Continuity of the solution map to parametric quasiequilibrium problems*

Lam Quoc Anh[1] and Phan Quoc Khanh[2]

[1]Department of Mathematics, Teacher College, Cantho University, Cantho, Vietnam
quocanh@ctu.edu.vn

[2]Department of Mathematics, International University of Hochiminh City,
Khu pho 6, Linh Trung, Thu Duc, Hochiminh City, Vietnam
pqkhanh@hcmiu.edu.vn

Abstract. In this article we consider a parametric vector quasiequilibrium problem in topological vector spaces. We establish sufficient conditions for the solution map to be lower and Hausdorff lower semicontinuous, upper semicontinuous and continuous. Our results improve recent existing ones in the literature.

Key Words. Parametric vector quasiequilibrium problems · Solution map · Lower semicontinuity · Hausdorff lower semicontinuity · Upper semicontinuity · Continuity · Generalized concavity · Generalized pseudomonotonicity

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1. Introduction

Throughout the paper, unless otherwise specified, let X, Y and Λ be Hausdorff topological vector spaces. Let $A \subseteq X$ be nonempty and $\Gamma \subset Y$ be a closed subset of Y with $\mathrm{int}\Gamma \neq \emptyset$ and $\Gamma \neq Y$. The problem under our investigation is as follows. Let $K : A \times \Lambda \to 2^A$ be a multifunction with nonempty convex values and $f : A \times A \times \Lambda \to Y$ be a function. For each parameter $\lambda \in \Lambda$ consider the following quasiequilibrium problem

(QEP) Find $\bar{x} \in K(\bar{x}, \lambda)$ such that, $\forall y \in K(\bar{x}, \lambda)$,

$$f(\bar{x}, y, \lambda) \in \Gamma.$$

If C is closed convex cone. Setting $\Gamma = Y \setminus -\mathrm{int}C$, then (QEP) becomes a parametric quasiequilibrium problem studied by many authors (see [5, 6]). Setting $\Gamma = C$, our problem becomes another problem investigated in e.g. [2, 5].

Let $h : X \to Y$ and $\theta \in Y$. We denote the $\theta.\Gamma$-level sets as follows

$$\mathrm{lev}_{\theta.\Gamma} = \{x \in X \mid h(x) \in \theta + \Gamma\}.$$

Definition 1.1. Let $g : X \times X \to Y$ be a function.

(i) g is called Γ-quasimonotone in $A \subseteq X$ if, $\forall x, y \in A : x \neq y$,

$$[g(x, y) \in \mathrm{int}\Gamma] \Longrightarrow [g(y, x) \notin \mathrm{int}\Gamma].$$

(ii) g is termed Γ-pseudomonotone in $A \subseteq X$ if, $\forall x, y \in A : x \neq y$,

$$[g(x, y) \in \Gamma] \Longrightarrow [g(y, x) \notin \mathrm{int}\Gamma].$$

Definition 1.2. Let X be a normed space, $A \subseteq X$ be nonempty, $b : A \to X^*$ be a mapping. The following terminology may be considered a special case of Definition 1.1.

(a) b is said to be quasimonotone in A if, $\forall x, y \in A$,

$$[\langle b(x), y - x\rangle > 0] \Longrightarrow [\langle b(y), x - y\rangle \leq 0].$$

(b) b is said to be pseudomonotone in A if, $\forall x, y \in A$,

$$[\langle b(x), y - x\rangle \geq 0] \Longrightarrow [\langle b(y), x - y\rangle \leq 0].$$

Definition 1.3. Let $g : X \to Y$ is called generalized Γ-concave in a convex set $A \subseteq X$ if, $\forall x, y \in X$, from $g(x) \in \Gamma$, $g(y) \in \mathrm{int}\Gamma$, it follows that, $\forall t \in (0, 1)$,

$$g((1-t)x + ty) \in \mathrm{int}\Gamma.$$

2. Upper semicontinuity of the solution set

In the sequel let, for $\lambda \in \Lambda$,

$$E(\lambda) = \big\{x \in A \mid x \in K(x, \lambda)\big\}$$

and $S(\lambda)$ be the solution set of problem (QEP) corresponding to λ. Since the solution existence of (QEP) has been intensively studied in the literature, we focus on the stability study, assuming always that $S(\lambda) \neq \emptyset$.

Theorem 2.1. *For problem* (QEP) *assume that*

(i) *E is usc at λ_0, $E(\lambda_0)$ is compact and K is lsc in $A \times \mathit{\Lambda}$;*

(ii) $\mathrm{lev}_{0.\Gamma} f(.,.,\lambda_0)$ *is closed in $K(A, \mathit{\Lambda}) \times K(A, \mathit{\Lambda})$;*

(iii) *$\forall x, y \in K(A, \mathit{\Lambda})$, $f(x, y, .)$ is $Y \setminus \Gamma$- usc at λ_0, uniformly with respect to $x, y \in K(A, \mathit{\Lambda})$ i.e., if $f(x, y, \lambda_0) \in Y \setminus \Gamma$, there is a neighborhood N of λ_0, (N do not depend on x, y), such that*

$$f(x, y, N) \subseteq Y \setminus \Gamma.$$

Then the solution map S is usc at λ_0.

