GLOBAL EXISTENCE OF DIFFUSIVE-DISPERSIVE TRAVELING WAVES FOR GENERAL FLUX FUNCTIONS

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ABSTRACT. We establish a global existence of traveling waves for diffusive-dispersive conservation laws for locally Lipschitz flux functions.

1. INTRODUCTION

We consider in this paper the existence of a certain kind of smooth solutions, called the *traveling waves*, of the following third-order partial differential equation

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = a\partial_{xx}u(x,t) + b\partial_{xxx}u(x,t), \quad x \in \boldsymbol{R}, t > 0, \tag{1.1}$$

where, a, b represent the diffusion and dispersion coefficients, respectively. Here, we assume a and b are positive constants.

When traveling waves of (1.1) exist, one is interested in their limit when $a, b \to 0+$. This is a certain kind of admissibility criteria for shock waves of the conservation law

$$\partial_t u + \partial_x f(u) = 0. \tag{1.2}$$

Conversely, when a shock wave of (1.2) exists, it has been shown that the corresponding traveling waves also exist, under certain circumstances, see [15].

Diffusive-dispersive traveling waves were discovered by Jacobs, McKinney, and Shearer [11] for the cubic function. The developments can be found in [9, 8, 5, 6, 4], etc. In [15], the relationship between the existence of traveling waves of (1.1) and the existence of *classical* and *nonclassical* shock waves was considered. A geometrical distinction between the classical shocks (see [22, 13, 20]) and nonclassical shocks is

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that in the case of classical shocks, the line connecting the two left-hand and right-hand states does not cross the graph of the flux function in the interval between these two states, while it is the case for nonclassical shocks. The reader is referred to [1, 2, 9, 10, 14, 10, 15, 18, 19, 17, 21], and the references therein for nonclassical shock waves. Recently, non-monotone traveling waves for van der Waals fluids with diffusion and dispersion terms were obtained in [3].

The organization of the paper is as follows. In Section 2, we will provide basic concepts and properties of traveling waves of (1.1) and the stability of equilibria of the associated differential equation. In Section 3, we will establish an invariance result concerning traveling waves of (1.1), relying on LaSalle's invariance principle. In Section 4 we will demonstrate that traveling waves of (1.1) exist whenever there is a Lax shock of the associate conservation law (1.2) satisfying Oleinik's entropy condition.

2. Traveling waves and stability of equilibria

Let us consider *traveling waves* of (1.1) i.e., smooth solution $u = u(y)$ depending on the re-scaled variable

$$y := \alpha \frac{x-\lambda t}{a} = \frac{x-\lambda t}{\sqrt{b}}. \tag{2.1}$$

for some constant speed λ and

$$\alpha = a/\sqrt{b}.$$

Substituting $u = u(y)$ to (1.1), after re-scaling, the traveling wave u connecting a left-hand state u_- to a right-hand state u_+ satisfies the ordinary differential equation

$$-\lambda \frac{du}{dy} + \frac{df(u)}{dy} = \alpha \frac{d^2u}{dy^2} + \frac{d^3u}{dy^3}, \quad y \in \boldsymbol{R}, \tag{2.2}$$

and the boundary conditions

$$\begin{aligned} &\lim_{y\to\pm\infty} u(y) = u_\pm, \\ &\lim_{y\to\pm\infty} \frac{du}{dy} = \lim_{y\to\pm\infty} \frac{d^2u}{dy^2} = 0. \end{aligned} \tag{2.3}$$

Integrating (2.2) and using the boundary condition (2.3), we find u such that

$$\frac{d^2u}{dy^2} + \alpha \frac{du}{dy} = -\lambda(u(y) - u_-) + f(u) - f(u_-), \quad y \in \boldsymbol{R}. \tag{2.4}$$

Using (2.3) again, we deduce from (2.4)

$$\lambda = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \tag{2.5}$$

Setting

$$v = \frac{du}{dy}$$

we can re-write the second-order differential equation (2.4) to the following second-order system

$$\begin{aligned} \frac{du(y)}{dy} &= v(y), \\ \frac{dv(y)}{dy} &= -\alpha v(y) - \lambda(u(y) - u_-) + f(u(y)) - f(u_-). \end{aligned} \tag{2.6}$$

