THE RIEMANN PROBLEM FOR SHALLOW WATER EQUATIONS WITH DISCONTINUOUS TOPOGRAPHY

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ABSTRACT. We solve the Riemann problem for shallow water equations with discontinuous topography. Since the system is non-strictly hyperbolic and not fully conservative, there are two-parameter wave sets. The selection of admissible waves therefore plays a key role. Our construction of Riemann solutions is explicit and simple and therefore can be easily expressed in computer code so that the exact solutions can be used for tests in related numerical schemes.

CONTENTS

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1. Introduction

Let us consider the following one-dimensional supplemented shallow water equations

$$\begin{aligned}
&\partial_t h + \partial_x(hu) = 0,\\
&\partial_t(hu) + \partial_x(h(u^2 + g\frac{h}{2})) = -gh\partial_x a, \qquad (1.1)\\
&\partial_t a = 0,
\end{aligned}$$

where h is the height of the water from the bottom to the surface, u is the velocity, g is the gravity constant, and a is the height of the river bottom from a given level. The third equation $\partial_t a = 0$ is a trivial equation, and it is supplemented to the first two equations describing the dynamics of shallow water, see [16] for an initial use.

Our interest is that a will be piece-wise constant

$$a(x) = \begin{cases} a_L, & x < 0,\\ a_R, & x > 0,\end{cases}$$

where a_L, a_R are two distinct constants.

We will consider the Riemann problem for (1.1), sophisticated in some sense, with three variables (h, u, a), which is the initial-value problem for (1.1) with the initial conditions of the form

$$(h, u, a)(x, 0) = \begin{cases} (h_L, u_L, a_L), & x < 0,\\ (h_R, u_R, a_R), & x > 0.\end{cases} \qquad (1.2)$$

Since a is discontinuous, the system (1.1) cannot be written in conservation form. The usual notion of weak solutions for systems of conservation laws does not apply. However, the equations still make sense as a measure within the framework introduced by Dal Maso, LeFloch, and Murat [6].

Why we add the trivial equation $\partial_t a = 0$ to the shallow water equations? As we will see later, this will transform a strictly hyperbolic system of balance laws in *nonconservative form* to a *non-strictly hyperbolic* system of balance laws in producing a linearly degenerate characteristic field.

On the other hand, suppose that we have a discontinuity of (1.1) propagating with a speed λ. Then, we make use the Rankine-Hugoniot relation associated with the third equation in (1.1), which reads

$$-\lambda[a] = 0, \qquad (1.3)$$

where $[a] := a_+ - a_-$ is the jump of the bottom level function a, and $a_\pm$ denotes its left- and right-hand traces.

Some conclusions can be derived from the equation (1.4):

(i) either the component a remains constant across the shock,
(ii) or a changes its levels across the discontinuity and the discontinuity is stationary, i.e., the speed λ vanishes.

The above conclusions motivate us to define all admissible elementary waves of the system (1.1). Let us assume first that the bottom level a remains constant across a discontinuity. Then, a should be constant in a neighborhood of the discontinuity. Eliminating a from (1.1), we obtain the following system of two conservation laws

$$\begin{aligned} &\partial_t h + \partial_x(hu) = 0,\\ &\partial_t(hu) + \partial_x(h(u^2 + g\frac{h}{2})) = 0, \end{aligned} \tag{1.4}$$

Thus, the left- and right-hand states are related by the Rankine Hugoniot relations corresponding to (1.4)

$$\begin{aligned} &-\lambda[h] + [hu] = 0,\\ &-\lambda[hu] + [h(u^2 + g\frac{h}{2})] = 0, \end{aligned} \tag{1.5}$$

where $[h] := h_+ - h_+$, etc.

Suppose next that the component a is discontinuous and that, therefore, the discontinuity speed vanishes. The solution is independent of time, ad it is natural to search for a solution as the limit of a sequence of time-independent smooth solutions of (1.1). And we will see how the variable components are related shortly.

Suppose that $(x,t) \mapsto (h,u,a)$ is a smooth solution of (1.1). Then, for this solution, the system (1.1) can be written in the following form as a system of conservation laws for conservative variables (h,u,a):

$$\begin{aligned} &\partial_t h + \partial_x(hu) = 0,\\ &\partial_t u + \partial_x\big(\frac{u^2}{2} + g(h+a)\big) = 0,\\ &\partial_t a = 0. \end{aligned} \tag{1.6}$$

Thus, time-independent solutions of (1.1) satisfy

$$\begin{aligned} &(hu)' = 0,\\ &\big(\frac{u^2}{2} + g(h+a)\big)' = 0, \end{aligned} \tag{1.7}$$

where "′" stands for the derivative with respect to x. Trajectories of the system of two differential equations (1.7) passing through each point

(h_0, u_0, a_0) can be obtained easily and satisfy

$$\begin{aligned} &hu = h_0 u_0, \\ &\frac{u^2}{2} + g(h+a) = \frac{u_0^2}{2} + g(h_0 + a_0). \end{aligned} \tag{1.8}$$

It is derived from (1.8) that the trajectories of (1.7) can be expressed in the form $u = u(h), a = a(h)$. Now, letting $h \to h_\pm$ and setting $u_\pm = u(h_\pm)), a_\pm = a(h_\pm)$, we see that the states $(h_\pm, u_\pm, a_\pm)$ satisfy the Rankine-Hugoniot relations associated with (1.6), but with zero shock speed:

$$\begin{aligned} &[hu] = 0, \\ &[\frac{u^2}{2} + g(h+a)] = 0, \end{aligned} \tag{1.9}$$

We therefore define elementary waves as follows.

Definition 1.1. *The admissible waves for the system (1.1) are the following ones:*

(a) *the* **rarefaction waves**, *which are smooth solutions of (1.1) with constant component a depending only on the self-similarity variable x/t;*
(b) *the* **shock waves** *which satisfy (1.5) and Lax shock inequalities (see* [15]*) and have constant component a;*
(c) *and the* **stationary waves** *which have zero propagation speed and satisfy (1.9).*

As seen later, the system (1.1) is *not strictly hyperbolic*, as in our previous work [17]. In [17], we considered the Riemann problem in a nozzle with variable cross-section and constructed all the Riemann solutions. Other related works can be seen from [18, 12, 11, 7, 1, 2]. Numerical methods for systems of balance laws with source terms have been considered by many authors, see [9, 10, 4, 5, 8, 3, 13, 14], etc.