Proof. Suppose that S is not usc at λ_0, i.e. there is an open superset U of $S(\lambda_0)$ such that there are nets $\lambda_\alpha \to \lambda_0$ and $x_\alpha \in S(\lambda_\alpha)$, $x_\alpha \notin U, \forall \alpha$. By the upper semicontinuity of E and the compactness of $E(\lambda_0)$ one can assume that $x_\alpha \to x_0$, for some $x_0 \in E(\lambda_0)$. If $x_0 \notin S(\lambda_0)$, there is $y_0 \in K(x_0, \lambda_0)$ such that $f(x_0, y_0, \lambda_0) \in Y \setminus \Gamma$. The lower semicontinuity of K in turn shows the existence of $y_\alpha \in K(x_\alpha, \lambda_\alpha)$ such that $y_\alpha \to y_0$. Since $\mathrm{lev}_{0.\Gamma} f(.,.,\lambda_0)$ is closed, one can assume that

$$f(x_\alpha, y_\alpha, \lambda_0) \in Y \setminus \Gamma.$$

Since $f(x, y, .)$ is $Y \setminus \Gamma$-usc at λ_0, there is neighborhood N of λ_0 such that

$$f(x_\alpha, y_\alpha, N) \subseteq Y \setminus \Gamma,$$

which is impossible as $x_\alpha \in S(\lambda_\alpha), \forall \alpha$. Thus, $x_0 \in S(\lambda_0) \subseteq U$, which is again a contradiction, since $x_\alpha \notin U$, $\forall \alpha$. □

Remark 2.1. When $K(x, \lambda) \equiv K$, the closedness assumption (ii) for $f(.,.,\lambda_0)$ can be relaxed to that for $f(., y, \lambda_0), \forall y \in K(A, \Lambda)$ and the uniformity with respect to $x, y \in K(A, \Lambda)$ in (iii) can be weakened to the uniformity with respect to $x \in K$. Therefore, Theorem 2.1 improves Theorem 3.1 in Bianchi and Pini (2003) and Theorem 2.1 in Bianchi and Pini (2006), since our assumptions are required only in $K(A, \Lambda)$ (not globally in A as in the mentioned theorems) and our semicontinuity assumption in (iii) is weaker than the corresponding in these theorems.

Assumption (iii) in Theorem 2.1 is essential as shown by the following example.

Example 2.1. Let $X = A = Y = l_2, \varLambda = [0,1], \Gamma = \{x \in l_2 \mid x_k \geq 0, k = 1,2,...\}, K(x,\lambda) = \{x \in l_2 \mid 0 \leq x_n \leq \frac{1}{n}\}, \lambda_0 = 0$ and

$$f(x,y,\lambda) = \begin{cases} x-y, & \text{if} \quad \lambda = 0, \\ x(x-y), & \text{otherwise,} \end{cases}$$

where $l_2 = \{x = (x_1, x_2, ...) \mid \sum_{n=1}^{\infty} x_n^2 < +\infty\}$. Then (i) is satisfied as $K(x,\lambda)$ is constant and compact. (ii) is fulfilled since $f(.,.,0)$ is continuous. It is clear that $S(0) = \{(1, \frac{1}{2}, ..., \frac{1}{n}, ...)\}$ and $S(\lambda) = \{(0,0,...), (1, \frac{1}{2}, ..., \frac{1}{n}, ...)\}$ and hence S is not usc at 0. The reason is that assumption (iii) is violated. Indeed, taking $x = (0,0,...), y = (\frac{1}{2}, \frac{1}{4}, ..., \frac{1}{2n}, ...)$, one has, for $\lambda \neq 0$,

$$f(x,y,0) = (-\frac{1}{2}, -\frac{1}{4}, ..., \frac{1}{2n}, ...) \in l_2 \setminus \Gamma,$$

$$f(x,y,\lambda) = (0,0,...) \notin l_2 \setminus \Gamma.$$

Although assumption (iii) cannot be dropped, we can replace it as follows.

Theorem 2.2. *Theorem 2.1 is still valid if we replace assumptions* (ii) *and* (iii) *by*

(ii') $\text{lev}_{0.\Gamma} f$ *is closed in* $K(A,\varLambda) \times K(A,\varLambda) \times \{\lambda_0\}$.

Proof. Reasoning ab absurdo, suppose the existence of an open subset $U \supseteq S(\lambda_0)$ and a net $(x_\alpha, \lambda_\alpha) \to (x_0, \lambda_0)$, such that $x_\alpha \in S(\lambda_\alpha) \setminus U, \forall \alpha$. If $x_0 \notin S(\lambda_0)$, there is $y_0 \in K(x_0, \lambda_0), f(x_0, y_0, \lambda_0) \in Y \setminus \Gamma$. Since K is lsc at (x_0, λ_0), there exists a net $y_\alpha \in K(x_\alpha, \lambda_\alpha), y_\alpha \to y_0$. As $x_\alpha \in S(\lambda_\alpha)$, $f(x_\alpha, y_\alpha, \lambda_\alpha) \in \Gamma$. From assumption (ii') we have $f(x_0, y_0, \lambda_0) \in \Gamma$, a contradiction. If $x_0 \in S(\lambda_0) \subseteq U$, one has another contradiction, as $x_\alpha \notin U, \forall \alpha$. □