The system (2.6) can be written in a more compact of autonomous differential equations

$$\frac{dU(y)}{dy} = F(U(y)), \quad y \in \boldsymbol{R}, \tag{2.7}$$

where $U = (u, v) \in \boldsymbol{R}^2$ and

$$F(U) = \big(v, -\alpha v + h(u)\big), \quad h(u) = -\lambda(u - u_-) + f(u) - f(u_-).$$

We observe that the function F is locally Lipschitz in $\boldsymbol{R}^2$ if f is locally Lipschitz in $\boldsymbol{R}$. From the local existence theory, it is not difficult to check the following result, which provides us with the global existence for (2.7).

Lemma 2.1. *Let f be locally Lipschitz in $\boldsymbol{R}$. Suppose that there exists a compact set $W \subset \boldsymbol{R}^2$ such that any solution of*

$$\frac{dU(y)}{dy} = F(U), \quad y > 0, \quad U(0) = U_0$$

lies entirely in W. Then, there is a unique solution passing through U_0 defined for all $y \geq 0$.

Next, we want to study the asymptotic behavior of trajectories of (2.7). For this purpose, we consider the stability of equilibria of (2.7). It is derived from (2.5) that

$$F(u_+, 0) = 0, \quad F(u_-, 0) = 0,$$

which means $(u_+, 0)$ and $(u_-, 0)$ are equilibrium points of the differential equation (2.7). By definition, a point U_0 is called an *equilibrium point* of (2.7) if

$$F(U_0) = 0.$$

Thus, any equilibrium point of (2.7) has the form $U = (u, 0)$, where u satisfies

$$h(u) = -\lambda(u - u_-) + f(u) - f(u_-) = 0.$$

The last equality means that u, u_-, λ satisfy the Rankine-Hugoniot relation for the associate conservation law (1.2). We therefore conclude that

Proposition 2.2. *A point U is an equilibrium point of the autonomous differential equation (2.7) if and only if U has the form $U = (u_+, 0)$ for some constant u_+ so that the states $u_\pm$ and the shock speed λ satisfy the Rankine-Hugoniot relation for the associate conservation law (1.2).*

Consequently, when $U = (u_+, 0)$ is an equilibrium point of (2.7), the function

$$u(x,t) = \begin{cases} u_-, & x < \lambda t, \\ u_+, & x > \lambda t, \end{cases} \tag{2.8}$$

is a weak solution of the conservation law (1.2). Conversely, a jump of (1.2) of the form (2.8) gives equilibria $(u_-, 0), (u_+, 0)$ of the differential equation (2.7).

Now, let us study the boundary conditions (2.3). We recall some basic concepts. The reader is referred to [12] for more details. An equilibrium point $U_0 = (u_0, 0)$ of (2.7) is positively (negatively) stable if for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$||U(0) - U_0|| < \delta \quad \Rightarrow \quad ||U(y) - U_0|| < \varepsilon, \quad \forall y \geq 0,$$

($\forall y \leq 0$,respectively). The equilibrium point U_0 is *positively (negatively) asymptotically stable* if it is positively (negatively) stable and δ can be chosen such that

$$||U(0) - U_0|| < \delta \quad \Rightarrow \quad \lim_{y\to\infty} U(y) = 0, \quad (\lim_{y\to-\infty} U(y) = 0 \quad \text{respectively}).$$

Whenever a *Lyapunov function* defined on a domain D containing the equilibrium point U_0 is found, the equilibrium point is stable. A Lyapunov function for (2.7) is a continuously differentiable function L satisfying

$$L(U_0) = 0, \quad L(U) > 0, \quad U \in D \setminus \{U_0\}$$

and

$$\dot{L}(U) = \nabla L(U) \cdot F(U) \leq 0, \quad U \in D.$$

Lyapunov's stability theorem says that if such a function exists, then the equilibrium point U_0 is stable in D. Moreover, if

$$\dot{L}(U) = \nabla L(U) \cdot F(U) < 0, \quad U \in D \setminus \{U_0\}$$

then the equilibrium point U_0 is asymptotically stable.