2. Backgrounds

2.1. **Non-strictly hyperbolic system.** Setting the variable $U = (h, u, a)$, we want to obtain the nonconservative form of the system (1.1). It is derived from (1.6) that for smooth solutions, the system (1.1) can be written as

$$\begin{aligned} &\partial_t h + u\partial_x h + h\partial_x u = 0, \\ &\partial_t u + g\partial_x h + u\partial_x u + g\partial_x a = 0, \\ &\partial_t a = 0. \end{aligned} \tag{2.1}$$

The system (2.1) can be written in the nonconservative form

$$\partial_t U + A(U)\partial_x U = 0, \tag{2.2}$$

where the Jacobian matrix $A(U)$ is given by

$$A(U) = \begin{pmatrix} u & u & 0 \\ g & u & g \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.3}$$

To find the characteristic fields, we find first for eigenvalues of A by solving

$$|A(U) - \lambda I| = 0 \tag{2.4}$$

which gives three eigenvalues

$$\lambda_1(U) := u - \sqrt{gh} < \lambda_2(U) := u + \sqrt{gh}, \quad \lambda_3(U) := 0, \tag{2.5}$$

so that the corresponding eigenvectors can be chosen as

$$\begin{aligned} &r_1(U) := (h, -\sqrt{gh}, 0)^t, \quad r_2(U) := (h, \sqrt{gh}, 0)^t, \\ &r_3(U) := (gh, -gu, u^2 - gh)^t. \end{aligned} \tag{2.6}$$

Thus, we can see that the first and the third characteristic fields may coincide:

$$(\lambda_1(U), r_1(U)) = (l_3(U), r_3(U)) \tag{2.7}$$

on a certain hyper-surface of the space (h, u, a), which can be identified as

$$\mathcal{C}_+ := \{(h, u, a)| \quad u = \sqrt{gh}\}. \tag{2.8}$$

Similarly, the second and the third characteristic fields may coincide:

$$(\lambda_2(U), r_2(U)) = (l_3(U), r_3(U)) \tag{2.9}$$

on a certain hyper-surface of the space (h, u, a), which can be identified as

$$\mathcal{C}_- := \{(h, u, a)| \quad u = -\sqrt{gh}\}. \tag{2.10}$$

Evidently, the third one (λ_3, r_3) is linearly degenerate. And we have

$$-\nabla\lambda_1(U) \cdot r_1(U) = \nabla\lambda_2(U) \cdot r_2(U) = \frac{3}{2}\sqrt{gh} \neq 0, \quad h > 0.$$

The last conclusion implies that the first and the second characteristic fields (λ_1, r_1), (λ_2, r_2) are genuinely nonlinear in the open half-space $\{(h, u, a)| \quad h > 0\}$.

Now, it is convenient to set

$$\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_- = \{(h, u, a)| \quad u^2 - gh = 0\}, \tag{2.11}$$

which is the hyper-surface on which the system fails to be strictly hyperbolic.

Consequently, we have

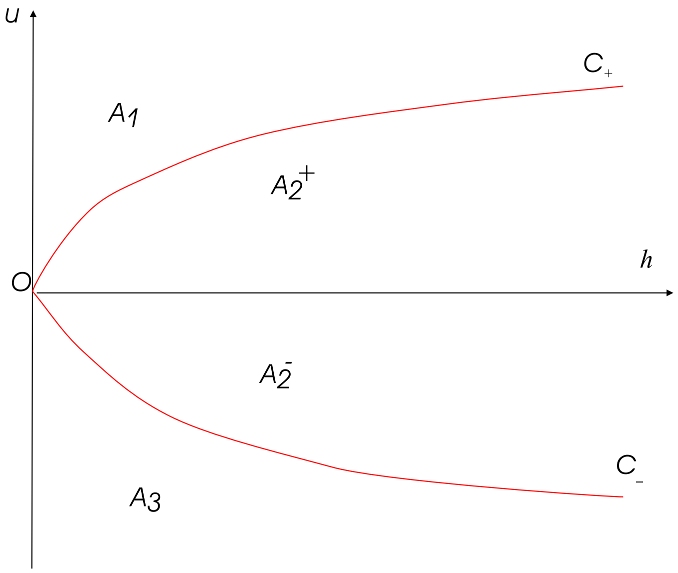

FIGURE 1. Projection of strictly hyperbolic areas in the (h, u)-plane

Proposition 2.1. *There exists a hyper-surface $\mathcal{C}_+$ of the space (h, u, a) on which the first and the third characteristic fields coincide and there exists a hyper-surface $\mathcal{C}_-$ of the space (h, u, a) on which the second and the third characteristic fields coincide. Consequently, the system (1.1) is non-strictly hyperbolic. (See Figure* **??***).*

We have seen that the system lacks the strict hyperbolicity only on the surface $\mathcal{C}$. However, this surface divides the phase domain into three sub-domains which are disjoint regions, or areas, denoted by A_1, A_2 and A_3, so that in each region the system is strictly hyperbolic. More precisely,

$$\begin{aligned} A_1 &:= \{(h,u,a) \in \boldsymbol{R}_+ \times \boldsymbol{R} \times \boldsymbol{R}_+ | \quad \lambda_2(U) > \lambda_1(U) > \lambda_3(U)\}, \\ A_2 &:= \{(h,u,a) \in \boldsymbol{R}_+ \times \boldsymbol{R} \times \boldsymbol{R}_+ | \quad \lambda_2(U) > \lambda_3(U) > \lambda_1(U)\}, \\ A_2^+ &:= \{(h,u,a) \in A_2 | \quad u > 0\}, \\ A_2^- &:= \{(h,u,a) \in A_2 | \quad u < 0\}, \\ A_3 &:= \{(h,u,a) \in \boldsymbol{R}_+ \times \boldsymbol{R} \times \boldsymbol{R}_+ | \quad \lambda_3(U) > \lambda_2(U) > \lambda_1(U)\}. \end{aligned} \tag{2.12}$$

As the strictly hyperbolic regions are not connected, the Riemann problems becomes challenging.

2.2. **The wave curves.** In this section we investigate properties of the curves of admissible waves. The admissible waves are defined earlier.

First, let us consider shock curves from a given left-hand state $U_0 = (h_0, u_0, a_0)$ consisting of all right-hand states $U = (h, u, a)$ that can be connected to U_0 by a shock wave. Thus, it is derived from (1.6) that U and U_0 are related by the Rankine-Hugoniot relations

$$\begin{aligned} &-\bar{\lambda}[h] + [hu] = 0,\\ &-\bar{\lambda}[hu] + [h(u^2 + g\frac{h}{2}] = 0, \end{aligned} \tag{2.13}$$

where $[h] = h - h_0$, etc, and $\bar{\lambda} = \bar{\lambda}(U_0, U)$ is the shock speed.

A straightforward calculation derives from the Rankine-Hugoniot relations (2.13) that the Hugoniot set consisting of two curves when restricting to the (h, u) plane starting from U_0 is given by

$$u = u_0 \pm \sqrt{\frac{g}{2}}(h - h_0)\sqrt{\Big(\frac{1}{h} + \frac{1}{h_0}\Big)}. \tag{2.14}$$

Moreover, along these two curves it holds

$$\frac{du}{dh} = \pm\sqrt{\frac{g}{2}}\Big\{\sqrt{\frac{1}{h} + \frac{1}{h_0}} - (h - h_0)\frac{1}{2h^2\sqrt{\frac{1}{h} + \frac{1}{h_0}}}\Big\} \to \pm\sqrt{\frac{g}{h_0}}$$

as $h \to h_0$. Thus, using the fact that the ith-Hugoniot curve is tangent to $r_i(U_0)$ at U_0, we conclude that the first Hugoniot curve associate with the first characteristic field is defined by

$$\mathcal{H}_1(U_0): \quad u := u_1(h, U_0) = u_0 - \sqrt{\frac{g}{2}}(h - h_0)\sqrt{\Big(\frac{1}{h} + \frac{1}{h_0}\Big)}, \quad h \geq 0, \tag{2.15}$$

and the second Hugoniot curve associate with the second characteristic field is defined by

$$\mathcal{H}_2(U_0): \quad u := u_2(h, U_0) = u_0 + \sqrt{\frac{g}{2}}(h - h_0)\sqrt{\Big(\frac{1}{h} + \frac{1}{h_0}\Big)}, \quad h \geq 0. \tag{2.16}$$