Theorem 2.3. *Theorem 2.2 is still valid if* (ii') *is replaced by the following three conditions* (ii") $\mathrm{lev}_{0.(Y\setminus \mathrm{int}\Gamma)}f$ *is closed in* $K(A,\Lambda)\times K(A,\Lambda)\times\{\lambda_0\}$ *and,* $\forall y\in K(A,\lambda_0)$, $\mathrm{lev}_{0.\Gamma}f(.,y,\lambda_0)$ *is closed;*

(iii") *there is a neighborhood* U *of* λ_0*, such that* $\forall\lambda\in U(\lambda_0)$, $f(.,.,\lambda)$ *is* Γ*-pseudomonotone in* $K(A,\lambda)\times K(A,\lambda)$;

(iv") $\forall x\in K(A,\lambda_0)$, $f(x,.,\lambda_0)$ *is generalized* $Y\setminus\mathrm{int}\Gamma$*-concave in* $E(\lambda_0)$ *and* $f(x,x,\lambda_0)\in\Gamma$.

Proof. We prove first that S is closed at λ_0. Consider $\lambda_\alpha\to\lambda_0$, $x_\alpha\in S(\lambda_\alpha)$, $x_\alpha\to x_0$. For each $y\in K(x_0,\lambda_0)$, since K is lsc at (x_0,λ_0), there is a net $y_\alpha\in K(x_\alpha,\lambda_\alpha)$ such that $y_\alpha\to y$. Since $x_\alpha\in S(\lambda_\alpha)$ and $f(.,.,\lambda_\alpha)$ is pseudomonotone one has

$$f(y_\alpha,x_\alpha,\lambda_\alpha)\in Y\setminus\mathrm{int}\Gamma.$$

By the closedness of $\mathrm{lev}_{0.(Y\setminus\mathrm{int}\Gamma)}f$ we have

$$f(y,x_0,\lambda_0)\in Y\setminus\mathrm{int}\Gamma. \tag{1}$$

We show that $f(x_0,y_0,\lambda_0)\in\Gamma$, $\forall y_0\in K(x_0,\lambda_0)$. For $t\in(0,1)$, $y_t=(1-t)x_0+ty_0\in K(x_0,\lambda_0)$, as $x_0\in K(x_0,\lambda_0)$. We claim that $f(y_t,y_0,\lambda_0)\in\Gamma$. Indeed, suppose $f(y_t,y_0,\lambda_0)\in Y\setminus\Gamma$. If $f(y_t,x_0,\lambda_0)\in Y\setminus\Gamma$, from the assumed $Y\setminus\mathrm{int}\Gamma$-concavity, one has $f(y_t,y_t,\lambda_0)\in Y\setminus\Gamma$, impossible. If $f(y_t,x_0,\lambda_0)\in\Gamma$. From (1) we have $f(y_t,x_0,\lambda_0)\in\mathrm{bd}\Gamma=\mathrm{bd}(Y\setminus\mathrm{int}\Gamma)\subseteq Y\setminus\mathrm{int}\Gamma$, where bd(.) denotes the boundary of set (.). By the generalized $Y\setminus\mathrm{int}\Gamma$-concavity, $f(y_t,y_t,\lambda_0)\in Y\setminus\Gamma$, a contradiction. So $f(y_t,y_0,\lambda_0)\in\Gamma$. Since $\mathrm{lev}_{0.\Gamma}f(.,y_0,\lambda_0)$ is closed, taking $t\to0^+$ we have $f(x_0,y_0,\lambda_0)\in\Gamma$, and hence $x_0\in S(\lambda_0)$, thus S is closed at λ_0.

Now we show that S is usc at λ_0. Suppose there is an open superset U of $S(\lambda_0)$ such that there are nets $\lambda_\alpha\to\lambda_0$ and $x_\alpha\in S(\lambda_\alpha)$, $x_\alpha\notin U,\forall\alpha$. By the upper semicontinuity of E and the compactness of $E(\lambda_0)$ one can assume that $x_\alpha \to x_0$, for some $x_0 \in E(\lambda_0)$. Since S is closed at λ_0, we have $x_0 \in S(\lambda_0) \subseteq U$, it is impossible since $x_\alpha \notin U, \forall \alpha$. $\square$

The following example shows that the assumption about the closedness of $\mathrm{lev}_{0.(Y\setminus \mathrm{int}\Gamma)} f$ in Theorem 2.3 cannot be dropped.

Example 2.2. Let $X = Y = A = R$, $\Lambda = [0,1], \Gamma = R_+, K(x,\lambda) = [0,1], \lambda_0 = 0$ and

$$f(x,y,\lambda) = \begin{cases} x - y, & \text{if} \quad \lambda = 0, \\ xy(x-y), & \text{otherwise.} \end{cases}$$

We check the assumed R_--concavity. If $f(x,y,0) \le 0$ and $f(x,z,0) < 0$, then $x \le y$ and $x < z$. So $f(x,(1-t)y + tz, 0) = x - (1-t)y - tz < 0, \forall t \in (0,1)$, i.e. $f(x,.,0)$ is R_--concave. It is easy to see that the assumptions of Theorem 2.3 are fulfilled except the closedness of $\mathrm{lev}_{0.R_-} f$. (Indeed, let $x_n = 1, y_n = 0$ and $\lambda_n = \frac{1}{n}$. Then $(x_n, y_n, \lambda_n) \to (1,0,0)$ and $f(x_n,y_n,\lambda_n) = 0$, but $f(1,0,0) = 1 > 0$.) It is clear that $S(0) = \{1\}$, $S(\lambda) = \{0,1\}, \forall \lambda \in (0,1]$, and hence S is not usc at 0. The reason is that $\mathrm{lev}_{0.R_-} f$ is not closed.