When an equilibrium point is asymptotically stable, it is interesting to see how far from U_0, the trajectories of (2.7) still converges to U_0 as y approaches infinity. The *domain of attraction* of an equilibrium point U_0 is the set of all points in D such that the solution of (2.7) starting from such a point will converge to U_0 as y approaches infinity.

Let us consider the stability of equilibria of (2.6), or (2.7). It follows from Proposition 2.2 that the set Γ of equilibria has the form $(u_i, 0), i \in I$ and that

$$\lambda = \frac{f(u_i) - f(u_-)}{u_i - u_-}, \quad \forall i \in I, u_i \neq u_-. \tag{2.9}$$

This yields

$$\begin{aligned} h(u) &= -\lambda(u - u_-) + f(u) - f(u_-) \\ &= -\lambda(u - u_i) + f(u) - f(u_i) \quad \forall i \in I. \end{aligned} \tag{2.10}$$

Geometrically, Γ is the intersection of the straight line connecting $(u_\pm, 0)$ and the graph of f.

The Jacobian matrix $DF(U)$ is given by

$$DF(U) = \begin{pmatrix} 0 & 1 \\ (f'(u) - \lambda) & -\alpha \end{pmatrix}.$$

The characteristic equation of $DF(U)$ is

$$|DF(U) - \beta I| = \beta^2 + \alpha\beta - (f'(u) - \lambda) = 0$$

which admits two roots as

$$\beta_1 = -\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 + f'(u) - \lambda}, \qquad \beta_2 = -\frac{\alpha}{2} - \sqrt{\frac{\alpha}{2}^2 + f'(u) - \lambda}. \tag{2.11}$$

Since we consider the asymptotic behaviors $u \to u_+$ and $y \to \infty$ and $u \to u_-$ as $y \to -\infty$, we have

Proposition 2.3. (i) *If $f'(u_+) < \lambda$ then $\beta_2 < \beta_1 < 0$. The point $(u_+, 0)$ is asymptotically stable.*

(ii) *If $f'(u_+) > \lambda$ then $\beta_2 < 0 < \beta_1$. The point $(u_+, 0)$ is a saddle.*

(iii) *If $f'(u_-) < \lambda$ then $\beta_2 < \beta_1 < 0$. The point $(u_-, 0)$ is unstable.*

(iv) *If $f'(u_-) > \lambda$ then $\beta_2 < 0 < \beta_1$. The point $(u_-, 0)$ is a saddle.*

Thus, traveling waves would exist in the cases of stable-to-saddle connection and saddle-to-saddle connection, only. However, generally, it is difficult to establish saddle-to-saddle connection. In the sequel we will study only the case stable-to-saddle connection.

3. Invariance result

We will establish the asymptotical behaviors of trajectories of (2.7) relying on *LaSalle's invariance principle.* We first provide some more definitions.

A set $M \subset D$ is said to be an *invariant set* with respect to (2.7) if

$$U(0) \in M \quad \Rightarrow \quad U(y) \in M, \quad \forall y \in \boldsymbol{R}. \tag{3.1}$$

A set $M \subset D$ is said to be a *positively invariant set* with respect to (2.7) if

$$U(0) \in M \quad \Rightarrow \quad U(y) \in M, \quad \forall y \geq 0. \tag{3.2}$$

And similarly for a *negatively invariant set.*

Therefore, a set M is invariant if and only if it is both positively and negatively invariant.