Along the Hugoniot curves $\mathcal{H}_1, \mathcal{H}_2$, the corresponding shock speeds are given by

$$\begin{aligned} \bar{\lambda}_{1,2}(U_0, U) &= \frac{hu_{1,2} - h_0 u_0}{h - h_0}\\ &= u_0 \mp \sqrt{\frac{g}{2}\Big(h + \frac{h^2}{h_0}\Big)}, \quad h \geq 0, \end{aligned} \tag{2.17}$$

The shock speed $\bar{\lambda}_i(U_0, U)$ is required to satisfy the Lax shock inequalities

$$\lambda_i(U) < \bar{\lambda}_i(U_0, U) < \lambda_i(U_0), \quad i = 1, 2. \tag{2.18}$$

Thus, we deduce that the 1-shock curve $\mathcal{S}_1(U_0)$ starting from a left-hand state U_0 consisting of all right-hand states U that can be connected to U_0 by a Lax shock associated with the first characteristic field is given by

$$\mathcal{S}_1(U_0): \quad u = u_1(h, U_0) = u_0 - \sqrt{\frac{g}{2}}(h - h_0)\sqrt{\Big(\frac{1}{h} + \frac{1}{h_0}\Big)}, \quad h > h_0, \tag{2.19}$$

Similarly, the 2-shock curve $\mathcal{S}_2(U_0)$ starting from a left-hand state U_0 consisting of all right-hand states U that can be connected to U_0 by a Lax shock associated with the second characteristic field is given by

$$\mathcal{S}_2(U_0): \quad u = u_1(h, U_0) = u_0 + \sqrt{\frac{g}{2}}(h - h_0)\sqrt{\Big(\frac{1}{h} + \frac{1}{h_0}\Big)}, \quad h < h_0, \tag{2.20}$$

We can summarize the above analysis in the following proposition.

Proposition 2.2. *Given a left-hand state U_0. The 1-shock curve $\mathcal{S}_1(U_0)$ consisting of all right-hand states U that can be connected to U_0 by a Lax shock is given by*

$$\mathcal{S}_1(U_0): \quad u = u_1(h, U_0) = u_0 - \sqrt{\frac{g}{2}}(h - h_0)\sqrt{\Big(\frac{1}{h} + \frac{1}{h_0}\Big)}, \quad h > h_0,$$

The 2-shock curve $\mathcal{S}_2(U_0)$ consisting of all right-hand states U that can be connected to U_0 by a Lax shock is given by

$$\mathcal{S}_2(U_0): \quad u = u_1(h, U_0) = u_0 + \sqrt{\frac{g}{2}}(h - h_0)\sqrt{\Big(\frac{1}{h} + \frac{1}{h_0}\Big)}, \quad h < h_0.$$

Reserving the order of Lax shock inequalities (2.18), we can also conclude that the backward 1-shock curve $\mathcal{S}_1^B(U_0)$ starting from a right-hand state U_0 consisting of all left-hand states U that can be connected to U_0 by a Lax shock associated with the first characteristic field is given by

$$\mathcal{S}_1^B(U_0): \quad u = u_1(h, U_0) = u_0 - \sqrt{\frac{g}{2}}(h - h_0)\sqrt{\Big(\frac{1}{h} + \frac{1}{h_0}\Big)}, \quad h < h_0, \tag{2.21}$$

Similarly, the backward 2-shock curve $\mathcal{S}_2^B(U_0)$ starting from a right-hand state U_0 consisting of all left-hand states U that can be connected to U_0 by a Lax shock associated with the second characteristic field is given by

$$\mathcal{S}_2^B(U_0): \quad u = u_1(h, U_0) = u_0 + \sqrt{\frac{g}{2}}(h - h_0)\sqrt{\Big(\frac{1}{h} + \frac{1}{h_0}\Big)}, \quad h > h_0, \tag{2.22}$$

Next, let us consider rarefaction waves , i.e., smooth self-similar solutions to the system (1.1) associate with the genuinely nonlinear characteristic fields. These waves satisfy the ordinary differential equations

$$\frac{dU}{d\xi} = \frac{r_i(U)}{\nabla\lambda_i \cdot r_i(U)}, \quad \xi = x/t, \quad i = 1, 2. \tag{2.23}$$

Thus, for waves in the first family, we have

$$\begin{aligned} \frac{dh(\xi)}{d\xi} &= -\frac{2h(\xi)}{3\sqrt{gh(\xi)}} = -\frac{2}{3\sqrt{g}}\sqrt{h(\xi)}, \\ \frac{du(\xi)}{d\xi} &= \frac{-2\sqrt{gh(\xi)}}{-3\sqrt{gh(\xi)}} = \frac{2}{3}, \\ \frac{da(\xi)}{d\xi} &= 0. \end{aligned} \tag{2.24}$$

It is derived from (2.24) that

$$\frac{du}{dh} = -\sqrt{\frac{g}{h}}. \tag{2.25}$$

Therefore, the trajectory passing through a given point $U_0 = (h_0, u_0, a_0)$ is given by

$$u = u_0 - 2\sqrt{g}(\sqrt{h} - \sqrt{h_0}). \tag{2.26}$$

Moreover, characteristic speed should be increasing through a rarefaction fan, i.e.,

$$\lambda_1(U) \geq \lambda_1(U_0), \tag{2.27}$$

which implies

$$h \geq h_0, \tag{2.28}$$

Thus, we can define a rarefaction curve $\mathcal{R}_1(U_0)$ starting from a given left-hand state U_0 consisting if all the right-hand states U that can be connected to U_0 by a rarefaction wave associate with the first characteristic field as

$$\mathcal{R}_1(U_0): \quad u = v_1(h, U_0) := u_0 - 2\sqrt{g}(\sqrt{h} - \sqrt{h_0}), \quad h \leq h_0. \quad (2.29)$$

Any 1-rarefaction wave can then be determined by

$$u = u_0 + \frac{2}{3}\Big(\frac{x}{t} - \frac{x_0}{t_0}\Big) \quad (2.30)$$

and h is determined by the equation (2.29), while the component a remains constant.

Similarly, we can define a rarefaction curve $\mathcal{R}_2(U_0)$ starting from a given left-hand state U_0 consisting if all the right-hand states U that can be connected to U_0 by a rarefaction wave associate with the second characteristic field as

$$\mathcal{R}_2(U_0): \quad u = v_2(h, U_0) := u_0 + 2\sqrt{g}(\sqrt{h} - \sqrt{h_0}), \quad h \geq h_0. \quad (2.31)$$

The u-component of any 2-rarefaction wave can then be determined by (2.30) and the h-component is given by (2.31).

We can summarize the above analysis on rarefaction waves in the following proposition.

