ON THE COMPACTIFICATION PROBLEMS FOR STEIN SURFACES II

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Abstract: Let M be a compactification of a Stein surface X and let Γ:=M \ X be the connected compact curve. Then we will show that X is biholomorphic to a toric surface **Š** := **C*** x **C*** iff X admits 2 algebraic compactifications M_i (resp. 2 non algebraic compactifications $\mathcal{M}_i$) for i= 1 and 2 which are not birationally equivalent (resp. not bimeromorphically equivalent) In contrast with **Š** , we shall show that there exist compactifiable Stein surfaces which do not admit any affine structure. Also, as applications, we shall characterize the algebraic structures of arbitrary compactifiable surfaces according to the topological type of Γ.

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§ 0.- **Preliminaries**

Unless the contrary is explicitly stated, all C-analytic spaces are assumed to be *non compact*. Also 2-dimensional connected C-analytic manifolds will be referred to simply as *surfaces*. All compact surfaces are assumed to be *minimal* i.e. free from *exceptional curves of the first kind*. For a given compact surface M, let us denote by a(M):= the transcendence degree of the field of global meromorphic functions on M over C. Also 1-dimensional C-analytic spaces will be referred to simply as *curves.*

(0.1) **Definition :** (a) A compact surface M is said to be an *analytic compactification* of a given surface X if there is given
(i) a compact **C**-analytic subvariety Γ ⊂ M and
(ii) an *analytic* isomorphism X ≅ M \ Γ
(b) A surface X is said to be *compactifiable* if it admits an analytic compactification M
(c) A compactifiable surface X is said to admit an *algebraic* (resp. a *non algebraic*) compactification if M is a *projective algebraic* (resp. a *non algebraic*) variety
(d) A surface X is said to admit an *affine structure* if there exists an affine variety $\mathcal{X}$ such that X≅$\mathcal{X}_h$ where $\mathcal{X}_h$ is the underlying C-analytic space associated to $\mathcal{X}$.
(e) Finally the *toric* surface X ≅ **C*** x **C***, where **C*** := **C** \{0} will be denoted, from now on by ***Š***

Our main concern here is the following
(0.2) **Problem:** To classify the compactifiable Stein surfaces ?
In [V3] it was shown that all compactifiable Stein surface are *quasi projective*, in particular they admit algebraic structure.
Furthermore one has:

(0.3) **Theorem**: [V3] *Let X a given Stein surface.*
Then all non algebraic compactifications of X are bimeromorphically equivalent, provided X≠ ***Š***

So naturally one would like to raise the following
(0.4) **Problem:** Does Theorem 0.3 hold if the"non algebraic" hypothesis is replaced by the 'algebraic" one ?
Notice that all known examples of compactifiable Stein surfaces, which are not affine do indeed admit some affine structure; hence another fundamental issue is
(0.5) **Problem:** Do compactifiable Stein surfaces always admit some affine structure ?
Finally there was the following
(0.6) **Problem:** (Hartshorne) To classify Stein surfaces which are not affine ?

This paper is the continuation of [V3]. Also it seeks to rectify and strengthen some results there. So the organization will be as follows. In section 1 we shall briefly review the intrinsic character of ***Š*** In section 2, we shall state concisely the algebraic structures of compactifiable Stein surfaces. In section 3, the uniqueness issue of compactifiable Stein surfaces will be taken up. Section 4 will be devoted to the proof of the Main Theorem which provides an affirmative answer to Problem 0.4. The affine structure of compactifiable Stein surface will be explored in section 5. Finally in section 6, some generalization problem of arbitrary compactifiable surfaces will be studied

§ 1.- **The toric surface**

(1.1) This venture, as well as many others, was inspired by the groundbreaking paper [Ho] and by the following pioneering observation [S] (p.108):
"For any given **C**-non singular elliptic curve $\mathcal{T}$ viewed as a Lie group, there exists a unique algebraic group $\mathcal{G}$ which is a non trivial extension

$$0 \;\rightarrow\; \mathbf{G}_a \;\rightarrow\; \mathcal{G} \;\rightarrow\; \mathcal{T} \;\rightarrow\; 0 \qquad (*)$$

where $\mathbf{G}_a$ is the 1-dimensional additive group. Consequently one can check ([H1] p.232) that
(a) $H^0(\mathcal{G}, \mathcal{O}) = \mathbf{C}$ where $\mathcal{O}$ is the algebraic structure sheaf of $\mathcal{G}$, and
(b) analytically $\mathcal{G}$ is isomorphic to $\mathbf{G}_m$ x $\mathbf{G}_m$ where $\mathbf{G}_m$ is the 1-dimensional multiplicative group ".