We also say that $U(y)$ approaches a set M as y approaches infinity, if for every $\varepsilon > 0$, there is $Y > 0$ such that the distance from a point p to a set M is less than ε:

$$\text{dist}(U(y), M) := \inf_{U \in M} ||p - U|| < \varepsilon, \quad \forall y > Y. \tag{3.3}$$

Suppose that there exists a continuous differentiable function $L : D \to \boldsymbol{R}$ such that

$$\dot{L}(U) := \nabla L(U) \cdot F(U) \leq 0, \quad U \in \Omega. \tag{3.4}$$

We defined

$$E = \{U \in \Omega \mid \quad \dot{L}(U) = 0\}. \tag{3.5}$$

The LaSalle's invariance principle states that: *If Ω is a compact set that is positively invariant with respect to (2.7), and M is the largest invariant set in E, then every solution starting in Ω approaches M as $y \to \infty$.*

Proposition 3.1. *Let f be locally Lipschitz. Suppose that there exists a compact set Ω that is positively invariant with respect to (2.7). Then, every trajectory of (2.7) starting in Ω approaches the set M of equilibria of (2.7) as $y \to \infty$.*

Proof. First, it is derived from Lemma 2.1 that any solution U of (2.7) starting in Ω exists globally for $y \geq 0$.

Next, we will establish the asymptotic behavior of trajectories of (2.7) starting in Ω. For this purpose, we will find a function L, and a set E satisfying (3.4) and (3.5). Set

$$L(U) = L(u,v) = \int_u^{u_+} h(v)dv + \frac{1}{2}v^2, \quad U = (u,v) \in \boldsymbol{R}^2, \tag{3.6}$$

where

$$h(u) := -\lambda(u-u_-) + f(u) - f(u_-).$$

The function $L(u,v)$ is continuously differentiable and satisfies

$$\dot{L}(u,v) = -h(u)v + v(-\alpha v + h(u)) = -\alpha v^2 \leq 0, \quad \forall (u,v) \in \boldsymbol{R},$$

so that $\dot{L}(u,v)$ is semi-negative definite. Thus, we define

$$E = \{(u,v) \in \boldsymbol{R}^2 \mid \quad \dot{L}(u,v) = 0\}.$$

The set E can be simplified as follows. Suppose

$$\dot{L}(u,v) = 0.$$

Then

$$\alpha v^2 = 0$$

or

$$v = 0.$$

Thus,

$$E = \{(u,v) \in \boldsymbol{R}^2 | \quad v = 0\}. \tag{3.7}$$

Applying LaSalle's invariance principle, we conclude that: every trajectory of (2.7) approaches the largest invariant set M of E as $y \to \infty$.

Let us next show that the largest invariant set M in E coincides with the set of equilibria. This can be done by proving that no solution can stay identically in E, except constant solutions $u(y) \equiv u_i, i \in I$. Indeed, let (u,v) be a solution that stays identically in E. Then,

$$\frac{du(y)}{dy} = v(y) \equiv 0,$$

which implies

$$u \equiv u_0 = \text{constant}.$$

Thus, $(u(y), v(y)) \equiv (u_0, 0)$ and coincides with some equilibrium point. We can therefore deduce that no solution can stay identically in E, except constant solutions. This implies that the largest invariant set M in E is the set of equilibria

$$M = \{(u_i, 0) \; i \in I\}. \tag{3.8}$$

Thus, every every trajectory of (2.7) starting any point in Ω must approach M as $y \to \infty$. □

4. Existence of traveling waves

In this section, we will show that the existence of a Lax shock of (1.2) between a left-hand u_- and a right-hand state u_+ implies the existence of a traveling wave of (1.1) connecting $(u_-, 0)$ and $(u_+, 0)$.

Recall that a shock wave of (1.2) is a weak solution of the form

$$u(x,t) = \begin{cases} u_-, & x < \lambda t, \\ u_+, & x > \lambda t, \end{cases} \tag{4.1}$$

satisfying the Oleinik entropy criterion

$$\frac{f(u) - f(u_-)}{u - u_-} > \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \quad \text{for any } u \text{ between } u_+ \text{ and } u_-. \tag{4.2}$$

The condition (4.2) is equivalent to

$$\frac{f(u) - f(u_+)}{u - u_+} < \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \quad \text{for any } u \text{ between } u_+ \text{ and } u_-. \tag{4.3}$$

Assume for definitiveness that $u_+ < u_-$. Geometrically, the inequality (4.2) means that in the interval $[u_+, u_-]$, the graph of f is lying below the straight line (Δ) connecting $(u_\pm, f(u_\pm))$. Without loss of generality, we may assume that this line never meets the graph of f elsewhere.

