A generalized distance and Ekeland's variational principle for vector functions on quasimetric spaces

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Abstract A weak τ-function as a generalized distance, a lower closedness of transitive relation and definitions of (e, K)-lower semicontinuity and (e, K)-lower semicontinuity from above are proposed to relax lower semicontinuity assumptions in Ekeland's variational principle for vector-valued functions. The obtained general results, when applied to particular cases, improve or coincide with many recent results in the literature.

Keywords Weak τ-functions · lower closed transitive relations · (e, K)-lower semicontinity · (e, K)-lower semicontinity from above · Ekeland's variational principle

1 Introduction

The famous Ekeland variational principle (EVP in short), a powerful tool in various fields of nonlinear analysis and optimization, was published in [1]. In the past three decades a great deal of efforts have been made to generalize this principle and its equivalent formulations. Recently, many authors have used generalized

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distances together with the original metric of the space to weaken the lower semicontinuity condition imposed by Ekeland [1]. The original principle of Ekeland in [1] says that for a bounded from below and lower semicontinuous scalar function f on a complete metric space X, a slightly perturbed function has a strictly minimum. Examining the assumptions we see that the completeness of X is crucial since the principle relies on a convergence to a desired point. The boundedness from below is inevitable since functions as regular as continuous linear functions are far from having a point as desired, since they are unbounded from below. So lower semicontinuity seems to be the only assumption which may be relaxed. This idea motivates our commitment in this paper. w-distance was introduced in Kada et al. [2] and used also in Park [3], Lin and Du [4]. Another generalized distance was proposed in Tataru [5]. τ-distance was defined in Suzuki [6], which is more general then both above-mentioned distances. In Lin and Du [7] τ-function, which is incomparable with τ-distance, was introduced. All these distances are used to weaken the mentioned lower semicontinuity assumption. In this paper we propose a notion of a weak τ-function, which is more relaxed than all the encountered generalized distances. We weaken the lower semicontinuity assumption by using this definition and also requiring this condition in fixed direction under consideration (see Definition 2.4). Moreover, we prove also a more general EVP by using a proposed general lower closed transitive relation (Theorems 2.4, 3.1). We derive general equivalent forms of the EVP as well.

The organization of the paper is as follows. In the remainder of this section we recall some preliminaries. In Section 2 we propose definitions of a weak τ-function, generalized lower semicontinuities of a vector function and lower closedness of a transitive relation. Section 3 is devoted to main results about the EVP and equivalent formulations. In the final Section 4 we show that our results improve or include as special cases the ones in [4, 7, 8, 9, 10, 11, 12].

Let X be a quasimetric space (i.e its distance needs not be symmetric) and Y be a topological vector space ordered by a convex cone K (containing zero). Let $f : X \to Y$ be a vector function. f is said to be K-lower semicontinuous (K-lsc) at x if for each $e \in Y$, each sequence $x_n \to x$ such that $f(x_n) + e \leq_K 0$, one has $f(x) + e \leq_K 0$, where $y \leq_K z$ means that $z \in y + K$. f is called K-lower semicontinuous from above (K-lsca) at x (see [13]) if, for each convergent sequence $x_n \to x$ such that $f(x_{n+1}) \leq_K f(x_n), \forall n,$, one has $f(x) \leq_K f(x_n), \forall n$. We always say that f has a property on $A \subseteq X$ if f has this property at every point of A. We omit "on A" if $A = \mathrm{dom} f := \{x \in X : \exists y \in Y, y = f(x)\}$. A subset $B \subseteq Y$ is called K-bounded from below if there is a bounded subset $M \subseteq Y$ such that $B \subseteq M + K$ and is called bounded from below if there is $y \in Y$ such that $B \subseteq y + K$. Note that boundedness from below implies K-boundedness from below but not vice versa. For a transitive relation $\Re$ in X, a subset $A \subseteq X$ is termed $\Re$-complete if any Cauchy sequence in A, which is $\Re$-decreasing, converges to a point in A.

2 Weak τ-functions, generalized lower semicontinuity of a function and lower closedness of a relation

Definition 2.1 ([7]) Let (X, d) be a quasimetric space. A function $p : X \times X \to R_+$ is called a τ-function if the following four conditions hold, for $x, y, z \in X$,

(τ1) (triangle inequality) $p(x, z) \leq p(x, y) + p(y, z)$;

(τ2) (lower semicontinuity) $\forall x \in X$, $p(x, .)$ is R_+-lsc;

(τ3) if $x_n, y_n \in X$ satisfies $\lim_{n\to\infty} p(x_n, y_n) = 0$ and $\lim_{n\to\infty} \sup\{p(x_n, x_m) :$

$m > n\} = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$;

($\tau 4$) $p(x, y) = 0$ and $p(x, z) = 0$ imply that $y = z$.

We propose a weaker notion as follows.

Definition 2.2 Let (X, d) be a quasimetric space. A function $p : X \times X \to R_+$ is said to be a weak τ-function if it satisfies three conditions ($\tau 1$), ($\tau 3$) and ($\tau 4$) of Definition 2.1.

Definition 2.3 ([6]) Let (X, d) be a quasimetric space. A function $p : X \times X \to R_+$ is called a τ-distance on X if there is a function $\eta : X \times R_+ \to R_+$ such that the following conditions are satisfied, for $x, y, z \in X$ and $t \in R_+$,

(τ_1) (triangle inequality) $p(x, z) \le p(x, y) + p(y, z)$;

(τ_2') (weak lower semicontinuity) if $x_n \to x$ and $\lim_{n\to\infty} \sup\{\eta(z_n, p(z_n, x_m)) : m \ge n\} = 0$ for some $z_n \in X$, then $p(w, x) \le \liminf_{n\to\infty} p(w, x_n)$ for all $w \in X$;

(τ_3') if $\lim_{n\to\infty} \eta(x_n, t_n) = 0$ and $\lim_{n\to\infty} \sup\{p(x_n, y_m) : m \ge n\} = 0$, then $\lim_{n\to\infty} \eta(y_n, t_n) = 0$;

(τ_4') $\lim_{n\to\infty} \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_{n\to\infty} \eta(z_n, p(z_n, y_n)) = 0$ imply that $\lim_{n\to\infty} d(x_n, y_n) = 0$;

(τ_5') $\eta(x, 0) = 0$, $\eta(x, t) \ge t$ and $\eta(x, .)$ is concave.