Proposition 2.3. *Given a left-hand state U_0. The 1-rarefaction curve $\mathcal{R}_1(U_0)$ consisting of all right-hand states U that can be connected to U_0 by a rarefaction wave associate with the first characteristic field is given by*

$$\mathcal{R}_1(U_0): \quad u = v_1(h, U_0) := u_0 - 2\sqrt{g}(\sqrt{h} - \sqrt{h_0}), \quad h \leq h_0.$$

The 2-rarefaction curve $\mathcal{R}_2(U_0)$ consisting of all right-hand states U that can be connected to U_0 by a rarefaction wave associate with the second characteristic field is given by

$$\mathcal{R}_2(U_0): \quad u = v_2(h, U_0) := u_0 + 2\sqrt{g}(\sqrt{h} - \sqrt{h_0}), \quad h \geq h_0.$$

Similarly, given a right-hand state U_0. The 1-rarefaction curve $\mathcal{R}_1^(U_0)$ consisting of all left-hand states U that can be connected to U_0 by a rarefaction wave associate with the first characteristic field is given by

$$\mathcal{R}_1^B(U_0): \quad u = v_1(h, U_0) := u_0 - 2\sqrt{g}(\sqrt{h} - \sqrt{h_0}), \quad h \geq h_0. \quad (2.32)$$

The 2-rarefaction curve $\mathcal{R}_2^B(U_0)$ consisting of all left-hand states U that can be connected to U_0 by a rarefaction wave associate with the second characteristic field is given by

$$\mathcal{R}_2^B(U_0): \quad u = v_2(h, U_0) := u_0 + 2\sqrt{g}(\sqrt{h} - \sqrt{h_0}), \quad h \le h_0. \tag{2.33}$$

It is convenient to define the wave curves

$$\begin{aligned} \mathcal{W}_1(U_0) &= \mathcal{S}_1(U_0) \cup \mathcal{R}_1(U_0), \\ \mathcal{W}_1^B(U_0) &= \mathcal{S}_1^B(U_0) \cup \mathcal{R}_1^B(U_0), \\ \mathcal{W}_2(U_0) &= \mathcal{S}_2(U_0) \cup \mathcal{R}_2(U_0), \\ \mathcal{W}_2^B(U_0) &= \mathcal{S}_2^B(U_0) \cup \mathcal{R}_2^B(U_0). \end{aligned} \tag{2.34}$$

The monotony property of the wave curves are given by

Lemma 2.4. *The wave curve $\mathcal{W}_1(U_0)$ can be parameterized as $h \mapsto u = u(h), h > 0$, which is a strictly convex and strictly decreasing function. The wave curve $\mathcal{W}_2(U_0)$ can be parameterized as $h \mapsto u = u(h), h > 0$, which is a strictly concave and strictly decreasing function.*

Proof. We need only to prove for the 1-wave curve $\mathcal{W}_1(U_0)$, since it is similar for $\mathcal{W}_2(U_0)$.

Actually, in the shock part $\mathcal{S}_1(U_0)$, we have

$$\frac{du}{dh} = -\sqrt{\frac{g}{2}}\frac{\dfrac{1}{2h} + \dfrac{1}{h_0} + \dfrac{h_0}{2h^2}}{\sqrt{\dfrac{1}{h} + \dfrac{1}{h_0}}} < 0.$$

And in the rarefaction part $\mathcal{R}_1(U_0)$, we have

$$\frac{du}{dh} = -\sqrt{\frac{g}{h}} < 0.$$

This establishes the monotony property of $\mathcal{W}_1(U_0)$.

The convexity of $\mathcal{W}_1(U)$ follows from the fact that du/dh is increasing, since

$$\frac{d^2u}{dh^2} > 0.$$

Indeed, on the shock part $\mathcal{S}_1(U_0)$, it holds

$$\frac{d^2u}{dh^2} = \sqrt{\frac{g}{2}}\frac{\left(\frac{1}{2h^2} + \frac{h_0}{h^3}\right)\sqrt{\frac{1}{h} + \frac{1}{h_0}} + \frac{1}{2h^2\sqrt{\frac{1}{h}+\frac{1}{h_0}}}\left(\frac{1}{2h} + \frac{1}{h_0} + \frac{h_0}{2h^2}\right)}{\frac{1}{h} + \frac{1}{h_0}} > 0$$

and on the rarefaction part $\mathcal{R}_1(U_0)$, it holds

$$\frac{d^2u}{dh^2} = \frac{\sqrt{g}}{2h^{3/2}} > 0.$$

This terminates the proof.

□

Next, let us consider the 3-curve from a state U_0 consisting of all states U that can be connected to U_0 by a stationary wave. As seen from (1.9), U and U_0 are related by the Rankine-Hugoniot relations

$$
\begin{aligned}
&[hu] = 0\\
&[\frac{u^2}{2} + g(h+a)] = 0.
\end{aligned}
\tag{2.35}
$$

This leads to the definition of a curve parameterized in h:

$$
\begin{aligned}
\mathcal{W}_3(U_0): \quad & u = u(h) = \frac{h_0 u_0}{h},\\
& a = a(h) = a_0 + \frac{u^2 - u_0^2}{2g} + h - h_0.
\end{aligned}
\tag{2.36}
$$

3. Admissibility conditions for stationary waves

3.1. **Two possible stationary jumps.** As seen from the last section, the two states of a stationary waves are constraint by the Rankine-Hugoniot relations (2.35). From a given left-hand state, we have to determine the right-hand state which has three component to be determined by two equations of (2.35). Moreover, since the component a changes only through stationary waves which propagate with zero speed, for given bottom levels $a_\pm$, we should solve for u and h in terms of a.

Thus, we can write (2.35) in the following form

$$
\begin{aligned}
&u = \frac{h_0 u_0}{h},\\
&a_0 - a + \frac{u^2 - u_0^2}{2g} + h - h_0 = 0.
\end{aligned}
\tag{3.1}
$$

Substitute for u, and re-arranging terms, we obtain

$$
\begin{aligned}
&u = \frac{h_0 u_0}{h},\\
&a_0 - a + \frac{u_0^2}{2g}\Big(\frac{h_0^2}{h^2} - 1\Big) + h - h_0 = 0.
\end{aligned}
\tag{3.2}
$$

Thus, we look for zeros of the function

$$
\varphi(h) = a_0 - a + \frac{u_0^2}{2g}\Big(\frac{h_0^2}{h^2} - 1\Big) + h - h_0. \tag{3.3}
$$

Set

$$
\begin{aligned}
h_{\min}(U_0) &:= \Big(\frac{u_0^2 h_0^2}{g}\Big)^{1/3},\\
a_{\min}(U_0) &:= a_0 + \frac{u_0^2}{2g}\Big(\frac{h_0^2}{h_{\min}^2} - 1\Big) + h_{\min} - h_0.
\end{aligned}
\tag{3.4}
$$

Properties of the function φ in (3.3) is given by the following lemma.

Lemma 3.1. *Suppose $u_0 \neq 0$. The function $\varphi(h), h > 0$ is smooth, convex, is decreasing in the interval $(-\infty, h_{\min})$ and is increasing in the interval $(h_{\min}, \infty)$, and satisfies the limit conditions*

$$
\lim_{h\to 0} \varphi(h) = \lim_{h\to\infty} \varphi(h) = \infty. \tag{3.5}
$$

Moreover, if $a \geq a_{\min}$, the function φ has two zeros $h_*(U_0), h^*(U_0)$ such that $h_*(U_0) \leq h_{\min}(U_0) \leq h^*(U_0)$. The inequalities are strict whenever $a > a_{\min}(U_0)$.

Proof. The smoothness of the function φ and the limit conditions are obvious. And

$$
\frac{d\varphi(h)}{dh} = -\frac{u_0^2 h_0^2}{g h^3} + 1
$$

(for $u_0 \neq 0$) is positive if and only if

$$
h > \Big(\frac{u_0^2 h_0^2}{g}\Big)^{1/3} = h_{\min}(U_0).
$$

This establish the monotonicity property of φ. Furthermore,

$$
\frac{d^2\varphi(h)}{dh^2} = \frac{3u_0^2 h_0^2}{g h^4} \geq 0.
$$

The last inequality establishes the convexity of φ. If $a > a_{\min}(U_0)$, then $\varphi(h_{\min}(U_0)) < 0$. The other conclusions follow immediately.