3. Lower semicontinuity of the solution set

As an auxiliary problem we consider also the following problem (QEP$_1$) together with (QEP):

(QEP$_1$) Find $\bar{x} \in K(\bar{x},\lambda)$ such that, $\forall y \in K(\bar{x},\lambda)$,

$$f(\bar{x},y,\lambda) \in \mathrm{int}\Gamma,$$

where X, Λ, A, K, Γ and f are as in Section 1. For $\lambda \in \Lambda$, let $S_1(\lambda)$ be the solution set of (QEP$_1$) corresponding to λ. Clearly $S_1(\lambda) \subseteq S(\lambda)$.

Theorem 3.1. *Assume for* (QEP) *that* $S_1(\lambda) \neq \emptyset$ *and that*

(i) *E is lsc at λ_0 and $E(\lambda_0)$ is convex; K is usc and compact-valued in $E(\lambda_0)\times\{\lambda_0\}$;*

(ii) *$\mathrm{lev}_{0.Y\setminus\mathrm{int}\Gamma} f$ is closed in $K(A,\Lambda)\times K(A,\Lambda)\times\{\lambda_0\}$;*

(iii) *for each $y\in K(A,\lambda_0)$, $f(.,y,\lambda_0)$ is generalized Γ-concave in $E(\lambda_0)$;*

(iv) *$\forall\bar{x}\in S(\lambda_0), \forall\bar{x}^1\in S_1(\lambda_0)$, $\forall y\in K(E(\lambda_0),\lambda_0)$,*

$$f(\bar{x},y,\lambda_0)\in\Gamma \quad \textit{and} \quad f(\bar{x}^1,y,\lambda_0)\in\mathrm{int}\Gamma.$$

Then S is lsc at λ_0.

Proof. We start by proving that S_1 is lsc at λ_0. Suppose to the contrary that $\exists x_0\in S_1(\lambda_0)$, $\exists\lambda_\alpha\to\lambda_0, \forall x_\alpha\in S_1(\lambda_\alpha), x_\alpha\not\to x_0$. Since E is lsc at λ_0, there is $\bar{x}_\alpha\in E(\lambda_\alpha)$, $\bar{x}_\alpha\to x_0$. By the above contradiction assumption, there must be a subnet $\bar{x}_\beta$ such that, $\forall\beta$, $\bar{x}_\beta\notin S_1(\lambda_\beta)$, i.e., for some $y_\beta\in K(\bar{x}_\beta,\lambda_\beta)$,

$$f(\bar{x}_\beta,y_\beta,\lambda_\beta)\in Y\setminus\mathrm{int}\Gamma. \tag{2}$$

As K is usc at (x_0,λ_0) and $K(x_0,\lambda_0)$ is compact one has $y_0\in K(x_0,\lambda_0)$ such that $y_\beta\to y_0$ (taking a subnet if necessary). By assumption (ii), (2) yields that $f(x_0,y_0,\lambda_0)\in Y\setminus\mathrm{int}\Gamma$, which is impossible since $x_0\in S_1(\lambda_0)$.

Now let us prove that

$$S(\lambda_0)\subseteq\overline{S_1(\lambda_0)} \tag{3}$$

Let $\bar{x}\in S(\lambda_0), \bar{x}^1\in S_1(\lambda_0)$ and $x_t=(1-t)\bar{x}+t\bar{x}^1$, where $t\in(0,1)$. Then $x_t\to\bar{x}$ as $t\to 0$. By assumption (iv), $\forall y\in K(x_t,\lambda_0)$, $f(\bar{x},y,\lambda_0)\in\Gamma$ and $f(\bar{x}^1,y,\lambda_0)\in\mathrm{int}\Gamma$. Since $f(.,y,\lambda_0)$ is generalized Γ-concave, we have $f(x_t,y,\lambda_0)\in\mathrm{int}\Gamma$, i.e.

$x_t \in S_1(\lambda_0)$. Therefore (3) holds. Now by the lower semicontinuity of S_1 at λ_0 we have

$$S(\lambda_0) \subseteq \overline{S_1(\lambda_0)} \subseteq \overline{\liminf S_1(\lambda_\alpha)} \subseteq \overline{\liminf S(\lambda_\alpha)}.$$

Since liminf is always closed, S is lsc at λ_0. □

The following example shows that the concavity of $f(., y, \lambda_0)$ is essential.