It is known [6, 7] that both the τ-function and τ-distance are w-distances, but the former two notions are incomparable. We show now that the weak τ-function is a more relaxed property than all the three as follows.

Lemma 2.1 *Any τ-distance in a quasimetric space is a weak τ-function.*

Proof If p is a τ-distance in a metric space, then p satisfies ($\tau 3$) by Lemma 3 of [6] and also ($\tau 4$) by Lemma 2 of [6]. Moreover, the proofs of these lemmas in [6] did not need the symmetry of the metric of the space and hence we are done. □

The following example gives a weak τ-function which is not a τ-function.

Example 2.1 Let (X, d) be a metric space, $\gamma > 0$ and $p : X \times X \to [0, +\infty)$ be defined by

$$p(x,y) = \begin{cases} d(x,y) + \gamma & \text{if} \quad x \neq y, \\ \frac{3}{2}\gamma & \text{if} \quad x = y. \end{cases}$$

Then p is not a τ-function since $p(x, .)$ is not lsc at x for any $x \in X$. To check that p is a weak τ-function we have to prove only condition ($\tau 1$), since $p(x, y) > \gamma > 0$ for any $x, y \in X$ and hence ($\tau 3$) and ($\tau 4$) are satisfied. For ($\tau 1$), direct verifications for each case of x, y, z: the three points are different, one pair coincide and the three points coincide are easily carried out. So p is a weak τ-function.

In the sequel we need also the following facts.

Lemma 2.2 ([7], Lemma 2.1) *Let p be a weak τ-function on a quasimetric space X. If a sequence x_n satisfies the condition* $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, *then x_n is a Cauchy sequence.*

Notice that in Lemma 2.1 of [7] it is assumed that X is a metric space and p is a τ-function. But the symmetry of the metric and condition ($\tau 2$) were not used in the proof. (In fact this proof in [7] is incomplete, since only $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ is verified. However, one can show that $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.)

Lemma 2.3 ([14], Lemma 3.4) *Let p be a weak τ-function on a quasimetric* *space* X, $x_n \to \overline{x}$ *and* $\Gamma : X \to 2^X$ *be a set-valued mapping such that the following conditions hold:*

(i) $x_{n+1} \in \Gamma(x_n)$ *and* $\Gamma(x_{n+1}) \subseteq \Gamma(x_n)$, *for all* $n \in \mathbb{N}$;

(ii) $\lim_{n\to\infty} \sup\{p(x_n, u) : u \in \Gamma(x_n)\} = 0$;

(iii) $\overline{x} \in \Gamma(x_n)$, *for all* $n \in \mathbb{N}$.

Then $\bigcap_{n\in\mathbb{N}} \Gamma(x_n) = \{\overline{x}\}$.

If, in addition,

(iv) $\Gamma(\overline{x}) \neq \emptyset$ *and* $\Gamma(\overline{x}) \subseteq \Gamma(x_n)$, *for all* $n \in \mathbb{N}$,

then $\overline{x}$ *is an invariant point of* Γ *(i.e.* $\Gamma(\overline{x}) = \{\overline{x}\}$*). Conversely, if* $p(x, x) = 0$ *for all* $x \in X$, *and* $\overline{x}$ *is an invariant point of* Γ, *then there is a sequence* x_n, *which converges to* $\overline{x}$ *and satisfies all conditions (i)-(iv).*

Now we pass to generalizing lower semicontinuity.

Definition 2.4 Let X be a quasimetric space, Y be a vector space ordered by a convex cone K, $f : X \to Y$ be a vector function and $e \in Y$.

(i) f is said to be (e, K)-lower semicontinuous ((e, K)-lsc in short) at x if for each $r \in R$, each sequence $x_n \to x$ with $f(x_n) + re \leq_K 0$, one has $f(x) + re \leq_K 0$.

(ii) f is called (e, K)-lower semicontinuous from above ((e, K)-lsca in short) at x if for each $r \in R$, each sequence x_n converging to x, from $f(x_0) + re \leq_K 0$ and from $f(x_{n+1}) + t_n e \leq_K f(x_n)$, for all $n \in \mathbb{N}$ and for some sequence $t_n \geq 0$, it follows that $f(x) + re \leq_K 0$.

Note that these generalized semicontinuities are defined for a vector space Y without any topological structure. Definition 2.4(i) is clear and shows that if f

is K-lsc at x then f is (e, K)-lsc at x for every $e \in K$. To see clearer the relation between K-lower semicontinuity from above and (e, K)-lower semicontinuity from above, we observe that f is K-lsca at x if and only if for each $e \in Y$, each sequence x_n converging to x, from $f(x_0) + e \leq_K 0$ and $f(x_{n+1}) \leq_K f(x_n)$, $\forall n$, one has $f(x) + e \leq_K 0$. Indeed, for the "if" assume that $f(x_{n+1}) \leq_K f(x_n)$, $\forall n$. For each fixed n, the sequence $\{x_{n+p}\}_p$ satisfies the conditions $f(x_{n+p+1}) \leq_K f(x_{n+p})$. Then by the assumption with $e = -f(x_n)$, one has $f(x) - f(x_n) \leq_K 0$, i.e. $f(x) \leq_K f(x_n)$, $\forall n$. The "only if" is obvious.

From this observation, it is evident that if f is K-lsca at x then f is (e, K)-lsca at x for every $e \in K$. Furthermore, (e, K)-lower semicontinuity implies (e, K)-lower semicontinuity from above.

Definition 2.5 A transitive relation $\Re$ on quasimetric space X is called lower closed if for any $\Re$-decreasing (i.e. $...\Re x_n \Re ... \Re x_2 \Re x_1$) sequence converging to $\overline{x}$ one has $\overline{x} \Re x_n, \forall n \in \mathbb{N}$.