First, we study trajectories approaching $(u_+, 0)$. So we consider the differential equation

$$\frac{dU(y)}{dy} = F(U(y)), \quad y \geq 0, \tag{4.4}$$

where

$$U = (u, v) \in D := \{(u, v) | \quad u < u_-\},$$
$$F(U) = \big(v, -\alpha v + h(u)\big), \quad h(u) = -\lambda(u - u_-) + f(u) - f(u_-).$$

It is derived from the assumptions that $(u_+, 0)$ is the unique equilibrium point of (4.3) in D.

We define

$$L(u, v) = \int_u^{u_+} h(v)dv + \frac{1}{2}v^2. \tag{4.5}$$

We claim that

$$\int_u^{u_+} h(v)dv > 0 \quad \forall u \neq u_+. \tag{4.6}$$

Indeed, it follows from the definition of λ that

$$\begin{aligned} h(u) &= -\lambda(u-u_-)+f(u)-f(u_-) \\ &= -\lambda(u-u_+)+f(u)-f(u_+) \\ &= (u-u_-)\Big(\frac{f(u)-f(u_-)}{u-u_-}-\lambda\Big) \\ &= (u-u_+)\Big(\frac{f(u)-f(u_+)}{u-u_+}-\lambda\Big). \end{aligned}$$

Thus

$$\begin{aligned} &h(u)<0 \quad u\in(u_+,u_-), \\ &h(u)>0 \quad u<u_+ \quad \text{or} \quad u>u_-. \end{aligned} \tag{4.7}$$

This establishes (4.6). Hence, the function P satisfies

$$L(u_+,0)=0, \qquad L(u,v)>0, \quad (u,v)\neq(u_+,0). \tag{4.8}$$

which means $L_1(u,v):=L(u+u_+,v)$ is positive definite and

$$\begin{aligned} \frac{dL(u,v)}{dy} &= \nabla L(u,v)\cdot(u',v') \\ &= -h(u)v+v(-\alpha v+h(u)) \\ &= -\alpha v^2\leq 0, \quad \forall(u,v)\in\boldsymbol{R}^2, \end{aligned} \tag{4.9}$$

which means $dL(u,v)/dy$ is semi-negative definite.

Choose an arbitrary $0<r<|u_- - u_+|$ and fix some $\gamma>0$, and set

$$E_r=\{(u,v)\in\boldsymbol{R}^2 \mid \ (u-u_+)^2+\gamma v^2\leq r^2\}\subset D. \tag{4.10}$$

Let ∂E_r be the boundary of E_r and let

$$\begin{aligned} m &= \min_{(u,v)\in\ \partial E_r} L(u,v) \\ &= \min_{(u,v)\in\ \partial E_r}\Big\{\int_{u_+}^{u}(v-u_+)\Big(\lambda-\frac{f(v)-f(u_+)}{u-u_+}\Big)dv+\frac{1}{2}v^2\Big\}. \end{aligned} \tag{4.11}$$

By (4.5), we have $m>0$. Take an arbitrary $c\in(0,m)$ and set

$$\Omega_c=\{(u,v)\in E_r : L(u,v)\leq c\}. \tag{4.12}$$

Obviously, Ω_c is a compact set. We claim that the set Ω_c is in the interior of E_r. Assume the contrary, then there is a point $U_0\in\Omega_c$ which lies on the boundary of E_r. Then, by definition of minimum

$$L(U_0)\geq m>c$$

which is a contradiction, since $U_0\in\Omega_c, L(U_0)\leq c$. Thus, the closed curve $L(u,v)=c$ lies entirely in the interior of E_r. Moreover, the fact that

$$\frac{dL(u(y),v(y))}{dy} \le 0$$

yields

$$L(u(y),v(y)) \le L(u(0),v(0)) \le c, \quad \forall y > 0.$$

The last inequality means that any trajectory starting in Ω_c cannot cross the closed curve $L(u,v) = c$. Therefore, the compact set Ω_c is positively invariant with respect to (4.3). According to Lemma 2.1, (4.3) has a unique solution for $y \ge 0$ whenever $U(0) \in \Omega_c$. It is derived from Proposition 3.1 that any trajectory U starting in Ω_c converges to $(u_+, 0)$ as $y \to \infty$.