□

Moreover, it is straightforward to check that

Lemma 3.2. *We have the following comparisons*

– *Comparisons for $h_{\min}$:*

$$
\begin{aligned}
h_{\min}(U_0) &> h_0, \quad \text{if} \quad U_0 \in A_1 \cup A_3,\\
h_{\min}(U_0) &< h_0, \quad \text{if} \quad U_0 \in A_2,\\
h_{\min}(U_0) &= h_0, \quad \text{if} \quad U_0 \in \mathcal{C},
\end{aligned}
\tag{3.6}
$$

– *Comparisons for the zeros $h_{1,2}$:*

(i) *If $a > a_0$, then*

$$
h_*(U_0) < h_0 < h^*(U_0). \tag{3.7}
$$

(ii) *If $a < a_0$, then*

$$\begin{aligned} h_0 &< h_*(U_0) \quad \textit{for} \quad U_0 \in A_1 \cup A_3, \\ h_0 &> h^*(U_0) \quad \textit{for} \quad U_0 \in A_2. \end{aligned} \tag{3.8}$$

– *Comparisons for $a_{\min}(U)$:*

$$\begin{aligned} a_{\min}(U) &< a, \quad (h,u) \in A_i,\ i = 1,2,3, \\ a_{\min}(U) &= a, \quad (h,u) \in \mathcal{C}_\pm, \\ a_{\min}(U) &= 0, \quad h = 0 \quad \textit{or} \quad u = 0. \end{aligned} \tag{3.9}$$

The states that can be connected by stationary waves are characterized as in the following proposition.

Theorem 3.3. *Given a left-hand state $U_0 = (h_0, u_0, a_0)$ with a right-hand bottom level a.*

(i) *If $u_0 \neq 0$, $a > a_{\min}(U_0)$, then there are two distinct right-hand states*

$$U_{1,2} := (h_{1,2}(U_0), u_{1,2}(U_0), a)$$

where $u_i(U_0) := h_0 u_0 / h_i(U_0), i = 1, 2$, that can be connected to U_0 by a stationary wave satisfying the Rankine-Hugoniot relations.

(ii) *If $u_0 \neq 0$, $a = a_{\min}(U_0)$, the two states in (i) coincide and we obtain a unique stationary wave.*

(iii) *If $u_0 \neq 0$, $a < a_{\min}(U_0)$, there is no stationary wave from U_- to a state with level a.*

(iv) *If $u_0 = 0$, there is only one stationary jump defined by*

$$u = u_0 = 0, \quad h = h_0 + a - a_0.$$

The following proposition provides us with location of the resulted states by stationary jumps.

Proposition 3.4. *For $u_0 \neq 0$, the state $(h_1(U_0), u_1(U_0))$ belongs to A_1 if $u_0 < 0$, and belongs to A_3 if $u_0 > 0$, while the state $(h_2(U_0), u_2(U_0))$ always belongs to A_2. Moreover, we have*

$$\begin{aligned} (h_{\min}(U_0), u = h_0 u_0 / h_{\min}(U_0)) &\in \mathcal{C}^+ \quad \textit{if} \quad u_0 > 0, \\ (h_{\min}(U_0), u = h_0 u_0 / h_{\min}(U_0)) &\in \mathcal{C}^- \quad \textit{if} \quad u_0 < 0, \end{aligned} \tag{3.10}$$

It is interesting that the shock speed in genuinely nonlinear characteristic fields will change sign along shock curves. Therefore, it interchanges the order with the linearly degenerate field, as we will see in the following theorem.

Theorem 3.5. *(a) If $U_0 \in A_1$, there exists $\tilde{U}_0 \in \mathcal{S}_1(U_0) \cap A_2^+$ corresponding to $h = \tilde{h} > h_0$ such that*

$$\begin{aligned}
&\bar{\lambda}_1(U_0, \tilde{U}_0) = 0, \\
&\bar{\lambda}_1(U_0, U) > 0, \quad U \in \mathcal{S}_1(U_0), h \in (h_0, \tilde{h}_0), \\
&\bar{\lambda}_1(U_0, U) < 0, \quad U \in \mathcal{S}_1(U_0), h \in (\tilde{h}_0, +\infty).
\end{aligned} \tag{3.11}$$

If $U_0 \in A_2 \cup A_3$, then

$$\bar{\lambda}_1(U_0, U) < 0, \quad \forall U \in \mathcal{S}_1(U_0). \tag{3.12}$$

(b) If $U_0 \in A_3$, there exists $\bar{U}_0 \in \mathcal{S}_2^B(U_0) \cap \in A_2^-$ corresponding to $h = \bar{h} > h_0$ such that

$$\begin{aligned}
&\bar{\lambda}_2(U_0, \bar{U}_0) = 0, \\
&\bar{\lambda}_2(U_0, U) > 0, \quad U \in \mathcal{S}_2^B(U_0), h \in (h_0, \bar{h}_0), \\
&\bar{\lambda}_2(U_0, U) < 0, \quad U \in \mathcal{S}_2^B(U_0), h \in (\bar{h}_0, +\infty).
\end{aligned} \tag{3.13}$$

If $U_0 \in A_1 \cup A_2$, then

$$\bar{\lambda}_2(U_0, U) > 0, \quad \forall U \in \mathcal{S}_2^B(U_0). \tag{3.14}$$

3.2. **Two-parameter wave sets.** From proposition 3.3, the the arguments from the earlier section, we can now perform composites of waves. It turns out that two-parameter wave sets can be constructed. To see this we will illustrate one case only.

Suppose $U_0 = (h_0, u_0, a_0) \in A_2^+$. A stationary wave from U_0 to a state $U_m = (h_m, u_m, a_m) \in A_2^+$ using h^*, followed by another stationary wave from U_m to $U \in A_1$ using the correspondence h_*, then continued by 1 waves, as in A_1 the characteristic speed is positive. As a_m can vary, the set of U forms a two-parameter set of composite waves between the first and the third waves.

To make the Riemann problem well-imposed, it is necessary to set up certain admissibility criterion which is capable to exclude the above situations.

3.3. **The monotonicity criterion.** As seen in the previous section that the Riemann problem for (1.1) may admit up to a one-parameter family of solutions. This phenomenon can be avoided by requiring Riemann solutions to satisfy a monotone condition on the component a.

(MC) (Monotonicity Criterion) Along any stationary curve $\mathcal{W}_3(U_0)$, the bottom level a is monotone as a function of h. The total variation of the bottom level component of any Riemann solution must not exceed (and, therefore, is equal to) $|a_L - a_R|$, where a_l, a_r are left-hand and right-hand cross-section levels.

A similar criterion was used by Isaacson and Temple [11, 12] and by LeFloch and Thanh [17], and by Goatin and LeFloch [7].

Under the transformation by the transformation if necessary

$$x \to -x, u \to -u,$$

a right-hand state $U = (h, u, a)$ will become a left-hand state of the form $U' = (h, -u, a)$. Therefore, it is not restrictive to assume that

$$a_L < a_R. \tag{3.15}$$

Lemma 3.6. *The Monotonicity Criterion implies that any stationary shock does not cross the boundary of strict hyperbolicity. In other words:*

(i) *If $U_0 \in A_1 \cup A_3$, then only the stationary shock based on the value $h_*(U_0)$ is allowed.*
(ii) *If $U_0 \in A_2$, then only the stationary shock using $h^*(U_0)$ is allowed.*

Proof. Recall that the Rankine-Hugoniot relations associate the linearly degenerate field (2.36) implies that the component a can be expressed as a function of h:

$$a = a(h) = a_0 + \frac{u^2 - u_0^2}{2g} + h - h_0,$$

where

$$u = u(h) = \frac{h_0 u_0}{h}.$$

Thus, taking the derivative of a with respect to h, we have

$$\begin{aligned} a'(h) &= \frac{uu'(h)}{g} + 1 = -u\frac{h_0 u_0}{gh^2} + 1 \\ &= -\frac{u^2}{gh} + 1 \end{aligned}$$

which is positive (negative) if and only if

$$u^2 - gh < 0$$

or $(h, u, a) \in A_2$ ($\in A_1$ or $\in A_3$). Thus, in order that a' keeps the same sign, the point (h, u, a) must remain on the same side as (h_0, u_0, a_0) with respect to $\mathcal{C}_\pm$. The conclusions of (i) and (ii) then follow.