Theorem 2.4 (minimal elements for lower closed relations) *Let $\Re$ be a lower closed transitive relation on a quasimetric space X with a weak τ-function p. For $x_0 \in X$ assume the $\Re$-sector of x_0, i.e. $S_{\Re}(x_0) = \{x \in X : x \Re x_0\}$, is nonempty and $\Re$-complete. Assume that any $\Re$-decreasing sequence $x_n \in X$ is asymptotic by p (i.e. $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$). Then, there exists $\overline{x} \in S_{\Re}(x_0)$ such that*

$$S_{\Re}(\overline{x}) = \emptyset \quad or \quad S_{\Re}(\overline{x}) = \{\overline{x}\}.$$

Moreover, if $\Re$ is reflexive, then $S_{\Re}(\overline{x}) = \{\overline{x}\}$.

Proof Starting by x_0 we construct a sequence $x_n \in S_{\Re}(x_0)$ as follows: having $x_n \in S_{\Re}(x_{n-1})$, we choose $x_{n+1} \in S_{\Re}(x_n)$ by the following rule:

(a) if $p_n := \sup\{p(x_n, x) : x \in S_{\Re}(x_n)\} < +\infty$, x_{n+1} is taken so that

$$p(x_n, x_{n+1}) \geq \frac{1}{2} p_n; \tag{1}$$

(b) if $p_n = +\infty$, x_{n+1} is taken so that $p(x_n, x_{n+1}) \geq 1$.

There is then $n_0 \in \mathbb{N}$ such that, $\forall n \geq n_0$, $p_n < +\infty$. Indeed, otherwise we would have an $\Re$-decreasing sequence x_n with $p(x_n, x_{n+1}) \not\to 0$, a contradiction. Now, we check the assumptions of Lemma 2.3 for $\Gamma = S_{\Re}$. By the transitivity of $\Re$, (i) holds. By (1) and the asymptoticity of the sequence x_n, (ii) is satisfied. Lemma 2.2 implies that x_n is a Cauchy sequence and hence converges to some $\overline{x}$. Since $\Re$ is lower closed, $\overline{x} \in S_{\Re}(x_n), \forall n$, i.e. (iii) is fulfilled. Lemma 2.3 yields that $\bigcap_{n\in\mathbb{N}} S_{\Re}(x_n) = \{\overline{x}\}$, i.e. $S_{\Re}(\overline{x}) = \emptyset$ or $S_{\Re}(\overline{x}) = \{\overline{x}\}$.

If $\Re$ is reflexive, then (iv) is also satisfied and $\overline{x}$ is an invariant of $S_{\Re}$. □

Now we discuss some particular cases, which will be considered in details later in connections with the EVP. From now on, unless otherwise specified, let X be a quasimetric space, Y be a Hausdorff locally convex space, $K \subseteq Y$ be a convex cone containing zero, $k_0 \in K \setminus -\text{cl}K$, Y^* be the topological dual of Y, K^+ be the positive polar of K, i.e.

$$K^+ := \{y^* \in Y^* : < y^*, k > \geq 0, \forall k \in K\},$$

and $z^* \in K^+$ such that $z^*(k_0) = 1$ (the existence of z^* is guaranteed by the separation theorem). We extend Y similarly as for the one-dimensional case by an additional element, denoted by $+\infty$, with the usual rules for addition of elements and multiplication with reals. We adopt that $y \leq_K +\infty, \forall y \in Y$, and avoid indeterminate expressions like $0.(+\infty)$. We consider a vector function $\Psi : X \times X \to Y \cup \{+\infty\}$ and impose the condition

(H) *if* $\Psi(x,y) \in -K$ *and* $\Psi(y,z) \in -K$, *then*

$$\Psi(x,z) \leq_K \Psi(x,y)+\Psi(y,z).$$

Let p be a weak τ-function on X. We define a relation $\leq_{k_0}$ on X by setting

$$y \leq_{k_0} x \quad \Leftrightarrow \quad \Psi(x,y)+p(x,y)k_0 \in -K. \tag{2}$$

Note that the relation $\leq_{k_0}$ is transitive. Indeed, if $z \leq_{k_0} y$ and $y \leq_{k_0} x$, then (2) implies that $\Psi(x,y) \in -K$ and $\Psi(y,z) \in -K$. By conditions (H) and (τ1) one obtains the transitivity, since

$$\begin{aligned}\Psi(x,z) + p(x,z)k_0 &\in (\Psi(x,y) + \Psi(y,z)) + (p(x,y) + p(y,z))k_0 - K \\ &\in -K.\end{aligned}$$

For $\leq_{k_0}$ we denote the sector of x, i.e. $\{x' \in X : x' \leq_{k_0} x\}$, by $S_{k_0}(x)$.

Lemma 2.5 *Let* X, Y, K, p, k_0 *and* Ψ *be as above.*

(i) *If* $S_{k_0}(x)$ *is closed for each* $x \in X$, *then* $\leq_{k_0}$ *is lower closed.*

(ii) *If* K *is closed,* p *satisfies* (τ2) *and* $\Psi(x,.)$ *is* (k_0,K)*-lsc, then* $S_{k_0}(x)$ *is closed. Hence, if this is satisfied for all* $x \in X$ *then* $\leq_{k_0}$ *is lower closed.*

(iii) *If* K *is closed,* p *satisfies* (τ2) *and* $\Psi(x,.)$ *is* (k_0,K)*-lsca for all* $x \in X$, *then* $\leq_{k_0}$ *is lower closed.*

Proof (i) It is clear that each transitive relation $\Re$ has this property.

(ii) For a fixed $x \in X$, assume that $x_n \in S_{k_0}(x)$ and $x_n \to \overline{x}$. We show that $\overline{x} \in S_{k_0}(x)$. By the lower semicontinuity of $p(x,.)$, $\forall i \in \mathbb{N}$ (large enough), $\exists Q(i) \in \mathbb{N}$, $\forall n > Q(i)$,

$$p(x,x_n) \geq p(x,\overline{x}) - \frac{1}{i}.$$

Therefore, one has

$$\Psi(x, x_n) + (p(x, \overline{x}) - 1/i)k_0 \in -K.$$

Since $\Psi(x, .)$ is (k_0, K)-lsc at $\overline{x}$, one obtains

$$\Psi(x, \overline{x}) + (p(x, \overline{x}) - 1/i)k_0 \in -K.$$

By the closedness of K, passing $i \to \infty$ one sees that $\overline{x} \leq_{k_0} x$, i.e. $\overline{x} \in S_{k_0}(x)$.

