}_\kappa/\!\!/\mathbb{F}_\lambda)_{\geq}$ is $\mathbb{F}_\mu$-steady if $\mu \leq \kappa$.*

Proof. Let $\mathcal{F} \in (\mathbb{F}_\kappa/\!\!/\mathbb{F}_\lambda)_{\geq}, \mathcal{E} \in \mathbb{F}_\kappa$ and $\mathcal{D} \in \mathbb{F}_\mu$ be such that $\mathcal{D}\#(\mathcal{E} \vee \mathcal{F})$. Then $(\mathcal{D} \vee \mathcal{E})\#\mathcal{F}$ and $\mathcal{D} \vee \mathcal{E} \in \mathbb{F}_\kappa$, because $\mu \leq \kappa$; thus there is $\mathcal{G} \in \mathbb{F}_\lambda$ such that $\mathcal{G} \geq \mathcal{D} \vee \mathcal{E} \vee \mathcal{F}$. □

Lemma 6.3. *The property $(\mathbb{F}_\kappa/\!\!/\mathbb{F}_\lambda)_{\geq}$ is $\mathbb{F}_\mu$-composable if $\mu \leq \kappa \wedge \lambda$.*

Proof. If $\mathcal{F} \in (\mathbb{F}_\kappa/\!\!/\mathbb{F}_\lambda)_{\geq}, \mathcal{E} \in \mathbb{F}_\kappa$ and $\mathcal{M} \in \mathbb{F}_\mu$ be such that $\mathcal{E}\#(\mathcal{M}\mathcal{F})$. Then $\mathcal{M}^-\mathcal{E}\#\mathcal{F}$ and $\mathcal{M}^-\mathcal{E} \in \mathbb{F}_\kappa$ provided that $\mu \leq \kappa$. As $\mathcal{F} \in (\mathbb{F}_\kappa/\!\!/\mathbb{F}_\lambda)_{\geq}$ there is $\mathcal{G} \in \mathbb{F}_\lambda$ such that $\mathcal{G} \geq \mathcal{M}^-\mathcal{E} \vee \mathcal{F}$. Thus $\mathcal{M}\mathcal{G} \geq \mathcal{M}(\mathcal{M}^-\mathcal{E} \vee \mathcal{F}) \geq \mathcal{E} \vee \mathcal{M}\mathcal{F}$ and $\mathcal{M}\mathcal{G} \in \mathbb{F}_\lambda$ provided that $\mu \leq \lambda$. □

Fréchetness, strong Fréchetness, productive Fréchetness and bisequentiality are other examples of local properties that can be expressed in terms of refinable and me-refinable filters with respect to various classes (see [13] and a pioneering paper [4]). A filter $\mathcal{F}$ is

(1) *Fréchet* $\Longleftrightarrow \mathcal{F} \in (\mathbb{F}_0/\!\!/\mathbb{F}_1)_{\geq}$: A filter $\mathcal{F}$ is *Fréchet* if for each set A such that $A\#\mathcal{F}$ there exists a countably based filter $\mathcal{E}$ such that $A \in \mathcal{E} \geq \mathcal{F}$.
(2) *strongly Fréchet* $\Longleftrightarrow \mathcal{F} \in (\mathbb{F}_1/\!\!/\mathbb{F}_1)_{\geq}$: A filter $\mathcal{F}$ is *strongly Fréchet* if for each countably filter $\mathcal{G}$ such that $\mathcal{G}\#\mathcal{F}$ there exists a countably based filter $\mathcal{E}$ such that $\mathcal{E} \geq \mathcal{F} \vee \mathcal{G}$.
(3) *productively Fréchet* $\Longleftrightarrow \mathcal{F} \in \left((\mathbb{F}_1/\!\!/\mathbb{F}_1)_{\geq}/\mathbb{F}_1\right)_{\geq}$: A filter $\mathcal{F}$ is *productively Fréchet* if for each Fréchet filter $\mathcal{G}$ such that $\mathcal{G}\#\mathcal{F}$ there exists a countably based filter $\mathcal{E}$ such that $\mathcal{E} \geq \mathcal{F} \vee \mathcal{G}$.
(4) *bisequential* $\Longleftrightarrow \mathcal{F} \in (\mathbb{F}/\mathbb{F}_1)_{\#\geq}$: A filter $\mathcal{F}$ is bisequential if for each filter $\mathcal{G}$ such that $\mathcal{G}\#\mathcal{F}$ there exists a countably based filter $\mathcal{E}$ such that $\mathcal{E} \geq \mathcal{F}$ and $\mathcal{E}\#\mathcal{G}$.