□

It follows from Lemma 3.6 that for a given $U_0 = (h_0, u_0, a_0) \in A_i, i = 1, 2, 3$, and a level a, we can define a unique point $U = (h, u, a)$ so that the two points U_0, U can be connected by a stationary wave satisfying the (MC) criterion. And so we have a mapping

$$\begin{aligned} SW(.,a) : [0,\infty) \times \boldsymbol{R} \times \boldsymbol{R}_+ &\to [0,\infty) \times \boldsymbol{R} \times \boldsymbol{R}_+ \\ U_0 = (h_0, u_0, a_0) &\mapsto SW(U_0, a) = \{U = (h, u, a)\}, \end{aligned} \tag{3.16}$$

such that U_0 and U can be connected by a stationary wave satisfying the (MC) condition. Observe that this mapping is single-valued except on the hypersurface $\mathcal{C}$, where it has two-values.

Let us use the following notation: $W_i(U_0, U)$ will stand for the ith wave from a left-hand state U_0 to the right-hand state U, $i = 1, 2, 3$. To represent the fact that the wave $W_i(U_1, U_2)$ is followed by the wave $W_j(U_2, U_3)$, we use the notation:

$$W_i(U_1, U_2) \oplus W_j(U_2, U_3). \tag{3.17}$$

4. The Riemann problem

In this section we will provide constructions of Riemann solutions consisting of Lax shock, rarefaction waves, and stationary waves satisfying the admissibility condition (MC).

In general, solutions of the Riemann problem are shown to exist in a small neighborhood of some given point. This is to say that the right-hand states U_R lies in a small neighborhood of the given left-hand state U_L. However, the the system (1.1), we can almost establish a full domain of existence for a given left-hand state. This can be done by determining the range of right-hand states so that a Riemann solution exists.

4.1. **Solutions without repetition in wave family.** In this subsection we will construct solutions containing only one wave corresponding to each characteristic field which is identified as each family of waves. This structure of solutions is known as usual in the theory of hyperbolic system of conservation laws. In the next subsection we will study solutions that contain up to two waves in the same family. The following theorem deals with the case where the left-hand state U_L may be in A_1.

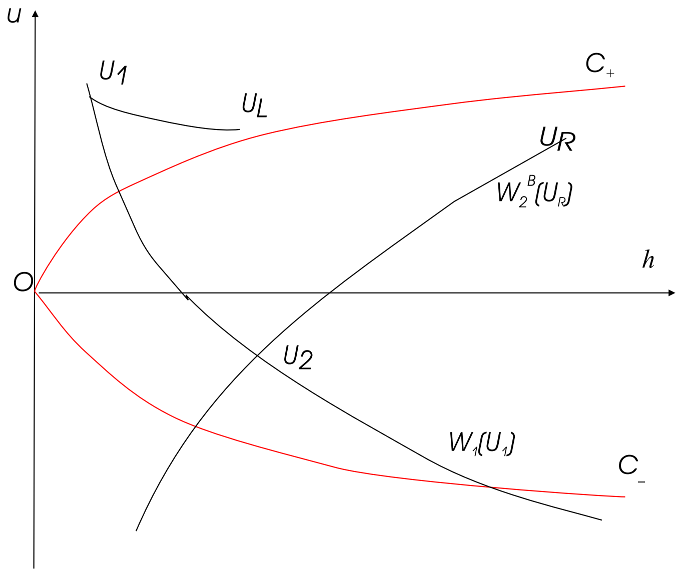

FIGURE 2. Solution for $U_L \in A_1$

Theorem 4.1. *Let* $U_L \in A_1$. *Set* $U_1 := SW(U_L, a_R)$, $\{U_2\} = \mathcal{W}_1(U_1) \cap \mathcal{W}_2^B(U_R)$. *The Riemann problem (1.1)-1.2 has a solution of the structure*

$$W_3(U_L, U_1) \oplus W_1(U_1, U_2) \oplus W_2(U_2, U_R), \tag{4.1}$$

provided $h_2 \leq \tilde{h}_1$. *(Figure 1).*

Proof. First let us establish the part (a). Observe that the the set of composite waves $SW(\mathcal{W}_1(U_L))$ consists of three monotone decreasing curves, each lies entirely in each region $A_i, i = 1, 2, 3$. The monotone increasing backward curve $\mathcal{W}_2^B(U_R)$ therefore may cut the three composite curves at a unique point, two point or does not meet the wave composite set. The Riemann problem therefore may admit a unique solution, two solutions, or has no solution.

The state U_L belongs to A_1 and in this region, the λ_3 is the smallest of the three characteristic speeds. A stationary wave from $U_L = (h_L, u_L, a_L)$ to $U_1 = (h_1, u_1, a_R)$ exists, since $a_L \leq a_R$. And, due to the (MC) criterion, $U_1 \in A_1$.

If $h_2 \leq h_1$, then the stationary wave is followed by a 1-rarefaction wave with positive speed, and then can be continued by a 2-wave $W_2(U_2, U_R)$. If $h_2 > h_1$, then the 1-wave in (3.17) is a shock wave. Since $h_2 \leq \bar{h}_1$ and $U_1 \in A_1$, the shock speed $\lambda_2(U_1, U_2) \geq 0$, and thus it can follow a stationary wave (with zero speed). Moreover, it is derived from (2.17) that

$$\begin{aligned}\bar{\lambda}_1(U_1, U_2) &= u_1 - \sqrt{\frac{g}{2}\Big(h_2 + \frac{h_2^2}{h_1}\Big)},\\ &= \frac{h_2u_2 - h_1u_1}{h_2 - h_1}\\ &= u_2 - \sqrt{\frac{g}{2}\Big(h_1 + \frac{h_1^2}{h_2}\Big)},\\ &\le u_2\\ &\le u_2 + \sqrt{\frac{g}{2}\Big(h_R + \frac{h_R^2}{h_2}\Big)} = \bar{\lambda}_2(U_2, U_R).\end{aligned} \tag{4.2}$$

This means the 1-shock $S_1(U_1, U_2)$ can always follow the 2-shock $S_2(U_2, U_R)$. Similar for rarefaction waves. Therefore, the solution structure (3.17) holds.

□

The following theorem deals with the case where the left-hand state U_L may be in $A_1 \in A_2$.