(iii) Assume that $x_{n+1} \leq_{k_0} x_n, \forall n \in \mathbb{N}$, and $x_n \to \overline{x}$. Fix n. For each sufficiently large $i \in \mathbb{N}$, by the lower semicontinuity of $p(x_n, .)$ one has $Q(i) \in \mathbb{N}$ such that, $\forall q > Q(i)$,

$$\Psi(x_n, x_{n+q}) + (p(x_n, \overline{x}) - 1/i)k_0 \in \Psi(x_n, x_{n+q}) + p(x_n, x_{n+q})k_0 - K.$$

Hence, as $x_{n+q} \leq_{k_0} x_n$,

$$\Psi(x_n, x_{n+q}) + (p(x_n, \overline{x}) - 1/i)k_0 \in -K. \tag{3}$$

We claim that, for $x, y \in S_{k_0}(x_n)$ with $y \leq_{k_0} x$, one has

$$\Psi(x, y) \geq_K \Psi(x_n, y) - \Psi(x_n, x). \tag{4}$$

Indeed, as $\Psi(x, y) \in -K$ and $\Psi(x_n, x) \in -K$, by condition (H) one has

$$\Psi(x_n, x) + \Psi(x, y) \geq_K \Psi(x_n, y),$$

which is (4). For $q \in \mathbb{N}$, we have $x_{n+q+1} \leq_{k_0} x_{n+q}$, i.e.

$$\Psi(x_{n+q}, x_{n+q+1}) + p(x_{n+q}, x_{n+q+1})k_0 \in -K.$$

Applying (4) to this inclusion we get

$$\Psi(x_n, x_{n+q+1}) - \Psi(x_n, x_{n+q}) + p(x_{n+q}, x_{n+q+1})k_0 \in -K,$$

which is rewritten as

$$\Psi(x_n, x_{n+q+1}) + p(x_{n+q}, x_{n+q+1})k_0 \leq_K \Psi(x_n, x_{n+q}). \tag{5}$$

Since $\Psi(x_n, .)$ is (k_0, K)-lsca at $\overline{x}$ and by (3), (5), we have

$$\Psi(x_n, \overline{x}) + (p(x_n, \overline{x}) - 1/i)k_0 \in -K.$$

Passing to the limit as $i \to \infty$ we obtain $\overline{x} \leq_{k_0} x_n$, by the closedness of K. $\square$

3 Main results

We continue to assume that $X, Y, K, k_0, p, z^*, \Psi$ and $\leq_{k_0}$ are defined as by the end of the previous section. The following result is a generalization of the EVP.

Theorem 3.1 *Let X, Y, K, k_0, p, Ψ and $\leq_{k_0}$ be as above. Assume that $\leq_{k_0}$ is lower closed and that, for $x_0 \in X$, $S_{k_0}(x_0)$ is nonempty and $\leq_{k_0}$-complete, and $\Psi(x_0, S_{k_0}(x_0))$ is K-bounded from below.*

Then there is $v \in S_{k_0}(x_0)$ such that, for each $x \neq v$,

$$\Psi(v, x) + p(v, x)k_0 \notin -K. \tag{6}$$

Proof To apply Theorem 2.4 it suffices to check that any $\leq_{k_0}$-decreasing sequence x_n in X is asymptotic by p. Suppose to the contrary the existence of $\delta > 0$ such that, $\forall n \in \mathbb{N}$, $p(x_n, x_{n+1}) \geq \delta$. Since this sequence decreases, one has

$$\delta k_0 \leq_K k_0 p(x_n, x_{n+1}) \leq_K -\Psi(x_n, x_{n+1}).$$

Since $\Psi(x_0, x_n) \in -K$ and $\Psi(x_n, x_{n+1}) \in -K$, by condition (H) one also has

$$-\Psi(x_n, x_{n+1}) \leq_K \Psi(x_0, x_n) - \Psi(x_0, x_{n+1}).$$

Therefore

$$\delta \leq z^*(-\Psi(x_n, x_{n+1}) \leq z^*(\Psi(x_0, x_n)) - z^*(\Psi(x_0, x_{n+1})).$$

Adding these inequalities from 0 to $n-1$, one obtains

$$n\delta \leq z^*(\Psi(x_0, x_0)) - z^*(\Psi(x_0, x_n)) \leq z^*(\Psi(x_0, x_0)) - \inf_{x \in S_{k_0}(x_0)} z^*(\Psi(x_0, x)).$$

Taking the K-boundedness of $\Psi(x_0, S_{k_0}(x_0))$ into account we get a contradiction.

Theorem 2.4 gives a point $v \in S_{k_0}(x_0)$ such that $S_{k_0}(v) = \emptyset$ or $S_{k_0}(v) = \{v\}$, which is the same as (6). □

The following form of the EVP is more traditional.

Theorem 3.2 *Assume, additionally to the assumptions of Theorem 3.1, that $\Psi(x_0, x) + \varepsilon k_0 \notin -K$, for some $\varepsilon > 0$ and all $x \in X$. Then, for each $\lambda > 0$, there is $v \in S_{k_0}(x_0)$ such that, for all $x \neq v$,*

(i) $p(x_0, v) \leq \lambda$;

(ii) $\Psi(x_0, v) + \frac{\varepsilon}{\lambda} p(x_0, v) k_0 \in -K$;

(iii) $\Psi(v, x) + \frac{\varepsilon}{\lambda} p(v, x) k_0 \notin -K$.

Proof Applying Theorem 3.1 with εk_0 in the place of k_0 and $\frac{1}{\lambda} p$ in the place of p we obtain (ii) and (iii). Suppose $p(x_0, v) > \lambda$. Then we would have

$$\Psi(x_0, v) + \varepsilon k_0 \in \Psi(x_0, v) + \frac{\varepsilon}{\lambda} p(x_0, v) k_0 - K \subseteq -K,$$

contradicting the property of ε. □

We collect some equivalent formulation of the EVP, mainly concerning fixed points, in the following result.