Example 3.1. Let $X = Y = A = R, \Lambda = [0,1], \Gamma = R+, K(x,\lambda) = [\lambda, \lambda + 3], \lambda_0 = 0$ and $f(x,y,\lambda) = x^2 - (\lambda+1)x$. Then, it is easy to see that $E(\lambda) = [\lambda, \lambda + 3], \forall \lambda \in [0,1]$, and assumptions (i), (ii) and (iv) of Theorem 3.1 are satisfied. But $S(0) = \{0\} \cup [1,3]$ and $S(\lambda) = [\lambda+1, \lambda+3], \forall \lambda \in (0,1]$, and hence $S(.)$ is not lsc at 0. The reason is that (iii) is violated. Indeed, let $x_1 = 0, x_2 = \frac{3}{2} \in E(0) = [0,3]$ and $t = \frac{1}{2}$. $\forall y \in K(A,0) = [0,3]$, we have $f(x_1, y, 0) = 0$, $f(x_2, y, 0) = \frac{3}{4}$, but

$$f\big(\frac{1}{2}x_1 + \frac{1}{2}x_2, y, 0\big) = -\frac{3}{16} \notin (0, +\infty).$$

Remark 3.1. If $K(x,\lambda) \equiv K(\lambda)$, we can omit assumption (iv). The following example shows that in the general case assumption (iv) cannot be dropped.

Example 3.2. Let $X = Y = R, \Lambda = [0,1], A = [0,6], \Gamma = R_+, K(x,\lambda) = [\lambda, 4\lambda + 2 - x]$, $\lambda_0 = 0$ and $F(x,y,\lambda) = \{x(x-y)\}$. Then we have $E(\lambda) = [\lambda, 2\lambda+1]$. It is easy to see that assumptions (i)-(iii) of Theorem 3.1 are satisfied. But $S(0) = \{0,1\}$, $S(\lambda) = \{2\lambda+1\}$ and hence $S(.)$ is not lsc at 0. The reason is that assumption (iv) is violated. Indeed, $K(E(0),0) = K([0,1],0) = [0,2]$. For $x = 1 \in S(0)$, $y = 2 \in K(E(0),0)$, we have $F(1,2,0) = -1 < 0$. So (iv) is not fulfilled.

Remark 3.2. Theorems 5.1-5.3 of [11] are incomparable with our Theorem 3.1. Assumption (vi) of the mentioned theorems is difficult to be checked, although it is weaker than assumption (ii) in Theorem 3.1. However, many of our assumptions are weaker than the corresponding ones of these theorems: we omit the concavity assumption of $K(.,\lambda)$; our concavity assumption of $f(.,y,\lambda_0)$ is imposed in $E(\lambda_0)$, while in the mentioned theorems the concavity property of f is imposed in $X \times X \times \Lambda$. The following example shows a case where Theorem 3.1 can be applied but Theorems 5.1-5.3 of [11] cannot.

Example 3.3. Let $X, Y, \Lambda, \Gamma, \lambda_0$ be as in Example 3.2, $A = [0,1], K(x,\lambda) = [0,\lambda]$ and

$$F(x,y,\lambda) = \begin{cases} -1, & \text{if} \quad y+\lambda = -1, \\ 1, & \text{otherwise.} \end{cases}$$

Then the assumptions of Theorem 3.1 are satisfied, (in fact $S(\lambda) = [0,\lambda], \forall \lambda \in [0,1]$ is lsc). But Theorems 5.1-5.3 in [11] cannot be applied since f is not concave as required in these theorems.

We now proceed to Hausdorff lower semicontinuity.

Theorem 3.2. *Assume the assumptions of Theorem 3.1, and the following additional conditions:*

(v) $K(.,\lambda_0)$ *is lsc in* $E(\lambda_0)$ *and* $E(\lambda_0)$ *is compact*;

(vi) $\mathrm{lev}_{0.\Gamma} f(.,.,\lambda_0)$ *is closed in* $K(A,\Lambda) \times K(A,\Lambda)$.

Then S is Hausdorff lower semicontinuous at λ_0.

Proof. We first show that $S(\lambda_0)$ is closed in X. Suppose that $x_\alpha \in S(\lambda_0), x_\alpha \to x_0$. Then $x_0 \in E(\lambda_0)$ by the compactness. If $x_0 \notin S(\lambda_0)$, there exists $y_0 \in K(x_0,\lambda_0)$ such that

$$f(x_0,y_0,\lambda_0) \in Y \setminus \Gamma. \tag{4}$$

Since $K(.,\lambda_0)$ is lsc at x_0, there is a net $y_\alpha \in K(x_\alpha,\lambda_0), y_\alpha \to y_0$. As $x_\alpha \in S(\lambda_0)$, we have

$$f(x_\alpha, y_\alpha, \lambda_0) \in \Gamma. \tag{5}$$

By the closedness of $\mathrm{lev}_{0.\Gamma} f(.,.,\lambda_0)$, we see a contradiction between (4) and (5). Thus, $S(\lambda_0)$ is closed and then compact.