Of course, in the first three conditions (but not in the fourth) the existence of a countably based filter $\mathcal{E}$ is equivalent to the existence of a *sequential filter* [6] $\mathcal{E}$. All these properties are $\mathbb{F}_0$-composable. Not all are $\mathbb{F}_1$-steady.

Proposition 6.4. *Classes of strongly Fréchet, productively Fréchet and bisequential filters are $\mathbb{F}_1$-steady; the class of Fréchet filters is not $\mathbb{F}_1$-steady. All the listed properties are $\mathbb{F}_0$-composable.*

Proof. All the cases are proved in [14] except for bisequential filters. Let $\mathcal{F}$ be bisequential and $\mathcal{E} \in \mathbb{F}_1$. If $\mathcal{D}$ is any filter such that $\mathcal{D}\#(\mathcal{E} \vee \mathcal{F})$, then $(\mathcal{D} \vee \mathcal{E})\#\mathcal{F}$, hence there is $\mathcal{G} \in \mathbb{F}_1$ such that $\mathcal{G} \geq \mathcal{F}$ and $\mathcal{G}\#(\mathcal{D} \vee \mathcal{E})$. The filter $\mathcal{G} \vee \mathcal{E} \in \mathbb{F}_1$ and $\mathcal{G} \vee \mathcal{E}$ meshes $\mathcal{D}$ and $\mathcal{G} \vee \mathcal{E} \geq \mathcal{G} \geq \mathcal{F}$. Let $\mathcal{F}$ be bisequential and A a relation.

[6] A filter is *sequential* if it is generated by the queues of a sequence.

If $\mathcal{D}$ is a filter such that $\mathcal{D}\#A\mathcal{F}$, then $A^{-}\mathcal{D}\#\mathcal{F}$, hence there is $\mathcal{H} \in \mathbb{F}_1$ such that $\mathcal{H}\#A^{-}\mathcal{D}$ and $\mathcal{H} \geq \mathcal{F}$. Thus $A\mathcal{H}\#\mathcal{D}$ and $A\mathcal{H} \geq A\mathcal{F}$.

If $\mathcal{F}$ is Fréchet but not strongly Fréchet, then there is $\mathcal{E} \in \mathbb{F}_1$ such that $\mathcal{G} \geq \mathcal{E} \vee \mathcal{F}$ for no $\mathcal{G} \in \mathbb{F}_1$. Hence $\mathcal{E} \vee \mathcal{F}$ is not Fréchet. □

7. Dictionary $X \longleftrightarrow \mathcal{O}_X$

Here there is a list of elementary equivalences that will be used to establish equivalences of more convoluted equivalences between properties of $C_\alpha(X, \$^*)$ and X. We consider only those collections α that are *finitely stable*, that is, $\mathcal{A}_0, \mathcal{A}_1 \in \alpha$ implies that $\mathcal{A}_0 \cap \mathcal{A}_1 \in \alpha$.

Let $Y \subset X$. A family $\mathcal{B}$ of (open) subsets of X is called an α-*cover* of Y if $\mathcal{B} \cap \mathcal{A} \neq \varnothing$ for every $\mathcal{A} \in \alpha$ such that $Y \in \mathcal{A}$. In particular, if $\alpha = \left\{\mathcal{O}(D) : D \in X^{<\aleph_0}\right\}$, then an α-cover is an ω-*cover*, that is, for each finite set D there is $B \in \mathcal{B}$ such that $D \subset B$.

Lemma 7.1. *A family $\mathcal{B}$ meshes $\mathcal{N}_{\alpha(X,\$^*)}(Y)$ if and only if $\mathcal{B}$ is an α-cover of Y.*

Proof. $\mathcal{B}$ meshes $\mathcal{N}_{\alpha(X,\$^*)}(Y)$ if and only if $\mathcal{B} \cap \mathcal{A} \neq \varnothing$ for each $\mathcal{A} \in \alpha$ such that $Y \in \mathcal{A}$. This means exactly that $\mathcal{B}$ is an α-*cover* of Y. □

Let $\mathcal{A}, \mathcal{B}$ be families of subsets of a given set. We say that $\mathcal{A}$ is *coarser* than $\mathcal{B}$ (equivalently, $\mathcal{B}$ is *finer* than $\mathcal{A}$)

$$\mathcal{A} \leq \mathcal{B}$$

if for every $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $B \subset A$. A collection of families of subsets of X can be considered as a family of subsets of 2^X. In this sense, we say that a collection is *finer* (*coarser*) than another collection. The following facts are just rewording of definitions, but we formulate them as lemmas for easy reference.