Theorem 3.3 *Let the assumptions of Theorem 3.1 hold. Then we have the following equivalent assertions, for any index set I.*

(i) (The EVP) *There exists $v \in S_{k_0}(x_0)$ such that, for $x \neq v$,*

$$\Psi(v, x) + p(v, x) k_0 \notin -K.$$

(ii) (Common fixed-point theorem for a family of multivalued maps) *Let* T_i*:* $S_{k_0}(x_0) \to 2^X$*,* $i \in I$*, be such that* $T_i(x) \neq \emptyset, \forall x \in X$*, and for each* $x \in S_{k_0}(x_0) \setminus T_i(x)$ *one has* $y \in S_{k_0}(x) \setminus \{x\}$*. Then there exists a common fixed point for* $T_i, i \in I$*, in* $S_{k_0}(x_0)$*.*

(iii) (Caristi's common fixed-point theorem for a family of multivalued maps) *Let* T_i*:* $S_{k_0}(x_0) \to 2^X$*,* $i \in I$*, be with nonempty values and such that for each* $x \in S_{k_0}(x_0)$ *one has* $y \in S_{k_0}(x)$*. Then the family* $\{T_i\}_{i\in I}$ *has a common fixed point in* $S_{k_0}(x_0)$*.*

(iv) (Caristi's common fixed-point theorem for a family of single-valued maps) *Let* T_i*:* $S_{k_0}(x_0) \to X$*,* $i \in I$*, be such that* $T_i(x) \in S_{k_0}(x)$ *for all* $x \in S_{k_0}(x_0)$*. Then* $\{T_i\}_{i\in I}$ *has a common fixed point in* $S_{k_0}(x_0)$*.*

(v) (Common invariant-point theorem) *Let* T_i*:* $S_{k_0}(x_0) \to 2^X$*,* $i \in I$*, be with nonempty values and such that* $T_i(x) \subseteq S_{k_0}(x)$ *for each* $x \in S_{k_0}(x_0)$*. Then* $\{T_i\}_{i\in I}$ *has a common invariant point in* $S_{k_0}(x_0)$*.*

(vi) (Maximal element theorem) *Let* T_i*:* $S_{k_0}(x_0) \to 2^X$*,* $i \in I$*, be such that for each* $x \in S_{k_0}(x_0)$ *with* $T_i(x) \neq \emptyset$*, there exists* $y \in S_{k_0}(x) \setminus \{x\}$*. Then there exists* $\overline{x} \in S_{k_0}(x_0)$ *such that* $T_i(\overline{x}) = \emptyset$ *for each* $i \in I$*.*

Proof (i) holds by Theorem 3.1. Now we show the equivalences. Note first that (i) is equivalent to saying that there exists $v \in S_{k_0}(x_0)$ such that $S_{k_0}(v) = \emptyset$ or $S_{k_0}(v) = \{v\}$.

"(i)⇒ (ii)" Suppose, for the above-mentioned v, $v \notin T_{i_0}(v)$ for some $i_0 \in I$. Then by the assumption, there is $y \in S_{k_0}(v) \setminus \{v\}$, a contradiction.

"(ii)⇒ (iii)" Suppose $\forall x \in S_{k_0}(x_0)$, $\exists i \in I$, $x \notin T_i(x)$. By the assumption, there is $y \in S_{k_0}(x)$. If all such y are equal to x, i.e. $S_{k_0}(x) = \{x\}$ for all

$x \in S_{k_0}(x_0)$, we are done. If there is $y \in S_{k_0}(x) \setminus \{x\}$, by (ii) we also arrive at the conclusion.

"(iii)$\Rightarrow$ (iv)" It is clear.

"(iv)$\Rightarrow$ (v)" Suppose that for each $x \in S_{k_0}(x_0)$ there are i and y with $y \in T_i(x) \setminus \{x\}$. By the assumption, $y \in T_i(x) \subseteq S_{k_0}(x)$. We define a family of single-valued maps $T_i^{'}$, $i \in I$, as follows. For $i \in I$, if there are x, y as above we put $T_i^{'}(x) = y$ for any such an y. Otherwise we set $T_i^{'}(x) = x^{'}$ for any $x^{'} \in S_{k_0}(x)$. By (iv) applied to the family of maps $T_i^{'}$ one has $\overline{x} \in S_{k_0}(x_0)$ such that $\overline{x} = T_i^{'}(\overline{x})$. By the contradiction assumption there is $i \in I$ with $T_i^{'}(\overline{x}) \in T_i(\overline{x}) \setminus \{\overline{x}\}$, a contradiction.

"(v)$\Rightarrow$(vi)" Suppose to the contrary that $\forall x \in S_{k_0}(x_0)$, $\exists i \in I$, $T_i(x) \neq \emptyset$. By the assumption, there is $y \in S_{k_0}(x) \setminus \{x\}$. We define a family of multivalued maps $T_i^{'}$ as follows. For each $i \in I$, if there are x, y as above, for each such an x we put $T_i^{'}(x)$ to be the set of all such y. Otherwise we put $T_i^{'}(x) = S_{k_0}(x)$. Then by (v) there is $\overline{x} \in S_{k_0}(x_0)$ such that $T_i^{'}(\overline{x}) = \{\overline{x}\}$, $\forall i \in I$. This contradicts the case where i satisfies the contradiction assumption.

"(vi)$\Rightarrow$ (i)" Suppose, ab absurdo, that $\forall x \in S_{k_0}(x_0)$, $\exists y \in S_{k_0}(x) \setminus \{x\}$. We define a multivalued map $T : S_{k_0}(x_0) \to 2^X$ by setting $T(x) = \{x\}$, $\forall x \in S_{k_0}(x_0)$. Applying (vi) to this family of one map one has $\overline{x} \in S_{k_0}(x_0)$ with $T(\overline{x}) = \emptyset$, a contradiction. □

Notice that Theorem 3.3 includes Theorem 1 of [3], where p is a w-distance and $\Psi(x, .)$ is assumed to be K-lsca for all $x \in X$.

Remark 3.1

(a) Lemma 2.5 provides sufficient conditions for the relation $\leq_{k_0}$ to be lower closed in terms of generalized lower semicontinuity of $\Psi(x, .)$ for all $x \in X$. These conditions may be easier to be checked, but they are less relaxed than the lower closedness of $\leq_{k_0}$ as shown by the following example.

