# On generalized Ekeland's variational principle and equivalent formulations for set-valued mappings

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**Abstract** We propose a very weak type of generalized distance called weak $\tau$-function and use it to weaken the assumptions about lower semicontinuity in existing formulations of Ekeland's variational principle for a kind of minimizers of a set-valued mapping, which is different from the Pareto minimizer, and in recent results which are equivalent to Ekeland's variational principle.

**Keywords** Quasimetric spaces, locally convex spaces, weak $\tau$-functions, Ekeland's variational principle, $K$-lower semicontinity from above, lower closedness.

## 1 Introduction

The celebrated Ekeland's variational principle (Ekeland 1974) (EVP, from now on) is one of the most important results and cornerstones of nonlinear analysis with applications in many fields of analysis, optimization and operations research. Its importance is emphasized by the fact that there are a number of equivalent formulations, all of which are well known with significant applications and many of which were discovered independently, namely the Caristi-Kirk fixed-point the-

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orem (Caristi 1976), the drop theorem of Daneš (Daneš 1972), the Takahashi theorem about the existence of minima (Takahashi 1991), the petal theorem of Penot (Penot 1986), the Krasnoselski-Zabrejko theorem on solvability of operator equations (Zabreiko and Krasnoselski 1971), Phelps' lemma (Phelps 1974), etc.

Over more than three decades a good deal of effort has been made to look for equivalent formulations or generalizations of the EVP.

The seminal EVP (Ekeland 1974) says roughly that, for a lower semicontinuous (lsc) and bounded from below function $f$ on a complete metric space $X$, a slightly perturbed function has a strictly minimum. Moreover, if $X$ is a Banach space and $f$ is Gateaux differentiable, then its derivative can be made arbitrarily small.

We can first observe generalizations of the EVP to vector minimization, i.e. to the case where $f$ is a mapping with a multidimensional range space $Y$, see e.g. Khanh (1989), Loridan (1984), Valyi (1985). Here $Y$ may be even an ordered vector space. Extensions of $X$ to the case of topological vector spaces, uniform spaces or $L$-spaces are investigated e.g. in Hamel (2003, 2005, 2006), Khanh (1989), Qui (2005). In this research direction, a general partial order is often proposed and a minimal point with respect to (wrt) this order is proved to be existence, leading to a type of the EVP, see also Göpfer et al. (2000), Hamel and Tammer (in press). Smooth variants of the EVP are studied e.g. in Borwein and Preiss (1987), Li and Shi (2000). The second conclusion of Ekeland in the seminal work (Ekeland 1974) that the Gateaux derivative of $f$ can be made arbitrarily small has been attracted also much attentions, see e.g. Ha (2003, 2005, 2006), Bao and Mordukhovich (in press). Here various kinds of generalized derivatives are discussed: the Fre'chet, Clarke and Mordukhovich coderivatives; the Fre'chet, Clarke and Mordukhovich subdifferentials. Fre'chet Hessians are also used to establish the Ekeland principle for second-order optimality conditions (Arutyunov 1997). Stability results for the EVP are obtained e.g. in Attouch (1993), Huang (2001, 2002). In connection with the EVP, existence conditions for optimal solutions for problems with noncompact feasible sets are dealt with in Ha (2003, 2006), Bao and Mordukhovich (in press), El Amrouss (2006) using generalizations of coercivity assumptions, the Palais-Smale condition or the Cerami condition.

One of the recent research interests is to consider the case where $X$ is a metric space but equipped with an additional generalized distance, based on which the semicontinuity assumption of Ekeland can be weakened. $w$-distance was introduced in Kada et al. (1996) and used also in Park (2000), Lin and Du (2007). In Tataru (1992) another distance was proposed to obtain a generalization of the EVP. In Suzuki (2001, 2005) $\tau$-distance, which is more general then both aforementioned distances, is introduced to improve the EVP. $\tau$-function is introduced and employed in Lin and Du (2006).

In this paper we propose a definition of weak $\tau$-functions to investigate the EVP and equivalent results for a multivalued mapping $F$ from a quasimetric (that is, not necessarily symmetric) space $X$ into a Hausdorff topological vector space $Y$ ordered by a convex cone $K$. Here $\min_{x \in X} F(x)$ is not understood in the usual Pareto sense, but in a meaning recently employed in Hamel (2006), Ha (2005), Kuroiwa (2001), see Definition 2.1. We improve recent existing results, e.g. in Göpfert et al. (2000), Ha (2005), Park (2000), Lin and Du (2006, 2007), Danneš et al. (1983), Bianchi et al. (2007). The layout of the paper is as follows. Section 2 is devoted to preliminaries needed in the sequel. In Section 3 we propose a generalized distance called weak $\tau$-function and discuss some properties. Our main results are presented in Section 4. Some applications are given in the final Section 5.

## 2 Preliminaries

We discuss first minimizer notions for multivalued mappings.

**Definition 2.1** (Kuroiwa 2001) Let $X$ be a set, $Y$ be a vector space, $K \subseteq Y$ be a nonempty convex cone and $F : X \to 2^Y$ be a multivalued mapping. Then $\overline{x} \in X$ is called a minimizer of $F$ if $F(\overline{x}) \subseteq F(x) + K$, for some $x \in X$, implies $F(x) \subseteq F(\overline{x}) + K$.

Recall that $\overline{x}$ is a Pareto minimizer of $F$ if there is $\overline{y} \in F(\overline{x})$ such that $F(X) \cap (\overline{y} - K) \subseteq \overline{y} + K \cap -K$. Hence, if $F$ is single-valued, $\overline{x}$ is a Pareto minimizer of $F$ if and only if $F(\overline{x}) \in F(x) + K$, for some $x \in X$, implies $F(x) \in F(\overline{x}) + K$. Therefore, the minimizer defined in Definition 2.1 may be roughly understood as a Pareto minimizer of $F$ consider as a single-valued mapping from $X$ into the space of all subsets of $Y$, i.e. each $F(x)$ is now dealt with rougher as a point in this space. However, the relation of the two above-mentioned minimizers is more interesting, and in fact the minimizer notion we adopt in this paper is not rougher, as illustrated by the Examples 2.1 and 2.2 below.

A notion stronger than minimizer is defined in the following natural way.

**Definition 2.2** (Ha 2005) Let $X, Y$ and $K$ be as in Definition 2.1 . Then $\overline{x} \in X$ is a strict minimizer of a multivalued map $F$ if $F(\overline{x}) \not\subseteq F(x) + K, \forall x \neq \overline{x}$.

A strict minimizer of a multivalued mapping $F$ is also corresponding to a Pareto strict minimizer of a single-valued mapping, characterized by $F(\overline{x}) \notin F(x) + K, \forall x \neq \overline{x}$.

*Example 2.1* Let $X = R$, $Y = R^2$, $K = R^2_+$ and $F$ be defined by, for $x \in X$,

$$F(x) = \{(x, y) \in R^2 : y = \lambda(x, 1) + (1 - \lambda)(0, 0), 0 \leq \lambda \leq 1\}.$$

Then, there is no minimizer of $F$ (in the sense of Definition 2.1), but each $x \in X$

is a Pareto minimizer.

*Example 2.2* Let $X$, $Y$ and $K$ be as in Example 2.1. Let $F$ be defined by, for $x \in X$,

$$F(x) = \{(x, y) \in R^2 : y > -x\}.$$

Then no Pareto minimizer exists, but each $x \in X$ is a minimizer and also a strict minimizer of $F$.