Theorem 4.2. *Let $U_L \in A_1 \cup A_2$. There is a region of U_R such that $SW(\mathcal{W}_1(U_L), a_R) \cap \mathcal{W}_2^B(U_R) \neq \emptyset$. In this case this intersection may contain either only one or both points $U_3 \in A_2$ and $U_4 \in A_3$. The Riemann problem (1.1)-1.2 therefore has a solution of the structure*

$$W_1(U_L, U_5) \oplus W_3(U_5, U_3) \oplus W_2(U_3, U_R), \tag{4.3}$$

where U_5 is the point such that $U_3 = SW(U_5, a_R) \in \mathcal{W}_1(U_L)$, and

$$W_1(U_L, U_6) \oplus W_3(U_6, U_4) \oplus W_2(U_4, U_R), \tag{4.4}$$

where $U_4 = SW(U_6, a_R) \in \mathcal{W}_1(U_L)$, if $h_4 \ge \bar{h}_R$ whenever $U_R \in A_2-$. (Figure 2).

Proof. The solution may begin with a 1-wave, either 1-shock with a negative shock speed to a state U_5, or a 1-rarefaction wave with $\lambda_1(U_5) \le 0$, followed by a stationary wave $W_3(U_5, U_3)$ from U_5 to U_3, then followed by a 2-wave $W_2(U_3, U_R)$ from U_3 to U_R. It is similar in the case of U_4. However, in order that the stationary wave $W_3(U_6, U_4)$, for some $U_6 \in \mathcal{W}_1(U_L)$ and $U_6 \in A_3$ obviously, to be followed by a 2-wave $W_2(U_4, U_R)$, it is required that the wave is a shock with the shock speed $\lambda_2(U_3, U_R)$ is nonnegative. This is equivalent to $h_4 \ge \bar{h}_R$.

□

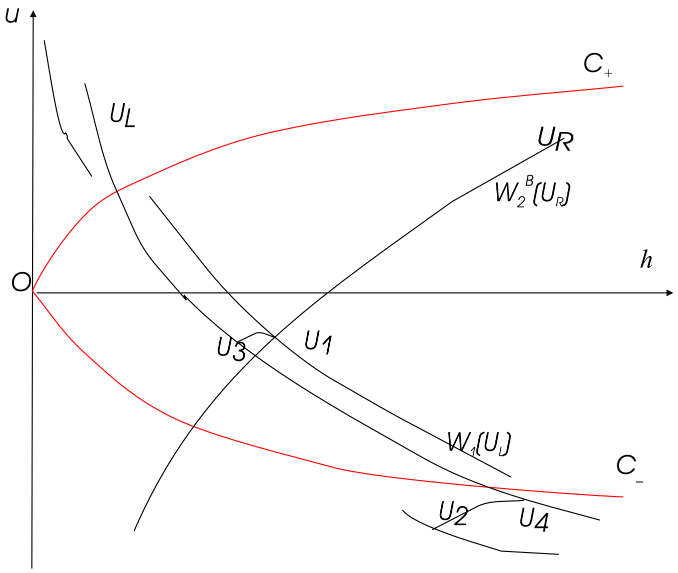

FIGURE 3. Solution for $U_L \in A_1 \cup A_2$

Theorem 4.3. *Let $U_L \in A_3$ and $U_R \in A_1 \cup A_2$. Then, set $U_1 = SW(\mathcal{W}_2^B(U_R), a_L) \cap \mathcal{W}_1(U_L)$, and let $U_2 = SW(U_1, a_R) \in \mathcal{W}_2^B(U_R)$.*

(i) *If $U_1 \in A_2^+ \cup \mathcal{C}_+ \cup \{u = 0\}$, the Riemann problem (1.1)-(1.2) has a solution with the structure*

$$W_1(U_L, U_1) \oplus W_3(U_1, U_2) \oplus W_2(U_2, U_R). \tag{4.5}$$

(ii) *If $U_1 \in A_2^- \cup \mathcal{C}_-$, provided $h_R \geq \bar{h}_2$, the Riemann solution (4.5) also exists.*

(iii) *If $U_1 \in A_1 \cup A_3$, the construction (4.5) does not make sense.*

(Figure 3).

Proof. If $U_1 \in A_2 \cup \mathcal{C}$, the non-positive speed wave $W_1(U_L, U_1)$ can be followed by a stationary wave $W_3(U_1, U_2)$. If

When $U_2 \in A_2-$, if $U_1 \in A_2^+ \cup \mathcal{C}_+ \cup \{u = 0\}$, then this stationary wave can always be followed by a 2-wave $W_2(U_2, U_R)$, since the wave speed of the 2-wave is positive. This establishes (i).

If $U_1 \in A_2^- \cup \mathcal{C}_-$, the wave speed of the 2-wave $W_2(U_2, U_R)$ is non-negative if and only if $h_R \geq \bar{h}_2$. This proves (ii).

If $U_1 \in A_1$, the 1-wave has positive speed. So it can not be followed by a stationary wave. If $U_1 \in A_3$, then $U_2 \in A_3$ by the (MC) criterion. So the 2-wave $W_2(U_2, U_R)$ has negative speed. So it can not be proceeded by a stationary wave. This proves (iii).

□

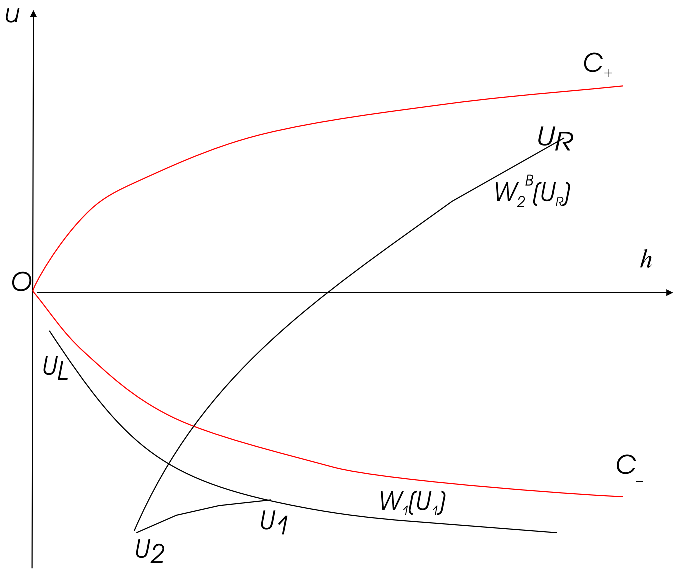

FIGURE 4. Solution for $U_L \in A_3$

The above theorem enables U_R to vary in each area A_1, A_2 and A_3. The next theorem enables U_L to vary in all the three areas.

Theorem 4.4. *Let $U_R \in A_3$. Set $U_1 = SW(U_R, a_L)$, $U_2 = \mathcal{W}_2^B(U_1) \cap \mathcal{W}_1(U_L)$. A Riemann solution exists and has the following structure*

$$W_1(U_L, U_2) \oplus W_2(U_2, U_1) \oplus W_3(U_1, U_R), \tag{4.6}$$

provided $h_2 \le \bar{h}_1$. (Figure 4).

Proof. The stationary wave $W_3(U_1, U_R)$ turns out to have the greatest wave speed. In order for this wave to be proceeded by the 2-wave $W_2(U_1, U_2)$, the wave speed of this 2-wave has to be non-positive. This is equivalent to the condition $h_2 \le \bar{h}_1$, according to Theorem (3.5). Similar to (4.4), we have

$$\lambda_1(U_L, U_2) \le \lambda_2(U_2, U_1).$$

so that the 1-wave $W_1(U_L, U_2)$ can follow the 2-wave $W_2(U_2, U_1)$. □

4.2. **Solutions with repetition in wave family.** It is of great interest to see that we may have solutions combined from four elementary waves even there are three characteristic fields. This explains the challenge of Riemann problem when the system is not strictly hyperbolic.