Example 3.1 Let $f : R \to R \cup \{+\infty\}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \neq 0, \\ +\infty & \text{if} \quad x = 0, \end{cases}$$

$K = R_+$, $k_0 = 1$, $\Psi(x, y) = f(y) - f(x)$, $p(x, y) = |y - x|$ and $\leq_{k_0}$ be defined by

$$y \leq_{k_0} x \Leftrightarrow f(y) - f(x) + |y - x| \leq 0.$$

Then $\Psi(x, .)$ is not (k_0, K)-lsca at 0 and $S_{k_0}(x) = \{y \in X : f(y) - f(x) + |y - x| \leq 0\}$ is not closed for each $x \neq 0$. However, in this case it is easy to check that $\leq_{k_0}$ is lower closed. Therefore, Theorem 3.1 shows the existence of a strict minimizer v of $f(.) + |. - v|$. It is easy to see directly that for each $v \in [-\frac{1}{2}, \frac{1}{2}] \setminus \{0\}$ and each $x \neq v$, one has $f(x) + |x - v| \not\leq f(v)$, i.e. v is a strict minimizer of $f(.) + |. - v|$.

The following example supplies a case where p is a true weak τ-function (not τ-function) and Theorem 3.1 can be applied but (k_0, K)-lsca assumptions are not satisfied.

Example 3.2 Let f, K, k_0 and Ψ be as in Example 3.1. Let

$$p(x, y) = \begin{cases} |y - x| + 1 & \text{if} \quad y \neq x, \\ \frac{3}{2} & \text{if} \quad y = x, \end{cases}$$

$$y \leq_{k_0} x \Leftrightarrow f(y) - f(x) + p(x, y) \leq 0.$$

Then it is easy to check that $\Psi(x, .)$ is not (k_0, K)-lsca and $S_{k_0}(x)$ is not closed for each $x \neq 0$, but $\leq_{k_0}$ is lower closed. Thus a strict minimizer v of $f(.) + p(v, .)$ is guaranteed by Theorem 3.1. Now we verify directly that each $v \in (-\frac{3}{2}, \frac{3}{2}) \setminus \{0\}$ is such a strict minimizer, i.e. for each $x \neq v$,

$$f(x) + |x - v| + 1 > v^2. \tag{7}$$

For $x = 0$, any $v \in R \setminus \{0\}$ satisfies this inequality. With $x \neq 0$, $v \neq 0$ and $x \neq v$ (7) is equivalent to

$$\begin{cases} x^2 - x - v^2 + v + 1 > 0 & \text{if} \quad x < v, \\ x^2 + x - v^2 - v + 1 > 0 & \text{if} \quad x > v. \end{cases}$$

By considering three subcases $v \leq -\frac{1}{2}$, $-\frac{1}{2} \leq v \leq \frac{1}{2}$ and $v \geq \frac{1}{2}$ one easily sees that each $v \in (-\frac{3}{2}, \frac{3}{2}) \setminus \{0\}$ satisfies (7).

(b) To the best of our knowledge all the papers in the literature about the EVP for a two-variable function $\Psi : X \times X \to Y$ impose the condition

(H') $\Psi(x, z) \leq_K \Psi(x, y) + \Psi(y, z)$ for all $x, y, z \in X$,

which is stronger than (H). They often conclude that the results include the corresponding ones for one-variable function $f : X \to Y$, since to prove the latter results one simply sets $\Psi(x, y) = f(y) - f(x)$ and applies the results for Ψ. In fact, the two cases are equivalent under condition (H'), if we impose K-boundedness and K-lower semicontinuity (or K-lower semicontinuity from above) assumptions. Indeed, to derive a result for Ψ from the corresponding one for f, one can set $f(.) = \Psi(x_0, .)$, where x_0 is a given point such that $\Psi(x_0, .)$ is K-bounded from below. Then the K-boundedness and K-lower semicontinuity (or K-lower semicontinuity from above) of $\Psi(x_0, .)$ clearly imply the same properties of $f(.)$. Hence we can derive from a result for f a corresponding one for Ψ (see also [14, 15]). However, if we impose a general and relaxed assumption about the lower closedness of the transitive relation $\leq_{k_0}$ defined by Ψ as in Theorems 3.1-3.3, these results are stronger than the corresponding ones for the relation $\leq^*_{k_0}$ defined by function $f : X \to Y$ as follows

$$y \leq^*_{k_0} x \Longleftrightarrow f(y) - f(x) + p(x, y)k_0 \in -K.$$

Namely, Theorem 3.1 implies the following

Theorem 3.1' *Let X, Y, K, k_0, p be as in Theorem 3.1 and $f : X \to Y$. Assume, for $x_0 \in X$, that $S^*_{k_0}(x_0) := \{x \in X : x \leq^*_{k_0} x_0\}$ is nonempty and $\leq^*_{k_0}$ complete. Assume further that $f(S^*_{k_0}(x_0))$ is K-bounded from below and $\leq^*_{k_0}$ is lower closed. Then there exists $v \in S^*_{k_0}(x_0)$ such that, for all $x \neq v$,*

$$f(x) - f(v) + p(v, x)k_0 \notin -K. \tag{8}$$

To prove this theorem we simply set $\Psi(x, y) = f(y) - f(x)$ to see that the assumptions of Theorem 3.1 are satisfied and then we can apply this theorem to derive (8).

By setting $f(.) = \Psi(x_0, .)$ we show now that Theorem 3.1' does not imply Theorem 3.1. By condition (H') it is clear that $y \leq_{k_0} x$ implies $y \leq^*_{k_0} x$. But the converse is not true in general. Indeed, the relation $y \leq^*_{k_0} x$ means that

$$f(y) - f(x) + p(x, y)k_0 \in -K,$$

i.e.

$$\Psi(x_0, y) - \Psi(x_0, x) + p(x, y)k_0 \in -K.$$

This and condition (H'), i.e. $\Psi(x_0, y) - \Psi(x_0, x) \leq_K \Psi(x, y)$, do not imply

$$\Psi(x, y) + p(x, y)k_0 \in -K,$$

which means that $y \leq_{k_0} x$. Hence from the lower closedness of $\leq_{k_0}$ we still do not have the lower closedness of $\leq^*_{k_0}$ in order to apply Theorem 3.1'.