Now suppose that S is not Hlsc at λ_0, i.e. $\exists B$ (a neighborhood of the origin in X), $\exists \lambda_\alpha \to \lambda_0$, $\forall \alpha, \exists x_{0\alpha} \in S(\lambda_0) \setminus (S(\lambda_\alpha) + B)$. Since $S(\lambda_0)$ is compact, we can assume that $x_{0\alpha} \to x_0 \in S(\lambda_0)$. Then there are α_1, a neighborhood B_1 of 0 in X with $B_1 + B_1 \subseteq B$ and $b_\alpha \in B_1$ such that, $\forall \alpha \geq \alpha_1$, $x_{0\alpha} = x_0 + b_\alpha$. Since S is lsc at λ_0, there is $z_\alpha \in S(\lambda_\alpha), z_\alpha \to x_0$ and then there is α_2 such that, $\forall \alpha \geq \alpha_2$,

$$z_\alpha \in x_0 - B_1,$$

i.e., there exists $b'_\alpha \in B_1, z_\alpha = x_0 - b'_\alpha$. Consequently, $\forall \alpha \geq \alpha_0 = \max\{\alpha_1, \alpha_2\}$,

$$x_{0\alpha} = x_0 + b_\alpha = z_\alpha + b'_\alpha + b_\alpha \in z_\alpha + B.$$

This is impossible due to the fact that $x_{0\alpha} \notin S(\lambda_\alpha) + B$. Thus, S is Hlsc at λ_0.□

The following example shows that the assumptions about E in (i) are essential.

Example 3.4. Let $X = A = R^2, Y = R, \Lambda = [0,1], \Gamma = R_+$, $\lambda_0 = 0$, and for $x = (x_1, x_2) \in R^2, K(x,\lambda) = \{(x_1, \lambda x_1)\}$, $f(x,y,\lambda) = 1 + \lambda$. Then $E(\lambda) = \{(x_1,x_2) \mid x_2 = \lambda x_1\}$. Clearly the assumptions of Theorem 3.2, but the compactness of $E(\lambda_0)$, are satisfied. Direct computations give $S(\lambda) = \{(x_1,x_2) \in R^2 \mid x_2 = \lambda x_1\}$ and then S is not Hlsc at 0 (although S is lsc at 0).

4. Continuity of the solution set

We can combine the results in Section 2 and Theorem 3.1 to derive sufficient conditions for the continuity of the solution map of (QEP). In this section we establish sufficient conditions without concavity assumptions.

Theorem 4.1. *Assume the assumptions of one of the Theorems 2.1-2.3. Assume further that*

(a) $f(.,.,\lambda_0)$ *is* Γ*-quasimonotone in* $K(A,\lambda_0)\times K(A,\lambda_0)$;

(b) $\forall x\in S(\lambda_0), \forall y\in S(\lambda_0)\setminus\{x\},\ f(x,y,\lambda_0)\in \mathrm{int}\Gamma$.

Then S *is continuous at* λ_0.

Proof. Assume first the assumptions of Theorems 2.1 or 2.2. It suffices to prove that S is lsc at λ_0. Suppose to the contrary that $\exists\lambda_\alpha\to\lambda_0, \exists x_0\in S(\lambda_0)$, $\forall x_\alpha\in S(\lambda_\alpha)$, $x_\alpha\not\to x_0$. Since E is usc and $E(\lambda_0)$ is compact, we can assume that $x_\alpha\to\bar{x}_0$ for some $\bar{x}_0\in E(\lambda_0)$.

From the proof of Theorem 2.1 or 2.2, we see that $\bar{x}_0\in S(\lambda_0)$. By the contradiction assumption we have $\bar{x}_0\neq x_0$. Due to assumption (b) one has

$$f(\bar{x}_0,x_0,\lambda_0)\in \mathrm{int}\Gamma \quad \text{and} \quad f(x_0,\bar{x}_0,\lambda_0)\in \mathrm{int}\Gamma,$$

which is impossible since $f(.,.,\lambda_0)$ is quasimonotone.

The proof for the case, where the assumptions of Theorem 2.3 are fulfilled, is similar. □

Theorem 4.2. *Assume the assumptions of one of Theorems 2.1-2.3 and assume further that*

(a') $f(.,.,\lambda_0)$ *is* Γ*-pseudomonotone in* $K(A,\lambda_0)\times K(A,\lambda_0)$;

(b') *if* $f(x,y,\lambda_0)\in \mathrm{bd}\Gamma$ *then* $x=y$*, where* $\mathrm{bd}(.)$ *denotes the boundary of the set* $(.)$;

(c') $\forall x, \bar{x} \in S(\lambda_0),\ f(x, \bar{x}, \lambda_0) \in \Gamma$.

Then S is continuous at λ_0.

Proof. By an argument similar to the first part of the proof of Theorem 4.1, we have $\bar{x}_0 \neq x_0$. (c') implies that $f(x, \bar{x}, \lambda_0) \in \Gamma$. By the pseudomonotonicity of $f(., ., \lambda_0)$, one has $f(\bar{x}_0, x_0, \lambda_0) \in Y \setminus \mathrm{int}\Gamma$.

On the other hand, since $x_0, \bar{x}_0 \in S(\lambda_0)$, $f(\bar{x}_0, x_0, \lambda_0) \in \Gamma$ and hence $f(\bar{x}_0, x_0, \lambda_0) \in \mathrm{bd}\Gamma$. By assumption (b'), we have $\bar{x}_0 = x_0$, a contradiction. □

5. Particular cases

Since equilibrium problems contain many problems as special cases, including variational inequalities, optimization problems, fixed point and coincidence point problems, complementarity problems, Nash equilibria problems, etc, we can derive from the results of Sections 2-4 consequences for such special cases. In this section we discuss only some corollaries of the typical results in Sections 2-4 for quasivariational inequalities and traffic network problems as examples.