Now we pass to lower-semicontinuity definitions. Let $X$ be a topological space, $Y$ be a topological vector space, $K \subseteq Y$ be a convex cone and $F : X \to 2^Y$. F is said to $K$-lower semicontinuous ($K$-lsc) if, $\forall e \in Y$, the set $\{x \in X : F(x) \cap (e - K) \neq \emptyset\}$ is closed. From the equality, for $A \subseteq X$,

$$\{x \in X : A \subseteq F(x) + K\} = \cap_{a \in A}\{x \in X : F(x) \cap (a - K) \neq \emptyset\}$$

it follows that $F$ is $K$-lsc on $X$ if and only if, $\forall A \subseteq Y$, the set $\{x \in X : A \subseteq F(x) + K\}$ is closed.

$F$ is called $K$-lower semicontinuous from above ($K$-lsca) at $\overline{x} \in X$ if for each convergent sequence $x_n \to \overline{x}$ with $F(x_n) \subseteq F(x_{n+1}) + K, \forall n \in \mathbb{N}$ (the set of natural numbers), one has $F(x_n) \subseteq F(\overline{x}) + K$. As any definition for a point is extended to a set, $F$ is called $K$-lsca on $A \subseteq X$ if $F$ is $K$-lsca at all $x \in A$. If $A = \mathrm{dom}F := \{x \in X : F(x) \neq \emptyset\}$ we omit "on $A$" in the statement. This notion was proposed in Chen et al. (2002). Of course if $\mathrm{dom}F = X$ then being $K$-lsc implies being $K$-lsca. The converse is not true as shown by

*Example 2.3* Let $X = Y = R$, $K = R_+$ and $f$ be the (single-valued) function

$$f(x) = \begin{cases} 3 - x, & \text{if} \quad x < 0, \\ 2, & \text{if} \quad x = 0, \\ (1 + x)^{-1}, & \text{if} \quad x > 0. \end{cases}$$

Then $f$ is $R_+$-lsca, but $f$ is not $R_+$-lsc at $\overline{x} = 0$.

We propose to extend the $K$-lower semicontinuity from above to the case of a transitive relation on a topological space $X$ as follows.

**Definition 2.3** A transitive relation $\Re$ on $X$ (i.e. $z\Re y$ and $y\Re x$ imply $z\Re x$) is said to be lower closed if for any $\Re$-monotone (i.e. $...\Re x_n\Re...\Re x_2\Re x_1$) convergent sequence $x_n \to \overline{x}$ one has $\overline{x}\Re x_n, \forall n \in \mathbb{N}$.

*Remark 2.1*

(i) If $S(x) := \{z \in X : z\Re x\}$ (called $\Re$-sector of $x$) is closed for all $x \in X$ then $\Re$ is lower closed (this motivates the term "lower closed"). But clearly the closedness of $S(x)$ for all $x \in X$ is stronger than the lower closedness of $\Re$ (see (ii)).

(ii) Assume that $F : X \to 2^Y$, where $X$ is a topological space, $Y$ is a topological vector space and $K \subseteq Y$ is convex. We define a relation $\Re$ on $X$ by

$$y\Re x \Leftrightarrow F(x) \subseteq F(y)+K. \tag{1}$$

Then $\Re$ is obviously transitive. $\Re$ is lower closed if and only if $F$ is $K$-lsca. On the other hand with $\Re$ defined by (1), one has

$$\begin{aligned} S(x) &= \{z \in X : z\Re x\} = \{z \in X : F(x) \subseteq F(z) + K\} \\ &= \{z \in X : F(z) \cap (e - K) \neq \emptyset, \forall e \in F(x)\}. \end{aligned}$$

Since $\emptyset$ is also closed, $S(x)$ is closed for all $x \in X$ means that $F$ is $K$-lsc.

The following example shows a case where $\Re$ defined by (1) is lower closed but $F$ is not $K$-lsc.

*Example 2.4* Let $X = Y = R$ and $K = R_+$ and $F$ be defined by

$$F(x) = \begin{cases} (1, 1 + x), & \text{if} \quad x > 0, \\ \{0\}, & \text{if} \quad x = 0, \\ \{-2 + (1 - x)^{-1}\}, & \text{if} \quad x < 0. \end{cases}$$

Then it is easy to see that $\Re$ defined by (1) is lower closed. But $F$ is not $R_+$-lsc, since the set $\{x \in A : (-1, 0) \subseteq F(x) + R_+\} = (-\infty, 0)$ is not closed.

Let $Y$ be a topological vector space ordered by a nonempty convex cone $K$. A subset $A \subseteq Y$ is said to be $K$-bounded from below if there is a bounded subset $M \subseteq Y$ such that $A \subseteq M + K$. $A$ is called $K$-closed if $A + K$ is closed. $A$ is called bounded from below if there is $\overline{y} \in Y$ such that $A \subseteq \overline{y} + K$. So $K$-boundedness (from below) implies boundedness (from below) but not vice versa as one can easily find an counterexample.

For a transitive relation $\Re$ in a complete metric space $X$, a subset $A \subseteq X$ is called $\Re$-complete if any Cauchy sequence in $A$, which is $\Re$-monotone, converges to a point of $A$.

## 3 Weak $\tau$-functions

We first recall the notion of $\tau$-functions.

**Definition 3.1** (Lin and Du 2006) Let $(X, d)$ be a quasi-metric space. A function $p : X \times X \to R_+$ is said to be a $\tau$-function if the following conditions hold:

($\tau$1) for all $x, y, z \in X$, $p(x, z) \leq p(x, y) + p(y, z)$;

($\tau$2) if $x \in X$ and $\{y_n\} \subseteq X$ with $\lim_{n\to\infty} y_n = y$ and $p(x, y_n) \leq M$ for some $M = M(x) > 0$, then $p(x, y) \leq M$;

($\tau$3) for any sequences $\{x_n\}$ and $\{y_n\}$ in $X$ with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$ and $\lim_{n\to\infty} p(x_n, y_n) = 0$, one has $\lim_{n\to\infty} d(x_n, y_n) = 0$;

($\tau$4) for $x, y, z \in X$, $p(x, y) = 0$ and $p(x, z) = 0$ imply $y = z$.

It is known Lin and Du (2006) that a $w$-distance is a $\tau$-function.

We propose a weaker notion as follows.

**Definition 3.2** Let $(X, d)$ be a quasi-metric space. A function $p : X \times X \to R_+$ is called a weak $\tau$-function if the conditions $(\tau 1)$, $(\tau 3)$ and $(\tau 4)$ hold.

Recall now the definition of $\tau$-distance.

**Definition 3.3** (Suzuki 2001) Let $(X, d)$ be a quasi-metric space, a function $p : X \times X \to R_+$ is called a $\tau$-distance on $X$ if there is a function $\eta : X \times R_+ \to R_+$ such that the following conditions are satisfied.

$(\tau_1)$ for all $x, y, z \in X$, $p(x, z) \leq p(x, y) + p(y, z)$;

$(\tau_2')$ $\eta(x, 0) = 0$, $\eta(x, t) \geq t$ and $\eta(x, .)$ is concave for all $x \in X$ and $t \in R_+$;

$(\tau_3')$ $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply $p(w, x) \leq \liminf_{n\to\infty} p(w, x_n)$ for all $w \in X$;

$(\tau_4')$ $\lim_{n\to\infty} \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_{n\to\infty} \eta(x_n, t_n) = 0$ imply $\lim_{n\to\infty} \eta(y_n, t_n) = 0$;

$(\tau_5')$ $\lim_{n\to\infty} \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_{n\to\infty} \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = 0$.