By Lemma 2.5, Theorem 3.1 implies the corresponding results for the cases, where (k_0, K)-lower semicontinuity or (k_0, K)-lower semicontinuity from above assumptions are imposed instead of the more general lower closedness assumption. The following example, shows however that the two afore mentioned cases of a two-variable setting and of an one-variable setting are not equivalent.

Example 3.3 Let $f : R^2 \to R^2 \cup \{+\infty\}$ be defined by

$$f(x_1, x_2) = \begin{cases} (x_1, x_2) & \text{if} \quad x_1 < 0, x_2 > x_1^2 \;\text{ or }\; x_1 \geq 0, x_2 \geq x_1^2, \\ +\infty & \qquad\qquad \text{otherwise,} \end{cases}$$

$K = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq x_1\}$, $k_0 = (1, 1)^T$ and $x^0 = (0, 1)^T$, where $+\infty$ is the additional element of R^2 (mentioned in Section 2) and we admit the conversion that $+\infty - (+\infty) = +\infty$. Then f is (k_0, K)-lsc, since for each r the set

$$\{(x_1, x_2) : f(x_1, x_2) + rk_0 \in -K\}$$

is closed. However $\Psi(x^0, .) = f(.) - (0, 1)^T$ is neither (k_0, K)-lsc nor (k_0, K)-lsca.

(c) The assumption that $\Psi(x_0, S_{k_0}(x_0))$ is K-bounded is weaker than the K-boundedness of $\Psi(x_0, X)$ as shown by the following example (we write $f(.)$ instead of $\Psi(x_0, .)$ and take $f(x_0) = 0$ for the sake of simplicity).

Example 3.4 Let $f : R^2 \to R^2 \cup \{+\infty\}$ be defined by

$$f(x_1, x_2) = \begin{cases} (x_1, x_2) & \text{if} \quad x_2 \leq -x_1^2, \\ +\infty & \text{otherwise,} \end{cases}$$

$K = \{(x_1, x_2) : x_2 \geq 0, x_1 \geq x_2\}$, $k_0 = (1, 1)^T$ and $x^0 = (0, 0)^T$. Then $f(S_{k_0}(x^0))$ is K-bounded from below, since $f(S_{k_0}(x^0)) \subseteq \{f(x) : f(x) \leq_K f(x^0)\}$ and the set $\{f(x) : f(x) \leq_K f(x^0)\}$ is K-bounded from below. However $f(R^2)$ is not K-bounded from below.

4 Particular cases

Corollary 4.1 (Lin and Du 2006, Theorem 2.1) *Let X be a complete quasimetric space and p be a τ-function. Let $f\colon X \to R \cup \{+\infty\}$ be proper, R_+-lsca and bounded from below. Let $\varphi\colon R \to (0, \infty)$ be nondecreasing. Then there exists $v \in X$ such that, for all $x \neq v$,*

$$p(v, x) > \varphi(f(v))(f(v) - f(x)).$$

Proof Setting $\Psi(x, y) = \varphi(f(x))(f(y) - f(x))$ we see that, $\forall x \in X$, $\Psi(x, .)$ is proper, R_+-lsca, bounded from below. To prove that Ψ satisfies condition (H) we see that if $\Psi(x, z) \leq 0$ and $\Psi(z, y) \leq 0$ then

$$f(y) \leq f(z) \leq f(x).$$

Hence

$$\begin{aligned}\Psi(x, z) + \Psi(z, y) &\geq \varphi(f(x))(f(z) - f(x)) + \varphi(f(x))(f(y) - f(z)) \\ &= \Psi(x, y).\end{aligned}$$

Now applying Lemma 2.5(iii) and Theorem 3.1 with $k_0 = 1$ one obtains $v \in X$ such that, for $x \neq v$,

$$\Psi(v, x) + p(v, x) > 0.$$

Therefore

$$p(v, x) > \varphi(f(v))(f(v) - f(x)). \qquad \square$$

Remark 4.1 As Theorem 3.1 is more general than Theorem 2.1 of Lin and Du (2006), Theorems 2.2-2.3 and Corollary 2.1 of Lin and Du (2006) are special cases of Theorem 3.3(iii), (iv), (vi).

Corollary 4.2 *Assume that X is a complete quasimetric space, p is a τ-function, f: $X \to R \cup \{+\infty\}$ is a proper, R_+-lsca and bounded from below function, φ: $R \to (0, \infty)$ is a nondecreasing function, $\varepsilon > 0$ and $x_0 \in X$ satisfies $f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon$. Then $v \in X$ exists such that, for each $x \neq v$,*

(i) $0 \leq f(x_0) - f(v) \leq \varepsilon$, *if* $p(x_0, x_0) = 0$;

(ii) $\varepsilon p(v, x) > \varphi(f(v))(f(v) - f(x))$.

Proof Setting $\Psi(x,y) = \varphi(f(x))(f(y) - f(x))$ and $k_0 = 1$. By Corollary 4.1 we have (ii) (taking εp for p in Corollary 4.1). Looking at the proof of Theorem 3.1 we see that $v \in S_{k_0}(x_0)$, which implies $\Psi(x_0, v) \leq 0$. Hence, $f(x_0) - f(v) \geq 0$. Since $f(x_0) \leq \inf_{x\in X} f(x) + \varepsilon \leq f(v) + \varepsilon$, we obtain (i). □

If X is a metric space and p is a w-distance, Corollary 4.2 coincides with Theorem 2.4, a main result, of Lin and Du (2007).

Corollary 4.3 *Let X, Y, K and k_0 be as in Theorem 3.1, with the additional completeness of X and closedness of K, and let p be a τ-function. Let $x_0 \in X$, $\varepsilon > 0$ and $\Psi \colon X \times X \to Y$ satisfy condition (H') and the following conditions*

(i) $\Psi(x_0, x_0) = 0$ *and* $z^*(\Psi(x_0, .))$ *is bounded from below, where* $z^* \in K^+$ *with* $z^*(k_0) = 1$;

(ii) $\Psi(x, .)$ *is* (k_0, K)*-lsca for all* $x \in X$.