Lemma 7.2. *A collection γ is finer than $\mathcal{N}_{\alpha(X,\$^*)}(Y)$ if and only if for each $\mathcal{A} \in \alpha$ such that $Y \in \mathcal{A}$ there is $\mathcal{G} \in \gamma$ such that $\mathcal{A} \subset \mathcal{G}$.*

Lemma 7.3. *A collection γ is coarser than $\mathcal{N}_{\alpha(X,\$^*)}(Y)$ if and only if for each $\mathcal{G} \in \gamma$ there is $\mathcal{A} \in \alpha$ such that $Y \in \mathcal{A} \subset \mathcal{G}$.*

In particular, a sequence $(\mathcal{G}_n)_n$ is finer than $\mathcal{N}_{\alpha(X,\$^*)}(Y)$ if for every $\mathcal{A} \in \alpha$ there is $n_{\mathcal{A}} < \omega$ such that $\mathcal{G}_n \subset \mathcal{A}$ for each $n \geq n_{\mathcal{A}}$.

7.1. Tightness.

Recall that (see e.g., [15]) the α-*Lindelöf number* of a topological space X is the least cardinal τ such that for each α-cover there exists an α-subcover of cardinality less than or equal to τ.[7]

By Lemma 7.1, [8]

Theorem 7.4. *The tightness of $C_\alpha(X, \$)$ is τ if and only if the the supremum of the α-Lindelöf numbers of open subsets of X is τ.*

Hence, by Theorem 5.3,

[7] More generally, if $\kappa \leq \lambda$ are cardinals, then we say that X is $\lambda/\kappa[\alpha]$-*compact* if for every open α-cover of X of cardinality $< \lambda$ there is an α-subcover of cardinality $< \kappa$ of X. In particular, a topological space is $[\alpha]$-*compact* if it is $\lambda/\aleph_0[\alpha]$-compact for each cardinal λ, *countably* $[\alpha]$-*compact* if it is $\aleph_1/\aleph_0[\alpha]$-compact, $[\alpha]$-*Lindelöf* if it is $\lambda/\aleph_1[\alpha]$-compact for every λ.

[8] Similar characterizations can be formulated for λ/κ-tightnes with $\kappa \geq \aleph_0$. We say that a filter $\mathcal{F}$ is λ/κ-*tight* if for each $H \in \mathcal{F}^{\#}$ with $\operatorname{card} H < \lambda$ there is $B \subset H$ such that $\operatorname{card} B < \kappa$ and $B \in \mathcal{F}^{\#}$. A topological space is λ/κ-*tight* if its every neighborhood filter is λ/κ-tight.

Theorem 7.5. *If X is completely regular and α is a compact web, then $C_\alpha(X,\mathbb{R})$ is τ-tight if and only if the α-Lindelöf number of X is τ.*

These facts specialize, in an obvious way, to *compact-open* topologies $C_k(X,Z)$, when $\alpha = \{\mathcal{O}(K) : K \in \mathcal{K}\}$ where $\mathcal{K}$ is the family of compact subsets od X, to *Isbell* topologies $C_\kappa(X,Z)$, when $\alpha = \kappa(X)$ is the collection of compact families. The case of $\alpha = \left\{\mathcal{O}_X(D) : D \in X^{<\aleph_0}\right\}$ has non-obvious interpretations. Corollary 7.5 specializes in this case with $\tau = \aleph_0$ to

Proposition 7.6. *If X is completely regular, then $C_p(X,\mathbb{R})$ is countably tight if and only if each ω-cover of X has a countable ω-cover of X.*

The following theorem is due to Arhangel'skii [1] and Pytkeev [17]:

Theorem 7.7. *If X is completely regular, then $C_p(X,\mathbb{R})$ is countably tight if and only if X^n is Lindelöf for every $n < \omega$.*