It is known that the definitions of a $\tau$-function and a $\tau$-distance are incomparable. We now show that the definition of a weak $\tau$-function is more general than both of $\tau$-function and $\tau$-distance.

We need the following

**Lemma 3.1** *Assume that $X$ is a quasimetric space with quasimetric $d$ and $p$ is a $\tau$-distance on $X$. Then* *(i) $p$ satisfies ($\tau$3);*

*(ii) $p$ satisfies ($\tau$4).*

*Proof* If $d$ is a metric, assertion (i) is Lemma 3 in Suzuki (2001) and (ii) is a part of Lemma 2 in Suzuki 2001. Moreover, the proofs of these lemmas in Suzuki (2001) do not use the symmetry $d$. Hence Lemma 3.1 holds. $\square$

**Lemma 3.2** *Any $\tau$-distance in a quasimetric space is a weak $\tau$-function.*

*Proof* It is clear from Lemma 3.1. $\square$

The following assertion, modified from Lemma 2.1 of Lin and Du (2006), will also be in use later

**Lemma 3.3** *Let $(X, d)$ be a quasimetric space and $p$ be a weak $\tau$-function on $X \times X$. If a sequence $\{x_n\}$ in $X$ with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a Cauchy sequence in $X$.*

Notice that in Lemma 2.1 of Lin and Du (2006) it is assumed that $d$ is a metric and $p$ is a $\tau$-function. However, the symmetry of $d$ and condition ($\tau$2) for $p$ are not used in the proof. (The proof in Lin and Du (2006) is incomplete, since only $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ is checked. However, one can show that $\lim_{n,\, q\to\infty} d(x_n, x_{n+q}) = 0$.)

The following lemma is crucial for our proof of the main results.

**Lemma 3.4** *Let $(X, d)$ be a quasimetric space and $p$ be a weak $\tau$-function on $X$. Let $\Gamma : X \to 2^X$ be a set-valued mapping and $\{x_n\} \subseteq X$ converge to $\overline{x} \in X$ such that the following conditions be satisfied:*

*(i) $x_{n+1} \in \Gamma(x_n)$ and $\Gamma(x_{n+1}) \subseteq \Gamma(x_n), \forall n \in \mathbb{N}$;*

*(ii)* $\lim_{n\to\infty} \sup\{p(x_n, u) : u \in \Gamma(x_n)\} = 0$;

*(iii)* $\overline{x} \in \Gamma(x_n), \forall n \in \mathbb{N}$.

*Then* $\bigcap_{n\in\mathbb{N}} \Gamma(x_n) = \{\overline{x}\}$.

*If, in addition,*

*(iv)* $\Gamma(\overline{x}) \neq \emptyset$ *and* $\Gamma(\overline{x}) \subseteq \Gamma(x_n), \forall n \in \mathbb{N}$,

*then* $\overline{x}$ *is invariant point of* $\Gamma$ *(i.e.* $\Gamma(\overline{x}) = \{\overline{x}\}$*). Conversely, if* $p(x, x) = 0$, $\forall x \in X$*, and* $\overline{x}$ *is an invariant point of* $\Gamma$*, then there is a sequence* $\{x_n\}$ *which converges to* $\overline{x}$ *and satisfies all conditions (i)-(iv).*

*Proof* By (iii), $\overline{x} \in \cap_{n\in\mathbb{N}}\Gamma(x_n)$. If $w \in \cap_{n\in\mathbb{N}}\Gamma(x_n)$, then $\lim_{n\to\infty} p(x_n, w) = 0$ by (ii). Because of (i), $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$. Putting now $y_n \equiv w, \forall n \in \mathbb{N}$, in ($\tau$3) we see that $\lim_{n\to\infty} d(x_n, w) = 0$. By the uniqueness of the limit we obtain that $w = \overline{x}$. If (iv) also holds then

$$\emptyset \neq \Gamma(\overline{x}) \subseteq \cap_{n\in\mathbb{N}}\Gamma(x_n) = \{\overline{x}\},$$

i.e. $\Gamma(\overline{x}) = \{\overline{x}\}$.

To see the "converse part" we take $x_n = \overline{x}$, $\forall n \in \mathbb{N}$. Then (i), (iii) and (iv) are fulfilled clearly. Since $p(\overline{x}, \overline{x}) = 0$, (ii) is also satisfied. □

*Remark 3.1* Lemma 3.4 strictly contains the following result of Daneš et al. (1983), which is applied in Ha (2005).

**Lemma 3.5** (Daneš et al. 1983) *Let* $(X, d)$ *be a complete metric space and* $\Gamma : X \to 2^X$ *satisfy the following conditions*

*(a)* $\forall x \in X$, $\Gamma(x)$ *is closed and* $x \in \Gamma(x)$;

*(b)* $\Gamma(y) \subseteq \Gamma(x), \forall y \in \Gamma(x)$;

*(c)* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, *if* $x_{n+1} \in \Gamma(x_n), \forall n$.

*Then* $\Gamma$ *has an invariant poit* $\overline{x}$.

*Proof* Indeed, we check (i)-(iv) under assumptions (a)-(c). By (c) we can take a sequence $\{x_n\}$ such that $x_n \in \Gamma(x_n)$ with $d(x_n, x_{n+1}) \leq \frac{1}{2^n}$. Then (i) is fulfilled by (b). Put $p(x, y) = d(x, y), \forall x, y \in X$. (ii) is satisfied by (c). We have, $\forall q \in \mathbb{N}$,

$$d(x_n, x_{n+q}) \leq \tfrac{1}{2^n} + ... + \tfrac{1}{2^{n+q-1}} \leq \tfrac{1}{2^{n-1}}$$

and then $x_n$ converges to some $\overline{x} \in X$. To see (iii) suppose $\overline{x} \notin \Gamma(x_{n_0})$ for some $n_0$. Since $\Gamma(x_{n_0})$ is closed there is a ball $B(\overline{x}, r)$ such that $B(\overline{x}, r) \cap \Gamma(x_{n_0}) = \emptyset$. Then by (b) and by the construction of $\{x_n\}$, $x_{n_0+q} \in \Gamma(x_n), \forall q \in \mathbb{N}$, contradicts the fact that $x_n \to \overline{x}$. Finally, (iv) is satisfied by (a) and (iii). □

The following example gives a case all (i)-(iv) of Lemma 3.4 are satisfied but we cannot apply Lemma 3.5.

*Example 3.1* Let $X = R$, $p(x, y) = d(x, y) = |x - y|$ and $\Gamma$ be defined by

$$\Gamma(x) = \begin{cases} [0, x), & \text{if} \quad x > 0, \\ \{0\}, & \text{if} \quad x = 0. \end{cases}$$

Then $x_n = \frac{1}{n}$ satisfies (i)-(iv) and $\overline{x} = 0$ is an invariant point of $\Gamma(0)$ but, for $x > 0$, $x \notin \Gamma(x)$ and $\Gamma(x)$ is not closed. Note that, since $\Gamma(x)$ is not closed, one cannot apply the Cantor theorem, which is often applied when proving variants of the EVP.

The following example show the essentialness of condition (iv).

*Example 3.2* Let $X = R$ and $p(x, y) = d(x, y) = |x - y|$.