Then there exists $v \in X$ such that, for all $x \neq v$,

(a) $\Psi(x_0, v) + \varepsilon p(x_0, v)k_0 \in -K$, *if* $p(x_0, x_0) = 0$;

(b) $\Psi(v, x) + \varepsilon p(v, x)k_0 \notin -K$.

Proof If $\Psi(x, .)$ is (k_0, K)-lsca for all $x \in X$, then by Lemma 2.5(iii), the relation $\leq_{k_0}$ is lower closed. Examining the proof of Theorem 3.1 we see that we can replace the assumed K-boundedness of $\Psi(x_0, .)$ by the weaker assumption that $z^*(\Psi(x_0, .))$ is bounded from below. In view of Theorem 3.1, we are done. □

Remark 4.6 Corollary 4.3 contains properly Theorem 3.1 of Ansari (2007) since in this theorem X is a complete metric space, p is a w-distance, and (i) is required to be fulfilled for all $x \in X$; Theorem 1 of Bianchi et al. (2007) since in this theorem X is a complete metric space, $p = d$, (i) is required to be fulfilled for all $x \in X$ and (ii) is replaced by the condition that $\Psi(x, .)$ is K-lsc for all $x \in X$; Theorem 2.1 of Bianchi et al. (2005), which is the special case with $Y = R$ of the preceding Theorem 1.

Corollary 4.4 *Let X, Y, K, p and k_0 be as in Corollary 4.3. Let Ψ: $X \times X \to Y$ satisfy conditions (H') and (ii) of Corollary 4.3.*

Define a binary relation $\leq'_{k_0}$ on X by

$$y \leq'_{k_0} x \quad \Leftrightarrow \quad y = x \quad \text{or} \quad y \leq_{k_0} x.$$

Assume that there exists a nonempty subset M of X such that

(i) *M is $\leq'_{k_0}$ complete;*

(ii) *there exists $x_0 \in M$ such that $\Psi(x_0, x_0) = 0$ and $z^*(\Psi(x_0, .))$ is bounded from below, where $z^* \in K^+$ with $z^*(k_0) = 1$.*

Then $\leq'_{k_0}$ is a quasi-order and there exists $v \in X$ such that, for all $x \neq v$,

(1) $\{y \in M : y \leq'_{k_0} v\} = \{v\}$;

(2) $\Psi(x_0, v) + \varepsilon p(x_0, v)k_0 \in -K$, *if* $p(x_0, x_0) = 0$;

(3) $\Psi(v, x) + \varepsilon p(v, x)k_0 \notin -K$.

This corollary is derived directly from Corollary 4.3 and properly includes Theorems 2.1-2.2 of Lin and Du (in press, 2007). Furthermore applying Theorem 3.3 we obtain also Theorem 3.1 of this paper.

The following corollary is a direct consequence of Theorem 3.1, by Lemma

2.5(i) and by replacing $\Psi(x,y) = f(y) - f(x)$. Note that the relation $\le_{k_0}$ defined by (2) now becomes

$$y \le_{k_0} x \quad \Leftrightarrow \quad f(y) + p(x,y)k_0 \le_K f(x).$$

Corollary 4.5 *Let X, Y, p, K and k_0 be as in Corollary 4.3. Assume that $f : X \to Y \cup \{+\infty\}$ be proper and K-bounded from below. Let $S(x){:=}\{x' \in X : f(x') + p(x,x')k_0 \le_K f(x)\}$ be closed for every $x \in X$. Then for every $x_0 \in domf$ there exists $v \in X$ such that, $\forall x \ne v$,*

$$f(v) + p(x_0, v)k_0 \le_K f(x_0), \tag{9}$$

$$f(x) + p(v,x)k_0 \not\le_K f(v). \tag{10}$$

This corollary properly includes Corollary 2 of Göpfert et al. (2000).

Corollary 4.6 *Let X, Y, p, K and k_0 be as in Corollary 4.3. Let $f : X \to Y \cup \{+\infty\}$ be proper and K-bounded from below. Assume that if $x_n \in domf, x_n \to x$ and $f(x_n)$ is $\le_K$-decreasing, then $f(x) \le_K f(x_n), \forall n \in \mathbb{N}$. Assume that K is closed in the direction k_0, i.e. $K \cap (y - R_+ k_0)$ is closed for all $y \in Y$. Assume further that $x_0 \in domf$ with $p(x_0, x_0) = 0$. Then there exists $v \in X$ such that, (9) and (10) hold for all $x \ne v$.*

Proof Since $\Psi(x,x) = f(x) - f(x) = 0$ and $p(x_0,x_0) = 0$, $S_{k_0}(x_0) \ne \emptyset$. To apply Theorem 4.3 we show the $\le_{k_0}$-completeness of $S_{k_0}(x_0)$. If $\{x_n\} \subseteq S_{k_0}(x_0)$ is $\le_{k_0}$-decreasing and Cauchy then $x_n \to x$, for some $x \in X$, and $f(x_n)$ is clearly $\le_K$-decreasing, and hence $f(x) \le_K f(x_n), \forall n \in \mathbb{N}$.

Now, fix n. For $i \in \mathbb{N}$, by $(\tau 2)$ there exists $Q(i) \in \mathbb{N}$ such that, $\forall q > Q(i)$,

$$p(x_n, x_{n+q}) \ge p(x_n, x) - \frac{1}{i}\,.$$

Consequently,

$$f(x) + p(x_n, x)k_0 \le_K f(x) + (p(x_n, x_{n+q}) + \tfrac{1}{i})k_0$$
$$\le_K f(x_{n+q}) + (p(x_n, x_{n+q}) + \tfrac{1}{i})k_0$$
$$\le_K f(x_n) + \tfrac{1}{i}k_0.$$

Passing $i \to \infty$, by the closedness of K in the direction k_0, we obtain that $f(x) + k_0 p(x_n, x) \le_K f(x_n)$, i.e. $x \le_{k_0} x_n, \forall n \in \mathbb{N}$. Hence $x \le_{k_0} x_0$, i.e. $x \in S_{k_0}(x_0)$ and $S_{k_0}(x_0)$ is $\le_{k_0}$-complete. Finally, (9) and (10) follow directly from the conclusion of Theorem 3.1. □

If f is not only K-bounded from below but also bounded from below and $p = d$, this corollary coincides with Corollary 3 of Göpfert et al. (2000).

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