On the other hand, on the boundary ∂E_r of E_r, we have

$$v^2 = \frac{1}{\gamma}(r^2 - (u-u_+)^2).$$

Thus,

$$\begin{aligned} m &= \min_{(u,v)\in\ \partial E_r} L(u,v) \\ &= \min_{(u,v)\in\ \partial E_r} \Big\{ \int_u^{u_+} h(v)dv + \frac{1}{2}v^2 \Big\}. \\ &= \min_{u\in[u_+-r,u_++r]} \Big\{ \int_u^{u_+} h(v)dv + \frac{1}{2\gamma}(r^2-(u-u_+)^2) \Big\}. \end{aligned}$$

So we reduce to consider the minimum of the function

$$\phi(u) = \int_u^{u_+} h(v)dv + \frac{1}{2\gamma}(r^2-(u-u_+)^2), \quad u \in [u_+-r, u_++r].$$

We have

$$\begin{aligned} \frac{d\phi(u)}{du} &= -h(u) - \frac{1}{\gamma}(u-u_+) \\ &= -\Big(\frac{1}{\gamma}+\lambda\Big)(u-u_+) + f(u) - f(u_+) \\ &= (u-u_+)\Big(\frac{f(u)-f(u_+)}{u-u_+} - \Big(\frac{1}{\gamma}+\lambda\Big)\Big) < 0 \end{aligned}$$

for $u \in [u_+-r, u_++r]$, provided

$$\frac{1}{\gamma} > |\lambda| + \mathrm{Lip}(f|_{[2u_+-u_-,u_-]}). \tag{4.13}$$

where $\mathrm{Lip}(f|_{[2u_+-u_-,u_-]})$ is the Lipschitz constant of f on the interval $[2u_+-u_-, u_-]$. Thus, the function $\phi(u)$ is decreasing and therefore attains minimum at $u = u_+ + r$. The value $u = u_+ + r$ corresponds to $v = 0$ to achieve m. Therefore

$$m = \int_{u_+ + r}^{u_+} h(v)dv = \int_{u_+}^{u_+ + r} -h(v)dv. \quad (4.14)$$

Since $h(u) < 0$ or $-h(u) > 0$ in (u_+, u_-), we have

$$c = \int_{u_+}^{u_+ + r - \delta/2} -h(v)dv < m,$$

for small $\delta > 0$. Moreover,

$$L(u, v = 0) = \int_{u_+}^{u} -h(v)dv \le c, \quad \forall u \in [u_+, u_+ + r - \delta/2].$$

Taking $r = u_- - u_+ - \delta/2$, we get $u_+ + r - \delta/2 = u_- - \delta$ and therefore

$$\Omega_c \supset [u_+, u_- - \delta] \times \{0\}. \quad (4.15)$$

Next, we consider the asymptotic behavior of solutions of (2.7) when $y \to -\infty$. Set $w(y) = u(-y), \quad y \ge 0$ to get

$$w_y(y) = -u_y(-y) \qquad w_{yy} = u_{yy}(-y).$$

Thus, still use the symbol u for w, the equation (2.4) for $y < 0$ becomes

$$\frac{d^2u(y)}{dy^2} - \alpha\frac{du(y)}{dy} = -\lambda(u(y) - u_-) + f(u(y)) - f(u_-), \quad y > 0. \quad (4.16)$$

Consider the linearized equation of (4.16)

$$\frac{d^2u(y)}{dy^2} - \alpha\frac{du(y)}{dy} - (f'(u_-) - \lambda)(u(y) - u_-) = 0, \quad y > 0. \quad (4.17)$$

or, by putting $w = u - u_-$

$$\frac{d^2w(y)}{dy^2} - \alpha\frac{dw(y)}{dy} - (f'(u_-) - \lambda)w(y) = 0, \quad y > 0.$$