7.2. **Character.** As an immediate consequence of Lemma 7.2,

Theorem 7.8. *The character of $C_\alpha(X,\$)$ is τ if and only if for every open subset Y of X there exists a subcollection γ of α such that* $\operatorname{card}\gamma \leq \tau$ *and for each $\mathcal{A} \in \alpha$ there is $\mathcal{G} \in \gamma$ such that $\mathcal{G} \subset \mathcal{A}$.*

Corollary 7.9. *If X is T_1, then $C_p(X,\$)$ is of countable character if and only if X is countable.*

Proof. By Theorem 7.8, the character of $\mathrm{C}_p(X,\$)$ is countable, if and only if for every open subset Y of X there is a sequence $(y_n)_n \subset Y$ such that $\{\mathcal{O}_X(\{x_1,\ldots,x_n\}) : n < \omega\}$ is finer than $\left\{\mathcal{O}_X(F) : F \in X^{<\aleph_0}\right\}$, that is, for every finite subset F of Y there is $n < \omega$ such that $\{x_1,\ldots,x_n\} \subset O$ implies $F \subset O$ for each open set O. Since X is T_1, this means that $F \subset \{x_1,\ldots,x_n\}$. □

Corollary 7.10. *If X is T_1, then $C_k(X,\$)$ is of countable character if and only if X is hereditarily hemicompact.*

Proof. Let Y be an open subset of X. The neighborhood filter $\mathcal{N}_{\mathcal{K}(X,\$^*)}(Y)$ is countably based if and only if there exists a sequence $(K_n)_n$ of compact subsets of Y such that for every $K \in \mathcal{K}_Y$ there exists n such that $\mathcal{O}_X(K_n) \subset \mathcal{O}_X(K)$, which, for a T_1-topology, is equivalent $K \subset K_n$. □

It is well-known that a (Hausdorff) topological vector space is metrizable if and only if it is of countable character. Therefore, we recover [15, p. 60]

Corollary 7.11. *If X is completely regular, then $C_p(X,\mathbb{R})$ is metrizable if and only if it is of countable character if and only if X is countable.*

Corollary 7.12. *If X is completely regular, then $C_k(X,\mathbb{R})$ is metrizable if and only if it is of countable character if and only if X is hemicompact.*

7.3. **Variants of Fréchetness.** Here we characterize some of the properties $(\mathbb{H}/\mathbb{E})_\geq$ of hyperspaces in terms of their underlying spaces.

Proposition 7.13. *$C_\alpha(X,\$)$ is $(\mathbb{F}_\kappa/\mathbb{F}_\lambda)_\geq$-based if and only if X enjoys the following property: For each open subset Y of X, for every collection γ of α-covers of Y with* $\operatorname{card}(\gamma) \leq \kappa$*, there exists a collection ζ of families of open sets with* $\operatorname{card}(\zeta) \leq \lambda$ *such that for every $\mathcal{A} \in \alpha$ with $Y \in \mathcal{A}$, and each $\mathcal{G} \in \gamma$ there exists $\mathcal{Z} \in \zeta$ such that $\mathcal{Z} \subset \mathcal{A} \cap \mathcal{G}$.*

As we have observed in a preliminary analysis, this property is (necessarily) hereditary for open sets.

Let $\kappa = 0$ and $\lambda = 1$ and $\alpha = \{\mathcal{O}_X(D) : D \in \mathcal{D}\}$, where $\mathcal{D}$ is finitely additive. Proposition 7.13 specializes as follows[9]:

Proposition 7.14. *$C_\alpha(X, \$)$ is Fréchet if and only if $\mathcal{G}$ is a family of open sets and for each $D \in \mathcal{D}$ with $D \subset Y$, there exists $G \in \mathcal{G}$ with $D \subset G$, then there exists a sequence $(G_n)_n \subset \mathcal{G}$ such that for each $D \in \mathcal{D}$ with $Y \subset D$, there is $n_D < \omega$, for which $D \subset G_n$ for every $n \geq n_D$.*

Of course, the sequence $(G_n)_n$ fulfills the condition above if and only if it converges to Y in $C_{\mathcal{D}}(X, \$^*)$. In the case of $\mathcal{D} = X^{<\aleph_0}$, it is equivalent to $Y \subset \underline{\mathrm{Lim}}_n G_n := \bigcup_{n<\omega} \bigcap_{k>n} G_k$ (the set-theoretic lower limit). In particular, for $Y = X$ the condition above is the condition (γ) of Gerlits and Nagy [8]: if $\mathcal{G}$ is an ω-cover of X, then there is a sequence $G_n \in \mathcal{G}$ with $\underline{\mathrm{Lim}}_n G_n = X$.