(a) Let $\Gamma$ be defined by $\Gamma(x) = [0, x)$ for $x > 0$. Take a sequence $\{x_n\}$ such that $x_{n+1} \in \Gamma(x_n)$ and $d(x_n, x_{n+1}) \leq \frac{1}{2^n}$, then $x_n \to 0$. We see that (i)-(iii) are satisfied but $\Gamma$ does not have any invariant point. The reason is that $\Gamma(0) = \emptyset$.

(b) Let $\Gamma$ be defined by

$$\Gamma(x) = \begin{cases} [0, x), & \text{if} \quad x > 0, \\ \{-1\}, & \text{if} \quad x = 0. \end{cases}$$

We take the same $\{x_n\}$ as in (a) to see that (i)-(iii) are satisfied but $\Gamma$ has no invariant point. The reason in this case is that $\Gamma(0) \not\subseteq \Gamma(x_n), \forall n \in \mathbb{N}$.

## 4 Main Result

From now on, unless specified otherwise, let $(X, d)$ be a complete quasimetric space, $p$ be a weak $\tau$-function on $X$, $Y$ be a Hausdorff locally convex space, $K \subseteq Y$ be a convex cone and $k_0 \in K \setminus -\text{cl}K$. Let $Y^*$ stand for the topological dual of $Y$ and $K^+$ is the positive polar of $K$, i.e.

$$K^+ := \{y^* \in Y^* : < y^*, y > \geq 0, \forall y \in K\}.$$

Take $z^* \in K^+$ such that $z^*(k_0) = 1$ (the existence of such a $z^*$ is seen by using a separation theorem for $k_0$ and $-\text{cl}K$). $K$ determines a relation $\leq_K$ on Y by

$$y_2 \leq_K y_1 \quad \Leftrightarrow \quad y_1 - y_2 \in K.$$

The convexity of $K$ implies the transitive of $\leq_K$. If $0 \in K$ then $\leq_K$ is reflexive and if $K$ is pointed, i.e. $K \cap (-K) = \{0\}$ then $\leq_K$ is antisymmetric. Let $F : X \to 2^Y$ be given. We define a relation $\leq_{k_0}$ on $X$ by

$$x_2 \leq_{k_0} x_1 \quad \Leftrightarrow \quad F(x_1) \subseteq F(x_2) + k_0 p(x_1, x_2) + K. \tag{2}$$

It is easy to see that $\leq_{k_0}$ is transitive. If $0 \in K$ and $p(x, x) = 0, \forall x \in X$, then $\leq_{k_0}$ is reflexive. For $x \in X$ let from now on $S(x) = \{x' \in X : x' \leq_{k_0} x\}$. The first result below is a generalization of the EVP.

**Theorem 4.1** *Let $X$, $Y$, $p$, $K$ and $k_0$ be as specified at the beginning of Section 4. Assume that $F : X \to 2^Y$ has $F(X)$ being $K$-bounded from below and the transitive relation $\leq_{k_0}$ (defined by (2)) is lower closed. Then there exists $v \in X$ such that, $\forall x \neq v$,*

$$F(v) \nsubseteq F(x) + k_0 p(v, x) + K,$$

*i.e. $v$ is a strict minimizer of $F(.) + k_0 p(v, .)$ if $p(x, x) = 0$, $\forall x \in X$.*

*Proof.* Reasoning ab absurdo, suppose for each $x \in X$ the existence of $x^{'} \neq x$ such that $x \leq_{k_0} x'$. Then for each $x \in X$, $S(x)$ is nonempty and different from $\{x\}$.

Let $z^* \in K^+$ be taken so that $z^*(k_0) = 1$. If $x^{'} \in S(x)$ then for any fixed $y \in F(x)$, $\exists y^{'} \in F(x^{'}), \exists k \in K$ such that $y = y^{'} + k_0 p(x, x^{'}) + k$. Since $F(X) \subseteq M + K$ with $M$ being bounded, we have

$$p(x, x^{'}) \leq z^*(y) - z^*(y^{'}) \leq z^*(y) - \inf z^*(M).$$

Thus, $\forall x \in X$,

$$\sup\{p(x, x') : x' \in S(x)\} < +\infty. \tag{3}$$

Fix any $x_0 \in \mathrm{dom}F$ and $y_0 \in F(x_0)$. We construct a sequence $\{(x_n, y_n)\} \subseteq S(x_0) \times Y$ in the following way: having $x_n \in S(x_{n-1})$ and $y_n \in F(x_n)$, by (3) we can choose $x_{n+1} \in S(x_n)$ such that

$$p(x_n, x_{n+1}) \geq \frac{1}{2} \sup \{p(x_n, x) : x \in S(x_n)\}. \tag{4}$$

As $x_{n+1} \in S(x_n)$, there is $y_{n+1} \in F(x_{n+1})$ with

$$y_n \in y_{n+1} + k_0 p(x_n, x_{n+1}) + K. \tag{5}$$

So we obtain a sequence $\{x_n\} \subseteq S(x_0)$ such that $S(x_{n+1}) \subseteq S(x_n)$, $\forall n \in \mathbb{N}$, and

$$... \leq_{k_0} x_n \leq_{k_0} ... \leq_{k_0} x_1 \leq_{k_0} x_0.$$

Suppose the existence of $\delta > 0$ such that, $\forall n \in \mathbb{N}$,

$$\delta \leq \sup\{p(x_n, x) : x \in S(x_n)\}.$$

From (4) and (5) it follows that

$$\frac{1}{2}\delta k_0 \leq_K k_0 p(x_n, x_{n+1}) \leq_K y_n - y_{n+1}$$

and hence

$$\frac{1}{2}\delta \leq z^*(y_n) - z^*(y_{n+1}).$$

Adding these inequalities from 0 to $n-1$ we obtains

$$\frac{1}{2}n\delta \leq z^*(y_0) - z^*(y_n) \leq z^*(y_0) - \inf z^*(M).$$

Passing to the limit as $n \to \infty$ one gets a contradiction. Therefore we have only two cases. The first case is $\sup\{p(x_{n_0}, x) : x \in S(x_{n_0})\} = 0$ for some $n_0 \in \mathbb{N}$. Since $S(x_{n_0}) \neq \emptyset$, this together with $(\tau 4)$ imply that $S(x_{n_0})$ is a singleton, say $\{v\}$. Then $\emptyset \neq S(v) \subseteq S(x_{n_0}) = \{v\}$, i.e. $S(v) = \{v\}$, which is impossible.

The remaining case is

$$\lim_{k\to\infty} \sup\{p(x_{n_k}, x) : x \in S(x_{n_k})\} = 0$$

for a subsequence $\{x_{n_k}\}$. Since $x_{n_m} \in S(x_{n_k})$ for all $m > k$,

$$\lim_{k\to\infty} \sup\{p(x_{n_k}, x_{n_m}) : m > k\} = 0.$$

Lemma 3.3 now implies that $\{x_{n_k}\}$ is a Cauchy sequence and hence converges to some $v \in X$. Since $\leq_{k_0}$ is lower closed, $v \in S(x_{n_k})$, $\forall k \in \mathbb{N}$. Now that all the assumptions of Lemma 3.4 for $S$ are satisfied, one has $S(v) = \{v\}$, a contradiction. □

Some sufficient conditions for $\leq_{k_0}$ to be lower closed, as needed in Theorem 4.1, are collected in the following

**Proposition 4.2** *Let* $X$, $Y$, $p$, $K$, $k_0$ *be as in Theorem 4.1.*

*(i) If* $S(x)$ *is closed for each* $x \in X$, *then* $\leq_{k_0}$ *is lower closed.*

*(ii) If* $K$ *is closed,* $F : X \to 2^Y$ *is* $K$*-lsca and* $K$*-closed valued and* $p$ *satisfies* $(\tau 2)$*, then* $\leq_{k_0}$ *is lower closed.*

*(iii) If* $K$ *is closed,* $F : X \to 2^Y$ *is* $K$*-lsc and* $K$*-closed valued and* $p$ *satisfies* $(\tau 2)$*, then* $S(x)$ *is closed for all* $x \in X$ *and hence* $\leq_{k_0}$ *is lower closed.*

*Proof* (i) It follows from Remark 2.1(i).