5.1. Quasivariational inequalities

Let X, A, Λ, K be as in Section 1, X^* be the dual space of X and $T : X \times \Lambda \to X^*$. We consider the following parametric quasivariational inequality, for each $\lambda \in \Lambda$,

(QVI) Find $\bar{x} \in K(\bar{x}, \lambda)$ such that, $\forall y \in K(\bar{x}, \lambda)$,

$$\langle T(y, \lambda), y - \bar{x} \rangle \geq 0,$$

where $\langle ., . \rangle$ denotes the pairing between X and X^*.

To convert (QVI) to a special case of (QEP) set $Y = R, \Gamma = R_+$ and $f(x, y, \mu) = \langle T(y, \mu), y - x \rangle$. Consequently, the following result is immediate from Theorem 5.1.

The three following results are derived from Theorems 2.1-2.3.

Corollary 5.1. *Assume for* (QVI) *that*

(i) *E is usc at λ_0, $E(\lambda_0)$ is compact and K is lsc in $K(A,\Lambda)\times K(A,\Lambda)$;*

(ii) *the set $\{(x,y)\in A\times A \mid \langle T(y,\lambda_0), y-x\rangle \geq 0\}$ is closed in $K(A,\Lambda)\times K(A,\Lambda)$;*

(iii) *$\forall x,y\in K(A,\Lambda)$, the function $\lambda\mapsto \langle T(y,\lambda), y-x\rangle$ is $(-\infty,0)$-usc at λ_0.*

Then the solution map S is usc at λ_0.

Remark 5.1.

(i) By Theorem 2.2, Corollary 5.1 is still valid if we replace assumptions (ii) and (iii) by

 (iii') The set $\{(x,y,\lambda) \mid \langle T(y,\lambda), y-x\rangle \geq 0\}$ is closed in $K(A,\Lambda)\times K(A,\Lambda)\times\{\lambda_0\}$.

(ii) Corollary 5.1 together with (i) include Theorems 2.2 and 2.3 of [9], Theorems 4.1 and 4.3 of [10].

(iii) Similarly, we can obtain direct corollaries of Theorems 3.1, 3.2 and these results are new for (QVI).

5.2. Traffic network problems

The notion of equilibrium flows for transportation network problem was introduced in Wardrop (1952) together with a basic traffic network principle. Since then, traffic network problems have raised a great interest and much developed in both theory and methodology view points. The variational approach to such traffic problems begins with Smith (1979), who proved that the Wardrop equilibrium can be expressed in terms of variational inequalities, see also Nagurney

(1993). In De Luca (1995) and Maugeri (1995), travel demands were proposed to depend on the equilibrium vector flow to met diverse practical situations. Then Wardrop equilibriums of the network problem are expressed as solutions of the corresponding quasivariational inequality. In Ait Mansour and Scrimali (online), the Hölder continuity of the solution sets of such parametric elastic traffic problems was considered. In this subsection, using results in Section 2 we establish continuity properties of the solution of an elastic traffic problem.

We first describe the problem. Let N be the set of nodes, L be that of links (or arcs), $W = (W_1, ..., W_l)$ be the set of origin-destination pairs (O/D pairs for short). Assume that the pair W_j, $j = 1, ..., l$, is connected by a set P_j of paths and P_j contains $r_j \geq 1$ paths. Let $F = (F_1, ..., F_m)$ be the path vector flow, where $m = r_1 + ... + r_l$. Following Giannessi (1980) the capacity of these paths must be taken into account in practice. So we assume that the capacity restriction is

$$F \in A := \{F \in R^m : 0 \leq \gamma_s \leq F_s \leq \Gamma_s, s = 1, ..., m\},$$

where A be a convex and compact subset of R^m. Assume further that the travel cost on the path flow $F_s, s = 1, ..., m$, depends on the whole path vector flow F and is $T_s(F) \geq 0$. Then we have the path cost vector $T(F) = (T_1(F), ..., T_m(F))$.

Following Wardrop (1952) a path vector flow H is said to be an equilibrium vector flow if $\forall W_j$, $\forall p \in P_j$, $\forall s \in P_j$,

$$[T_p(H) < T_s(H)] \Longrightarrow [H_s = \gamma_s \quad \text{or} \quad H_p = \Gamma_p].$$

Now assume that the perturbation on the traffic expresses by parameter c of a metric space C. Assume further that the travel demand g_j of the O/D pair W_j depends on $c \in C$ and also on the equilibrium vector flow H as explained in De Luca (1995), and Maugeri (1995). Denote the travel vector demand by

$g = (g_1, ..., g_l)$ and set

$$\phi_{js} = \begin{cases} 1, & \text{if} \quad s \in P_j, \\ 0, & \text{if} \quad s \notin P_j, \end{cases}$$

$$\phi = \{\phi_{js}\}, j = 1, ..., l; s = 1, ..., m.$$

Then the path vector flows meeting the travel demands are called the feasible path vector flows and form the constraint set

$$K(H, c) = \{F \in A \mid \phi F = g(H, c)\}.$$

ϕ is called the O/D pair - path incidence matrix.

Assume further that the path costs are also perturbed, i.e. depend on a perturbation parameter b of a metric space B: $T_s(F, b), s = 1, ..., m$.