The characteristic equation of the last second-order linear differential equation is given by

$$\beta^2 - \alpha\beta - (f'(u_-) - \lambda) = 0. \quad (4.18)$$

Assume that the shock satisfying the Lax shock inequalities. Then,

$$f'(u_-) > \lambda.$$

Thus, the equation (4.18) admits two real distinct roots

$$\beta_2 = \frac{\alpha}{2} - \sqrt{\left(\frac{\alpha}{2}\right)^2 + (f'(u_-) - \lambda)} < 0 < \frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 + (f'(u_-) - \lambda)} = \beta_1.$$

The general solution of (4.17) is therefore given by

$$u(y) = Ae^{\beta_1 y} + Be^{\beta_2 y} + u_-, \quad y > 0,$$

where A, B are constant. Therefore, the stable trajectories have the form

$$u(y) = Be^{\beta_2 y} + u_-, \quad y > 0. \tag{4.19}$$

We re-write the second-order equation (4.18) in the form of a second-order system and get the solutino of its linearized system thank to (4.19)

$$\begin{aligned} u' &= v, \\ v' &= \alpha v - \lambda(u - u_-) + f(u) - f(u_-). \end{aligned} \tag{4.20}$$

Since $\beta_2 < 0 < \beta_1$, $(u_-, 0)$ is a saddle of (4.20). Thus, we can expect that for (u_0, v_0) near the saddle $(u_-, 0)$, the trajectories of the nonlinear system (4.20) behave like the ones of the linearized system. From (4.19), we can see that the stable trajectories of the linearized system of (4.20) near $(u_-, 0)$ starting at the point $U_\varepsilon := (u_- - \varepsilon, -\beta_2\varepsilon)$ have the form

$$u(y) = -\varepsilon e^{\beta_2 y} + u_-, \quad v(y) = -\varepsilon\beta_2 e^{\beta_2 y}, \quad y > 0, \tag{4.21}$$

for ε sufficiently closed to 0. Since $u(y) \to (u_-, 0)$ as $y \to \infty$ we can expect that the trajectory of (4.20) starting at U_ε also tends to $(u_-, 0)$.

Moreover, from (4.15) we can see that the domain of attraction of $(u_+, 0)$, which contains the set Ω_c, cover the semi-closed interval $[u_+, u_-) \times \{0\}$. By taking $\varepsilon > 0$, we may hope that the starting point $(u_- - \varepsilon, -\varepsilon\beta_2)$ belongs to the domain of attraction of $(u_+, 0)$. This can be done under the following condition

(A) There exists $\varepsilon > 0$ such that the trajectory starting at $U_\varepsilon = (u_- - \varepsilon, -\beta_2\varepsilon)$ tends to $(u_-, 0)$ as $y \to -\infty$. Moreover,

$$\int_{u_-}^{u_- - \varepsilon} h(v)dv > \frac{1}{2}\Big(\frac{\alpha}{2} - \sqrt{\Big(\frac{\alpha}{2}\Big)^2 + (f'(u_-) - \lambda)}\Big)^2 \varepsilon^2.$$

Under the assumption (A), the starting point $(u_- - \varepsilon, -\beta_2\varepsilon)$ belongs to Ω_c, for δ chosen small enough. Using this point as the staring point for the equation (4.4), we obtain a traveling wave of (1.1).

The above argument leads to the following theorem which provides us with the existence of traveling waves associate with shock waves.

Theorem 4.1. *Let the function f be locally Lipschitz. Suppose there is a shock wave of (1.2) connecting the left-hand and right-hand states u_-, u_+ with the shock speed λ satisfying the Oleinik entropy condition and the Lax shock inequalities*

$$f'(u_+) < \lambda < f'(u_-).$$

Under the assumption (A), there exists a traveling wave of (1.1) connecting the states u_-, u_+.

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