(ii) Let $x_{n+1} \leq_{k_0} x_n, \forall n \in \mathbb{N}$, and $x_n \to \overline{x}$. Fix $n$. For $i \in \mathbb{N}$, by $(\tau 2)$ we have $Q(i) \in \mathbb{N}$ such that, $\forall q > Q(i)$,

$$p(x_n, x_{n+q}) \geq p(x_n, \overline{x}) - \frac{1}{i}\,. \tag{6}$$

Indeed, if $\forall Q \in N, \exists q > Q$ such that

$$p(x_n, x_{n+q}) < p(x_n, \overline{x}) - \frac{1}{i}\,,$$

then $(\tau 2)$ implies that

$$p(x_n, \overline{x}) \leq p(x_n, \overline{x}) - \frac{1}{i}\,,$$

a contradiction. By virtue of (6) one has

$$F(x_n) - (p(x_n, \overline{x}) - \frac{1}{i})k_0 \subseteq F(x_n) - p(x_n, z_{n+q})k_0 + K.$$

As $x_{n+q} \leq_{k_0} x_n$, one gets also

$$F(x_n) - p(x_n, x_{n+q})k_0 + K \subseteq F(x_{n+q}) + K.$$

Hence,

$$F(x_n) - (p(x_n, \overline{x}) - \frac{1}{i})k_0 \subseteq F(x_{n+q}) + K.$$

Since $F$ is $K$-lsca and $x_{n+q}$ is $\leq_{k_0}$-monotone and converges to $\overline{x}$ as $q \to \infty$ one has $F(x_{n+q}) \subseteq F(\overline{x}) + K, \forall q \in \mathbb{N}$. Thus, $\forall i \in \mathbb{N}$,

$$F(x_n) - (p(x_n, \overline{x}) - \frac{1}{i})k_0 \subset F(\overline{x}) + K,$$

i.e.

$$F(x_n) + \frac{1}{i}k_0 \subseteq F(\overline{x}) + p(x_n, \overline{x})k_0 + K.$$

As $F(\overline{x}) + p(x_n, \overline{x})k_0 + K$ is closed, this leads to $\overline{x} \leq_{k_0} x_n$, $\forall n \in \mathbb{N}$.

(iii) Assume that, for a fixed $x \in X$, $\{x_n\} \subseteq S(x)$ and $x_n \to \overline{x}$. We have to show that $\overline{x} \in S(x)$. Similarly as proving (6), for each $i \in \mathbb{N}$, we have $Q(i) \in \mathbb{N}$ such that, $\forall n > Q(i)$,

$$p(x, x_n) \geq p(x, \overline{x}) - \frac{1}{i}\,.$$

Since $x_n \in S(x)$, we have, $\forall n > Q(i)$,

$$F(x) - (p(x, \overline{x}) - \frac{1}{i})k_0 \subseteq F(x) - p(x, x_n)k_0 + K \subseteq F(x_n) + K.$$

As $x_n \to \overline{x}$ and $F$ is $K$-lsc, we have further, $\forall i \in \mathbb{N}$,

$$F(x) - (p(x, \overline{x}) - \frac{1}{i})k_0 \subset F(\overline{x}) + K,$$

i.e.

$$F(x) + \frac{1}{i}k_0 \subseteq F(\overline{x}) + p(x, \overline{x})k_0 + K.$$

By the assumption about closedness, we obtain in the limit as $i \to \infty$

$$F(x) \subseteq F(\overline{x}) + p(x, \overline{x})k_0 + K,$$

i.e. $\overline{x} \in S(x)$. □

**Theorem 4.3** *Assume, in addition to the assumptions of Theorem 4.1, that* $x_0 \in domF$, $S(x_0) \neq \emptyset$ *and* $S(x_0)$ *is* $\leq_{k_0}$*-complete. Then there exists* $v \in S(x_0)$ *such that,* $\forall x \neq v$,

$$F(v) \nsubseteq F(x) + k_0 p(v, x) + K, \tag{7}$$

*i.e.* $v$ *is a strict minimizer of* $F(.) + k_0 p(v, .)$ *if* $p(x, x) = 0$, $\forall x \in X$.

*Proof* For an arbitrary $y_0 \in F(x_0)$, starting from $(x_0, y_0)$ we construct a sequence $\{(x_n, y_n)\} \subseteq S(x_0) \times Y$ in the following way: having $x_n \in S(x_{n-1})$ and $y_n \in F(x_n)$ we choose $x_{n+1} \in S(x_n)$ as in the proof of Theorem 4.1. If there is $n_0$ such that $S(x_{n_0}) = \emptyset$, then $v = x_{n_0}$ satisfies (7). Otherwise, $S(x_n) \neq \emptyset, \forall n \in \mathbb{N}$. As in the proof of Theorem 4.1, by the completeness of $S(x_0)$, we always arrive at a point $v \in S(x_0)$ such that $S(v) = \{v\}$. Hence $\forall x \neq v$, $x \notin S(v)$, i.e. (7) holds. □

Traditionally, the statement of the EVP, say for a scalar function $f$ in a metric space, includes an $\varepsilon > 0$ such that $f(x_0) < \inf_{x \in \mathrm{dom} F} f(x) + \varepsilon$ and an estimate of $d(x_0, v)$. We can get a corresponding statement for our case by modifying Theorem 4.3 as follows.

**Theorem 4.4** *Assume, additionally to the assumptions of Theorem 4.3 that* $F(x_0) \nsubseteq F(x) + \varepsilon k_0 + K$, *for some* $\varepsilon > 0$ *and all* $x \in X$. *Then,* $\forall \lambda > 0$, $\exists v \in X$ *such that,* $\forall x \neq v$,

*(i)* $p(x_0, v) \leq \lambda$;

*(ii)* $F(x_0) \subseteq F(v) + \frac{\epsilon}{\lambda} p(x_0, v) k_0 + K$;

*(iii)* $F(v) \nsubseteq F(x) + \frac{\epsilon}{\lambda} p(x, v) k_0 + K$.

*Proof* By replacing $k_0$ by $\varepsilon k_0$ and $p$ by $\dfrac{1}{\lambda}p$ in the proof of Theorem 4.1, this theorem yields $v \in X$ such that (ii) and (iii) hold. We claim that $p(x_0, v) \leq \lambda$. Indeed, otherwise, with $p(x_0, v) > \lambda$ we would have

$$F(x_0) \subseteq F(v) + \frac{\epsilon}{\lambda}p(x_0, v)k_0 + K \subseteq F(v) + \varepsilon k_0 + K.$$

which contradicts the property of $x_0$. □

*Remark 4.1*

(i) If $0 \in K$ and $p(x_0, x_0) = 0$, then condition $S(x_0) \neq \emptyset$ is satisfied, since $x_0 \in S(x_0)$.