Remark 5.2. The above traffic model is formulated in terms of path flow variables. Another way to describe the traffic problem is using link flow variables. But the latter model can be employed only if the travel cost is additive, i.e. any path cost is the sum of the link costs for all the links involved in the path. The "path model" we use here does not need this additivity.

Our traffic network problem is equivalent to a quasivariational inequality as follows.

Lemma 5.2. (See De Luca (1995), Smith (1979)). *A path vector flow $H \in K(H, a)$ is an equilibrium flow if and only if it is a solution of the following quasivariational inequality*

(TNP) *Find $H \in K(H, c)$ such that, $\forall F \in K(H, c)$,*

$$\langle T(H, b), F - H \rangle \geq 0.$$

We need the following simple assertions

Lemma 5.3. (See Ait Mansour and Scrimali, online, Lemma 1) *Let A be an $m \times n$ matrix, a_1 and a_2 be given vectors in R^m. The solution set of the linear equality $Ax = a_i$, for $i = 1, 2$, is denoted by S_i. Then, there exists $\delta = \delta(A) > 0$ such that for each $x_1 \in S_1$ there exists $x_2 \in S_2$ satisfying*

$$\|x_1 - x_2\| \leq \delta \|a_1 - a_2\|.$$

Lemma 5.4. *Assume that g is continuous at (H_0, c_0). Then K is continuous at (H_0, c_0) and convex, compact-valued.*

Proof. Let $c \in V(c_0)$ and $H \in W(H_0)$, where $V(c_0)$ and $W(H_0)$ be neighborhoods of c_0 and H_0, respectively. Consider the system

$$\phi F = g(H_0, c_0),$$

$$\phi F = g(H, c).$$

By Lemma 5.3, there exists $\delta = \delta(\phi)$ such that for each $F_0 \in K(H_0, c_0)$, there exists $F \in K(H, c)$ satisfying

$$\|F - F_0\| \leq \delta \|g(H, c) - g(H_0, c_0)\|.$$

Since g is continuous at (H_0, c_0), K is lsc at (H_0, c_0). Suppose that K is not usc at (H_0, c_0), i.e., there are a neighborhood U of $K(H_0, c_0)$ and a net $(H_n, c_n) \to (H_0, c_0)$ such that, for each n, there exists $F_n \in K(H_n, c_n) \backslash U$. By the compactness of A, we can assume that $F_n \to F_0$. According to Lemma 5.3, there is $F_n^0 \in K(H_0, c_0)$ such that

$$\|F_n - F_n^0\| \leq \delta \|g(H_n, c_n) - g(H_0, c_0)\|.$$

Hence,

$$\|F_n^0 - F_0\| \leq \|F_n^0 - F_n\| + \|F_n - F_0\|.$$

Consequently, $F_n^0 \to F_0$. Since $\phi F_n^0 = g(H_0, c_0)$, we have $\phi F_0 = g(H_0, c_0)$, i.e., $F_0 \in K(H_0, c_0) \subseteq U$, a contradiction. □

Setting $X = R^m, \Lambda = C \times B$ and, for each $\lambda = (c, b) \in \Lambda$,

$$K_1(H, \lambda) = K(H, c),$$

$$f(x, y, \lambda) = \langle T(x, b), y - x\rangle.$$

Then (TNP) becomes a special case of (QEP).

The following results are implied directly from Theorems 2.2, 4.1 and 4.2.

Corollary 5.5. *For problem* (TNP) *assume that*

(i) g *is continuous in* $K(A, c_0) \times \{c_0\}$;

(ii) *the set* $\{(H, F, c) \mid \langle T(H, c), F - H\rangle \geq 0\}$ *is closed in* $A \times A \times \{c_0\}$.

Then the solution set S *is usc at* (c_0, b_0).

Proof. It is derived from Lemma 5.4 and Theorem 2.2. □

Corollary 5.6. *Assume the assumptions of Corollary 5.5 and assume further that*

(a) T *is quasimonotone in* $K(A, c_0)$;

(b) $\forall H \in S(c_0, b_0)$, $\forall H' \in S(c_0, b_0) \setminus \{H\}, \langle T(H, c_0), H' - H\rangle > 0$.

Then S *is continuous at* (c_0, b_0).

Proof. It is clear from Theorem 4.1. □

Corollary 5.7. *Assume* (i) *and* (ii) *of Corollary 5.5 and further more*

(a') T *is pseudomonotone in* $K(A, c_0)$;

(b') *if* $\langle T(H_1, c_0), H_2 - H_1\rangle = 0$ *then* $H_2 = H_1$;

(c') $\forall H_1, H_2 \in S(c_0, b_0), \langle T(H_1, c_0), H_2 - H_1\rangle \geq 0$.

Proof. It is a direct consequence of Theorem 4.2. □

Remark 5.3. Corollary 5.7 improves Theorem 4.1 of Li et al. [12], since here (c') needs to be fulfilled only at $x \in S(\lambda_0)$ and assumption (ii) is weaker than the continuity assumption of T required in this theorem. Corollaries 5.5 and 5.6 are new. We note further that the results in Subsection 5.1 can be applied for (TNP). But Theorems 3.1 - 3.3 in [10] cannot, since assumption (iii) in these theorems is not fulfilled in this case.

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