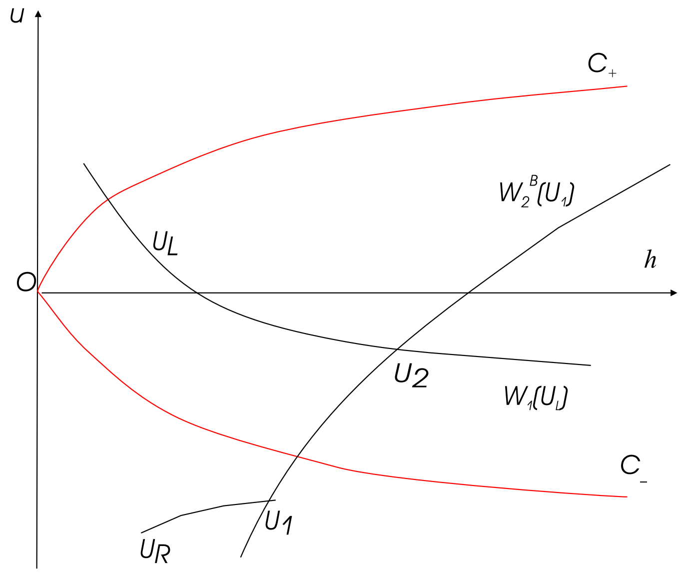

FIGURE 5. U_L may be anywhere

Theorem 4.5. *Let $U_L \in A_2 \cup A_3$. Set $U_+ = \mathcal{W}_1(U_L) \cap \mathcal{C}_+, \{U_1\} = SW(U_+, a_R) \cap A_1, \{U_2\} = \mathcal{W}_1(U_1) \cap \mathcal{W}_2^B(U_R)$. The Riemann problem (1.1)-(1.2) has a solution with the structure*

$$R_1(U_L, U_+) \oplus W_3(U_+, U_1) \oplus W_1(U_1, U_2) \oplus W_2(U_2, U_R), \tag{4.7}$$

provided $h_2 \leq \tilde{h}_1$. (Figure 5).

Theorem 4.6. *For any U_L, set $\{U_1\} = SW(\mathcal{C}_-, a_R) \cap \mathcal{W}_2^B(U_R) \cap A_2$, let $U_2 = (h_2, u_2, a_L) \in \mathcal{C}_-$ such that $U_1 = SW(U_2)$, and let $\{U_3\} = \mathcal{W}_2^B(U_2) \cap \mathcal{W}_1(U_L)$. The Riemann problem (1.1)-(1.2) has a solution with the structure*

$$W_1(U_L, U_3) \oplus R_2(U_3, U_2) \oplus W_3(U_2, U_1) \oplus W_2(U_1, U_R), \tag{4.8}$$

provided $h_R \geq \bar{h}_1$ and $h_3 \leq h_2$. (Figure 6).

Thus, we see from the last theorem that the Riemann problem (1.1)-(1.2) has a solution consisting of a 1-, a 3-, and two 2-waves.

It is interesting that there are solutions also satisfying the (MC) criterion which contain three waves with the same speed zero. This is the case when a stationary wave jumps from the level $a = a_L$ to an intermediate level a_m between a_L and a_R, followed by an "intermediate" k-shock with zero speed at the level a_m, $k = 1, 3$, and then followed by another stationary wave jumping from the level a_m to a_R. Thus, there are only two possibilities:

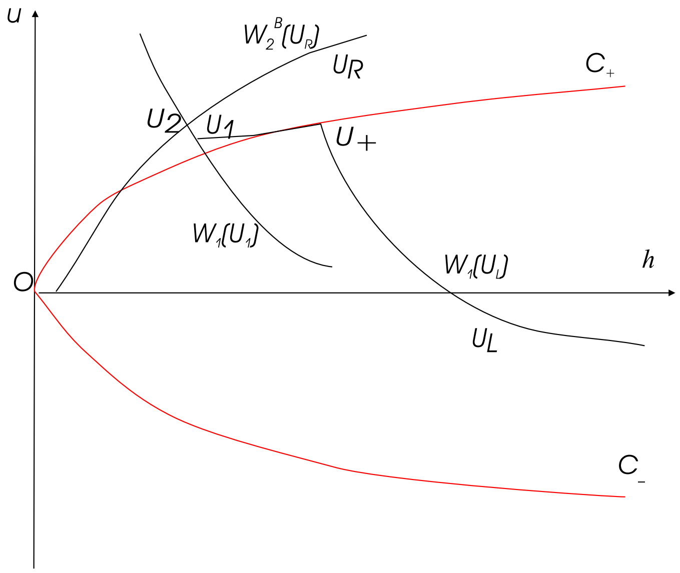

FIGURE 6. Solution with repeated two 1-waves

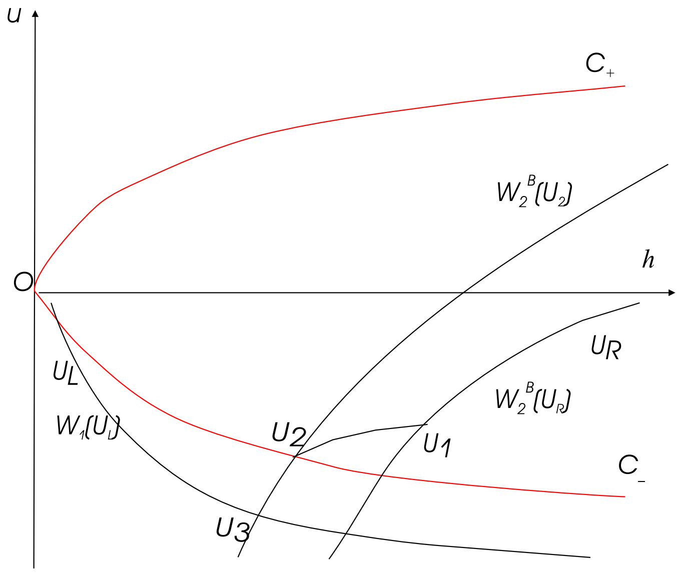

FIGURE 7. Solution with repeated two 2-waves

(i) U_L belongs to A_1 and a 1-shock with zero speed is used.
(ii) U_R belongs to A_3 and a 2-shock with zero speed is used.

We just describe the first case (i), as the second case is similar.

Recall from Theorem 3.5 that for any $U \in A_1$, there exists a unique point denoted by $\tilde{U} \in \mathcal{W}_1(U) \cap A_2$ such that

$$\bar{\lambda}_1(U, \tilde{U}) = 0.$$

Theorem 4.7. *Let $U_L \in A_1$. Set*

$$SW(U_L, [a_L, a_R]) := \cup_{a \in [a_L, a_R]} SW(U_L, a),$$
$$\widetilde{SW}(U_L, [a_L, a_R]) := \{\tilde{U} |\ U \in SW(U_L, [a_L, a_R])\}.$$

Whenever

$$\widetilde{SW}(U_L, [a_L, a_R]) \cap \mathcal{W}_2^B(U_R) \neq \emptyset$$

there are a value $a_m \in [a_L, a_R]$, a state $U_1 = SW(U_L, a_m)$, a state $U_2 \in \widetilde{SW}(U_L, [a_L, a_R]) \cap \mathcal{W}_2^B(U_R)$ that defines a solution with the structure

$$W_3(U_L, U_1) \oplus S_1(U_1, \tilde{U}_1) \oplus W_3(\tilde{U}_1, U_2) \oplus W_2(U_2, U_R). \tag{4.9}$$

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