(ii) By the $K$-boundedness from below of $F(X)$, $\forall \varepsilon > 0$, $\exists x_0 \in X$,

$$F(x_0) \nsubseteq F(x) + \varepsilon k_0 + K,$$

see Ha (2005), Proposition 3.1.

(iii) Since minimizers and Pareto minimizers are incomparable (see Examples 2.1 and 2.2) we see no direct comparison between Theorems 4.1, 4.3 and 4.4 with the results for Pareto minimizers. We observe only paper Ha (2005) which deals with minimizers. For the special case where $p = d$, a (complete) metric, Theorem 4.4 strictly contains Theorem 3.1, the main result of Ha (2005), by Proposition 4.2 (iii) and Example 2.3, since in Ha (2005) $F$ is assumed to be $K$-lsc instead of our assumption about lower closedness of $\leq_{k_0}$.

(iv) Several authors (see e.g. Park 2000, Oetli and Théra 1993, Bianchi et al. 2007) consider mapping $\Phi$ defined on $X \times X$ (of two variables) with the property (written here for the scalar case) $\Phi(x, z) \leq \Phi(x, y) + \Phi(y, z)$

for any $x, y, z \in X$ and argue that by setting $\Phi(x, y) = F(y) - F(x)$ this consideration contains the usual study of mapping of one variable. However, with the above-mentioned property, putting $F(.) = \Phi(x_0, .)$ we can derive the statements for $\Phi$ from the theorems for $F$, compare [39] and Section 5. So there is no need of considering such $\Phi$.

The following theorem collects some equivalent formulations of the EVP.

**Theorem 4.5** *Under the same assumptions of Theorem 4.3, the following assertions, which are equivalent to Theorem 4.3, hold.*

*(i)* *There exists* $v \in S(x_0)$ *such that* $S(v) = \emptyset$ *or* $S(v) = \{v\}$.

*(ii)* *There exists* $v \in S(x_0)$ *such that,* $\forall x \in X \setminus \{v\}$,

$$F(v) \not\subseteq F(x) + k_0 p(v, x) + K.$$

*(iii)* *Let* $I$ *be an index set. For* $i \in I$, *let* $T_i$: $S(x_0) \to 2^X$ *be a multivalued map with nonempty values such that, for each* $x \in S(x_0)$ *with* $x \notin T_i(x)$, *there exists* $y = y(x, i) \in S(x_0)$ *with* $y \neq x$ *such that* $y \leq_{k_0} x$. *Then* $\{T_i\}_{i \in I}$ *has a common fixed point in* $S(x_0)$.

*(iv)* *Let* $I$ *be an index set. For* $i \in I$, *let* $T_i$: $S(x_0) \to 2^X$ *be a multivalued map with nonempty values such that, for each* $x \in S(x_0)$, *there exists* $y = y(x, i) \in T_i(x)$ *with* $y \leq_{k_0} x$. *Then* $\{T_i\}_{i \in I}$ *has a common fixed point in* $S(x_0)$.

*(v)* *Let* $I$ *be an index set. For* $i \in I$, *let* $T_i$: $S(x_0) \to X$ *be a single-valued map such that* $T_i(x) \leq_{k_0} x$ *for all* $x \in S(x_0)$. *Then* $\{T_i\}_{i \in I}$ *has a common fixed point in* $S(x_0)$.

*(vi) Let $I$ be an index set. For $i \in I$, let $T_i$: $S(x_0) \to 2^X$ be a multivalued map with nonempty values such that, for each $x \in S(x_0)$, $y \leq_{k_0} x$ for all $y \in T_i(x)$. Then $\{T_i\}_{i\in I}$ has a common stationary point $\overline{x}$ in $S(x_0)$; that is, $T_i(\overline{x}) = \{\overline{x}\}$ for each $i \in I$.*

*(vii) Let $I$ be an index set. For $i \in I$, let $T_i$: $S(x_0) \to 2^X$ be a multivalued map. Suppose that, for each $(x, i) \in S(x_0) \times I$ with $T_i(x) \neq \emptyset$, there exists $y = y(x, i) \in S(x_0)$ with $y \neq x$ such that $y \leq_{k_0} x$. Then there exists $\overline{x} \in S(x_0)$ such that $T_i(\overline{x}) = \emptyset$ for all $i \in I$.*

*Proof* "(i)⇔ (ii)" It is obvious.

"(i)⇒ (iii)" By (i), there exists $v \in S(x_0)$ such that $S(v) = \emptyset$ or $S(v) = \{v\}$. Suppose $v \notin T_{i_0}(v)$ for some $i_0 \in I$, then, by hypothesis, there exists $w = w(v, i_0) \in S(x_0)$ with $w \neq v$ such that $w \leq_{k_0} v$, which leads to a contradiction. Hence $v$ is a common fixed point of $\{T_i\}_{i\in I}$.

"(iii)⇒ (i)" Suppose, for each $x \in S(x_0)$ $S(x) \neq \emptyset$ and $S(x) \neq \{x\}$. Therefore, for each $x \in S(x_0)$, there exists $y \in X$ with $y \neq x$ such that $y \leq_{k_0} x$. Then for each $x \in S(x_0)$, we can define a multivalued map $T : S(x_0) \to 2^X \setminus \{\emptyset\}$ by

$$T(x) = \{y \in X : y \neq x, y \leq_{k_0} x\}.$$

Clearly, $x \notin T(x)$ for all $x \in X$, contradicting (iii).

"(i)⇒ (iv)" By (i), there exists $v \in S(x_0)$ such that $S(v) = \emptyset$ or $S(v) = \{v\}$. Suppose that $v \notin T_{i_0}(v)$ for some $i_0 \in I$, then, by hypothesis, there exists $w = w(v, i_0) \in T_{i_0}(v)$ such that $w \leq_{k_0} v$. Since $w(v, i_0) \in T_{i_0}(v)$ and $v \notin T_{i_0}(v)$ then $w(v, i_0) \neq v$, which leads to a contradiction.

"(iv)⇒ (v)" It is clear.

"(v)$\Rightarrow$ (i)" Suppose, for each $x \in S(x_0)$, $S(x) \neq \emptyset$ and $S(v) \neq \{v\}$. Therefore, for each $x \in S(x_0)$, there exists $y \in X$ with $y \neq x$ such that $y \leq_{k_0} x$. Choose $T(x)$ to be one of such $y$. Then $T : S(x_0) \to X$ has no fixed point by its definition and $T(x) \leq_{k_0} x$ for all $x \in S(x_0)$, contradicting (v).

"(i)$\Rightarrow$ (vi)" By (i), there exists $v \in S(x_0)$ such that $S(v) = \emptyset$ or $S(v) = \{v\}$. By the hypothesis of (vi), $w \leq_{k_0} v$ for all $w \in T_i(v)$ for all $i \in I$. Then $\emptyset \neq T_i(v) \in S(v)$. Hence $S(v) = \{v\}$. Therefore, we have $T_i(v) = \{v\}$ for all $i \in I$.

"(vi)$\Rightarrow$ $(i)$" The proof is similar to "(iii)$\Rightarrow$ (i)"

"(i)$\Rightarrow$ (vii)" By (i), there exists $v \in S(x_0)$ such that $S(v) = \emptyset$ or $S(v) = \{v\}$. Suppose to the contrary that $T_{i_0}(v) \neq \emptyset$ for some $i_0 \in I$. Then, by the hypothesis of (vii), there exists $w = w(v, i_0) \in S(x_0)$ with $w \neq v$ such that $w \leq_{k_0} v$, a contradiction. Hence $T_i(v) = \emptyset$ for all $i \in I$.