Let $\kappa = \lambda = 1$ and $\alpha = \{\mathcal{O}_X(D) : D \in \mathcal{D}\}$, where $\mathcal{D}$ is finitely additive.

Then Proposition 7.13 specializes as follows[10]:

Proposition 7.15. *$C_\alpha(X, \$)$ is strongly Fréchet if and only if $\mathcal{G}_k \supset \mathcal{G}_{k+1}$ is sequence of families of open sets and for every k and each $D \in \mathcal{D}$ with $D \subset Y$, there exists $G \in \mathcal{G}_k$ with $D \subset G$, then there exists a sequence $G_n \in \mathcal{G}_n$ such that for each $D \in \mathcal{D}$ with $Y \subset D$, there is $n_D < \omega$, for which $D \subset G_n$ for every $n \geq n_D$.*

As we have seen in proposition 6.4, Fréchetness is not $\mathbb{F}_1$-steady. Nevertheless, it is known that a Fréchet topological group is strongly Fréchet (see [16]). Therefore

Theorem 7.16. *If α is a compact web on a completely regular space X, then $C_\alpha(X, \mathbb{R})$ is Fréchet if and only if it is strongly Fréchet if and only if for every α-cover $\mathcal{P}$ of X there is a sequence $(P_n)_n \subset \mathcal{P}$ such that for each $\mathcal{A} \in \alpha$ there is $n_{\mathcal{A}} < \omega$ such that $P_n \in \mathcal{A}$ for each $n \geq n_{\mathcal{A}}$.*

References

[1] A. V. Arhangel'skii. Construction and classification of topological spaces and cardinal invariants. *Uspehi Mat. Nauk*, 33:29–84, 1978.

[2] B. J. Day and G. M. Kelly. On topological quotient maps preserved by pullbacks or products. *Proc. Camb. Phil. Soc.*, **67**:553–558, 1970.

[3] S. Dolecki. Convergence-theoretic methods in quotient quest. *Topology Appl.*, **73**:1–21, 1996.

[4] S. Dolecki. Active boundaries of upper semicontinuous and compactoid relations; closed and inductively perfect maps. *Rostock. Math. Coll.*, **54**:51–68, 2000.

[5] S. Dolecki and F. Mynard. Interrelations between convergence function spaces, hyperspaces and underlying spaces. to appear.

[6] S. Dolecki and F. Mynard. Convergence-theoretic mechanisms behind product theorems. *Topology Appl.*, **104**:67–99, 2000.

[7] D. N. Georgiou, S. D. Iliadis, and B. K. Papadopouplos. On dual topologies. *Topology Appl.*, 140:57–68, 2004.

[8] J. Gerlits and Zs. Nagy. Some properties of $C_p(X)$. *Topology Appl.*, **14**:151–161, 1982.

[9] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M.W. Mislove, and D.S. Scott. *Continuous lattices and domains.* Cambridge Univ. Press, 2003.

[10] G. Gruenhage. Products of Fréchet spaces. to appear.

[11] F. Jordan. Productive local properties of function spaces. *Topology Appl.*, 154:870–883, 2007.

[12] F. Jordan, I. Labuda, and F. Mynard. Finite products of filters that are compact relative to a class of filters. to appear in Applied General Topology, 2002.

[9] We use the defintion in terms of sequences.

[10] We use the defintion in terms of sequences.

[13] F. Jordan and F. Mynard. Productively Fréchet spaces. *Czechoslovak Math. J.*, **54** (129):981–990, 2004.

[14] F. Jordan and F. Mynard. Compatible relations of filters and stability of local topological properties under supremum and product. *Topology Appl.*, **153**:2386–2412, 2006.

[15] R. A. McCoy and I. Ntantu. *Topological Properties of Spaces of Continuous Functions.* Springer-Verlag, 1988.

[16] P. Nyikos. Metrizability and the Fréchet-Urysohn property in topological groups. *Proc. Amer. Math. Soc.*, **83**(4):793–801, 1981.

[17] E. G. Pytkeev. On sequentiality of spaces of continuous functions. *Communications Moscow Math. Society*, 1982.

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