"(vii)$\Rightarrow$ (i)" Suppose, for each $x \in S(x_0)$, $S(x) \neq \emptyset$ and $S(v) \neq \{v\}$. Therefore, for each $x \in S(x_0)$, there exists $y \in X$ with $y \neq x$ such that $y \leq_{k_0} x$. Then for each $x \in S(x_0)$, we can define a multivalued map $T : S(x_0) \to 2^X$ by

$$T(x) = \{y \in X : \, y \neq x, y \leq_{k_0} x\}.$$

Clearly, $T(x) \neq \emptyset$ for all $x \in X$. This constricts (vii). Thus, (i) holds. $\square$

**Theorem 4.6** *Under the same assumptions of Theorem 4.3, the following also holds:*

*(viii) if* $\forall x \in S(x_0)$, $\forall y \in F(x)$ *with* $z^*(y) > \inf z^*(F(S(x_0)))$, *there exists* $w \in S(x_0)$ *with* $w \neq x$ *such that* $w \leq_{k_0} x$, *then there exist* $\overline{x} \in S(x_0)$ *and* $y_{\overline{x}} \in F(\overline{x})$ *such that* $z^*(y_{\overline{x}}) = \inf z^*(F(S(x_0)))$.

*In fact, any of (i)-(vii) implies (viii). Conversely, (viii) implies any of (i)-(vii) if* $p(x, y) = 0$ *implies* $x = y$.

*Proof* "(i)⇒(viii)" By (i), there exists $v \in S(x_0)$ such that $S(v) = \emptyset$ or $S(v) = \{v\}$. We will show that there is $y_v \in F(v)$ such that $z^*(y_v) = \inf z^*(F(S(x_0)))$. Suppose to the contrary that, $\forall y \in F(v)$, $z^*(y) > \inf z^*(F(S(x_0)))$. Then, by the hypothesis of (viii), there exists $w \in S(x_0)$ with $w \neq v$ such that $w \leq_{k_0} v$. Then it leads to a contradiction and (viii) holds.

"(viii)⇒(i)" Suppose that, for each $x \in S(x_0)$, $S(x) \neq \emptyset$ and $S(v) \neq \{v\}$. For each $x \in S(x_0)$, there exists then $w \in X$ with $w \neq x$ such that $w \leq_{k_0} x$. Hence, by (viii), there exist $a \in S(x_0)$ and $y_a \in F(a)$ such that $z^*(y_a) = \inf z^*(F(S(x_0)))$. By our hypothesis, there exists $b \in S(x_0)$ with $b \neq a$ such that $b \leq_{k_0} a$. Consequently, $F(a) \subseteq F(b) + k_0 p(a, b) + K$. Then $y_b \in F(b)$ exists such that

$$y_a \in y_b + k_0 p(a, b) + K.$$

Therefore,

$$p(a, b) \leq z^*(y_a) - z^*(y_b) \leq 0.$$

Hence $p(a, b) = 0$, which leads to a contradiction. □

*Remark 4.2* Consider the special case where $F$ is a single-valued scalar function. Then Theorems 4.5 and 4.6 contain properly Theorems 1, 1' and 2 of Park (2000) due to Remark 4.1(iv) and the fact that any $w$-distance is a weak $\tau$-function but not vice versa.

## 5 Some corollaries for the single-valued case

In this section we will discuss some corollaries of our main results for the single-valued (vector) case. Note that in this special case the minimizer definition coincides with that of the Pareto minimizer. To have a generalization similar to the extended real line $R \cup \{+\infty\}$, we extend $Y$ by an additional element, denoted also by $+\infty$, with the usual rules for addition and multiplication with reals. We avoid indeterminate expressions like $0.(+\infty)$ and adopt that $y \leq_K +\infty, \forall y \in Y$. Now we consider a mapping $f : X \to Y \cup \{+\infty\}$ and denote $\mathrm{dom} f := \{x \in X : f(x) \neq +\infty\}$. We say that $f$ is proper if $\mathrm{dom} F \neq \emptyset$. Note that the relation $\leq_{k_0}$ defined by (2) now becomes

$$x_2 \leq_{k_0} x_1 \quad \Leftrightarrow \quad f(x_2) + k_0 p(x_1, x_2) \leq_K f(x_1)$$

The following corollary is a direct consequence of Theorem 4.3, by Proposition 4.2(i) and Remark 4.1(i).

**Corollary 5.1** *Let $X$, $Y$, $p$, $K$ and $k_0$ be as specified in Section 4. Assume that $f : X \to Y \cup \{+\infty\}$ be proper and $K$-bounded from below. Let $S(x)$:=$\{x' \in X : f(x') + k_0 p(x, x') \leq_K f(x)\}$ be closed for every $x \in X$. Then for every $x_0 \in \mathrm{dom} f$ there exists $v \in X$ such that, $\forall x \neq v$,*

$$f(v) + k_0 p(x_0, v) \leq_K f(x_0), \tag{8}$$

$$f(x) + k_0 p(v, x) \not\leq_K f(v). \tag{9}$$

This corollary properly includes Corollary 2 of Göpfert et al. (2000), since $p \neq d$ and $f$ may not be bounded from below.

**Corollary 5.2** *Let $X$, $Y$, $p$, $K$, $k_0$ and $f$ be as in Corollary 5.1. Assume that if $x_n \in \mathrm{dom} f, x_n \to x$ and $f(x_n)$ is $\leq_K$ decreasing, then $f(x) \leq_K f(x_n), \forall n \in \mathbb{N}$. Assume that $0 \in K$ and $K$ is closed in the direction $k_0$, i.e. $K \cap (y - R_+ k_0)$ is closed for all $y \in Y$. Assume further that $x_0 \in \mathrm{dom} f$ and $p$ is a $\tau$-function. Then there exists $v \in X$ such that, $\forall x \neq v$, (8) and (9) hold.*

*Proof* By Remark 4.1(i), $S(x_0) \neq \emptyset$. To apply Theorem 4.3 we show the $\leq_{k_0}$-

completeness of $S(x_0)$. If $\{x_n\} \subseteq S(x_0)$ is $\leq_{k_0}$ decreasing and Cauchy then $x_n \to x$, for some $x \in X$, and $f(x_n)$ is clear $\leq_K$ decreasing, and hence $f(x) \leq_K f(x_n), \forall n \in \mathbb{N}$.

Now, fix $n$. For $i \in \mathbb{N}$, by $(\tau 2)$ there exists $Q(i) \in \mathbb{N}$ such that, $\forall q > Q(i)$,

$$p(x_n, x_{n+q}) \geq p(x_n, x) - \frac{1}{i}\,.$$

Consequently,

$$\begin{aligned} f(x) + k_0 p(x_n, x) &\leq_K f(x) + k_0(p(x_n, x_{n+q}) + \tfrac{1}{i}) \\ &\leq_K f(x_{n+q}) + k_0 p(x_n, x_{n+q}) + \tfrac{1}{i} k_0 \\ &\leq_K f(x_n) + \tfrac{1}{i} k_0. \end{aligned}$$

Passing $i \to \infty$, by the closedness of K in the direction $k_0$, we obtain that $f(x) + k_0 p(x_n, x) \leq_K f(x_n)$, i.e. $x \leq_{k_0} x_n, \forall n \in \mathbb{N}$. Hence $x \leq_{k_0} x_0$, i.e. $x \in S(x_0)$ and $S(x_0)$ is $\leq_{k_0}$ complete. Finally, (8) and (9) follow directly from the conclusion of Theorem 4.3. □

If $f$ is not only $K$-bounded from below but also bounded from below and $p = d$, this corollary coincides with Corollary 3 of Göpfert et al. (2000).

**Corollary 5.3** *Let $X$, $Y$, $K$ and $k_0$ be as in Corollary 5.1, with the additional closedness of $K$. Let $p$ be a $\tau$-function. Let $\Phi\colon X \times X \to Y \cup \{+\infty\}$ satisfy the assumptions*

*(i) there is $x_0 \in X$ such that $\Phi(x_0, x_0) = 0$ and $\Phi(x_0, .)$ is $K$-lsca and $K$-bounded from below;*

*(ii) if $\Phi(x, z) \in -K$ and $\Phi(z, y) \in -K$, then*

$$\Phi(x, y) \leq \Phi(x, z) + \Phi(z, y).$$

*Then there exists* $v \in X$ *such that,* $\forall x \neq v$,

(a) $\Phi(x_0, v) \in -K$, *if* $p(x_0, x_0) = 0$.

(b) $\Phi(v, x) + k_0 p(v, x) \notin -K$.

*Proof* Set $f(.) = \Phi(x_0, .)$. Then the assumptions of Proposition 4.2(ii) are clearly satisfied. Hence, by Theorem 4.1, there exists $v \in X$ such that, $\forall x \neq v$,

$$f(x) + k_0 p(v, x) \notin f(v) - K, \tag{10}$$

$$f(v) + k_0 p(x_0, v) \in f(x_0) - K. \tag{11}$$

As $f(x_0) = \Phi(x_0, x_0) = 0$, (11) implies that $\Phi(x_0, v) \in -K$. For any $x \in X \setminus \{v\}$, if $\Phi(v, x) \notin -K$ then (b) is fulfilled. If $\Phi(v, x) \in -K$, (10) implies that

$$\Phi(x_0, x) - \Phi(x_0, v) + k_0 p(v, x) \notin -K,$$

and hence

$$\Phi(v, x) + k_0 p(v, x) \notin -K,$$

since $\Phi(x_0, x) - \Phi(x_0, v) \in \Phi(v, x) - K$ by (ii). □

**Corollary 5.4** (Lin and Du 2006, Theorem 2.1) *Let* $X$ *be a complete quasimetric space and* $p$ *be a* $\tau$*-function. Let* $f\colon X \to R \cup \{+\infty\}$ *be a proper,* $R_+$*-lsca and bounded from below. Let* $\varphi\colon R \to (0, \infty)$ *be nondecreasing. Then there exists* $v \in X$ *such that,* $\forall x \neq v$,

$$p(v, x) > \varphi(f(v))(f(v) - f(x)).$$

*Proof* Setting $\Phi(x, y) = \varphi(f(x))(f(y) - f(x))$. We see that, $\forall x \in X$, $\Phi(x, .)$ is proper, $R_+$-lsca, bounded from below and $\Phi(x, x) = 0$. We claim that $\Phi$ satisfies (ii) of Corollary 5.3. Indeed, if $\Phi(x, z) \leq 0$ and $\Phi(z, y) \leq 0$ then

$$f(y) \leq f(z) \leq f(x).$$

Hence, as $\varphi(f(z)) \le \varphi(f(x))$

$$\begin{aligned}\Phi(x,z)+\Phi(z,y) &\ge \varphi(f(x))(f(z)-f(x))+\varphi(f(x))(f(y)-f(z))\\ &= \Phi(x,y).\end{aligned}$$

Now applying Corollary 5.3 with $k_0 = 1$ one obtains $v \in X$ such that, $\forall x \neq v$,

$$\Phi(v,x)+p(v,x) > 0.$$

Therefore

$$p(v,x) > \varphi(f(v))(f(v)-f(x)). \qquad \square$$

**Corollary 5.5** *Let $X$ be a complete quasimetric space and $p$ be a $\tau$-function. Let $f\colon X \to R\cup\{+\infty\}$ be proper, $R_+$-lsca and bounded from below. Let $\varphi\colon R \to (0,\infty)$ be nondecreasing. Let $\varepsilon > 0$ and $x_0 \in X$ satisfy $f(x_0) \le \inf_{x\in X} f(x)+\varepsilon$. Then $v \in X$ exists such that, $\forall x \neq v$,*

*(i)* $0 \le f(x_0) - f(v) \le \varepsilon$, *if* $p(x_0,x_0) = 0$;

*(ii)* $\varepsilon p(v,x) > \varphi(f(v))(f(v)-f(x))$.

*Proof* Setting $\Phi(x,y) = \varphi(f(x))(f(y)-f(x))$. By Corollary 5.4 we have (ii) (taking $\varepsilon p$ for $p$ in Corollary 5.4) and $\Phi(x_0,v) \le 0$. Hence, $f(x_0)-f(v) \ge 0$. Since $f(x_0) \le \inf_{x\in X} f(x)+\varepsilon \le f(v)+\varepsilon$, we obtain (i). $\square$

For the special case, where $X$ is a metric space and $p$ is a $w$-distance, Corollary 5.5 coincides with Theorem 2.4 of Lin and Du (2007).

**Corollary 5.6** *Let $X$, $Y$, $K$, $p$ and $k_0$ be as in Corollary 5.3. Let $x_0 \in X$, $\varepsilon > 0$ and $\Phi\colon X\times X \to Y$ satisfy the conditions*

*(i)* $\Phi(x_0,x_0) = 0$ *and* $z^*(\Phi(x_0,.))$ *is bounded from below, where* $z^* \in K^+$ *such that* $z^*(k_0) = 1$;

*(ii)* $\Phi(x_0, .)$ *is* $K$*-lsca;*

*(iii) for any* $x, y, z \in X$, $\Phi(x, y) \leq \Phi(x, z) + \Phi(z, y)$.

*Then there exists* $v \in X$ *such that,* $\forall x \neq v$,

*(a)* $\Phi(x_0, v) + \varepsilon k_0 p(x_0, v) \in -K$, *if* $p(x_0, x_0) = 0$*;*

*(b)* $\Phi(v, x) + \varepsilon k_0 p(v, x) \notin -K$.

*Proof* Without loss of generality assume $\varepsilon = 1$ (by regarding $\varepsilon p$ as a new $\tau$-function). Setting $f(.) = \Phi(x_0, .)$ we see that $f(x_0) = 0$, $f(.)$ is $K$-lsca and $K$-closed valued. By Proposition 4.2(ii), the relation $\leq_{k_0}$ is lower closed. Examining the proof of Theorem 4.1 we see that we can replace the assumed $K$-boundedness of $F(X)$ by a weaker assumption that $z^*(\Phi(x_0, .))$ is bounded from below (by using $\inf z^*(F(X))$ instead of $\inf z^*(M)$). In view of Theorem 4.1 there exists $v \in X$ such that, $\forall x \neq v$,

$$f(x) + k_0 p(v, x) \notin f(v) - K.$$

Consequently, by (iii), we obtain (b). Conclusion (a) is obvious from the proof of Theorem 4.1 if we use $x_0$ given in the assumptions of Corollary 5.6 to start the construction of $\{x_n\}$. □

Corollary 5.7 contains properly Theorem 1 of Bianchi et al. (2007), since in this theorem $X$ is a complete metric space, $p = d$, (i) is required to be fulfill for all $x \in X$ (instead of for $x_0$) and (ii) is replaced by $\Phi(x, .)$ is $K$-lsc for all $x \in X$.

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