(1.2) We infer readily that ***Š*** admits both *affine* algebraic and *non affine* algebraic structures.
In other words, ***Š*** admits 2 distinct families of algebraic compactifications:
(a) the non rational ones, namely elliptic ruled surfaces $\pi\colon\ \mathcal{E} \rightarrow \mathcal{T}$ and
(b) the rational ones , namely $\mathbf{P}_2$ and Hirzebruch surfaces, $\mathbf{F}_n$ for any $n \geq 0$ and $n \neq 1$

(1.3) **Remark**: (1) [Sh][Si][U] Those 2 families are the only algebraic compactifications of ***Š***
(2) Although all rational structures of ***Š*** are birationally equivalent, the novelty of (*) stems from the fact that it inherits ***Š*** with *infinitely* many different (i.e. non birationally equivalent) algebraic structures

(1.4) This phenomenon shows a sharp contrast with the case when dim X = 1 [Si] or when X is a compact **C**-analytic space which admits, in view of the GAGA principle at most one algebraic structure. Also notice that such a construction was also established in [MM](p.145)
(1.5) " Let $\mathcal{G}$ be the rank 2 group of (2x2) diagonal matrices with complex entries. Hence $\mathcal{G} \cong \mathbf{C}^*$x $\mathbf{C}^*$. Let $\mathcal{A}$ be the subgroup of $\mathcal{G}$, consisting of those matrices of the form

$$\begin{bmatrix} \exp z & 0 \\ 0 & \exp iz \end{bmatrix}$$

with $z \in \mathbf{C}$. Obviously $\mathcal{A}$ is a closed subgroup of $\mathcal{G}$. Since $\mathcal{A} \cong \mathbf{C}$**,** we infer that $\mathcal{G}$ is a topologically trivial principal bundle over $\mathcal{G}/\mathcal{A}$ with structural group $\mathcal{A}$. Since **C** is contractible, one has an isomorphism of fundamental groups $\pi(\mathcal{G}) \cong \pi(\mathcal{G}/\mathcal{A})$. Since $\mathcal{G}/\mathcal{A}$ is 1-dimensional and its fundamental group is abelian with 2 generators, it follows readily that $\mathcal{G}/\mathcal{A}$ is an elliptic curve."

(1.6) This result tells us that the toric surface **Š** admits a structure of an *affine principal line* bundle of *degree zero* over an elliptic curve.

Also notice that few years earlier, the structure of such bundle, also known as **A**-bundle of degree zero, was thoroughly investigated in [A]. Apparently, it was not aware that the latter is indeed biholomorphic to C* x C*, until [S] and [MM] came along

§ 2.- The Existence of an algebraic structure

Our main goal here is the following:

(2.1) **Problem:** To classify the Stein surfaces X which admit non algebraic compactifications ?

First of all , let us recall some fundamental constructions (see [E1,2])

(2.2) Let $m \geq 1$ and $k \geq 1$ be fixed integers. Let $\alpha \in$ C with $0 < |\alpha| < 1$, let $t := (t_1, \dots, t_k) \in \mathbf{C}^k$ and let $v \in \mathbf{C}^*$. Now, let $\tau := \Sigma_{1 \leq j \leq k} \, t_j v^{j-1}$ and let us define a holomorphic automorphism

$$g_{k,\alpha,\tau} : \mathbf{C} \times \mathbf{C}^* \to \mathbf{C} \times \mathbf{C}^* \quad \text{by} \quad (u, v) \mapsto (v^k u + \tau, \ \alpha v)$$

(2.3) Let $\mathcal{A}_{k,\alpha,\tau}$ be the quotient surface $\mathbf{C} \times \mathbf{C}^* / < g_{k,\alpha,\tau} >$. Then one can check that

(a) $\mathcal{A}_{k,\alpha,\tau}$ is a bundle of affine lines with structural group the affine group, over the elliptic curve $\mathcal{C}_\alpha := \mathbf{C}^*/<\alpha>$

(b) Its linear part **L** is actually a holomorphic line bundle over $\mathcal{C}_\alpha$ such that $c_1(\mathbf{L}) = -k$

(c) In the case where $\tau \neq 0$, $\mathcal{A}_{k,\alpha,\tau}$ will be referred to, from now on, as *generic* affine C-bundles of degree - k over $\mathcal{C}_\alpha$

Let us mention some intrinsic properties of affine C-bundles

(2.4) **Theorem**: [E1,2] *Let $\mathcal{A}$ be an affine C-bundle of degree – k over some elliptic curve* $\mathbf{C}^*/<\alpha>$. *Then $\mathcal{A}$ is equivalent as an affine C-bundle to some $\mathcal{A}_{k,\alpha,\tau}$ for some* $\tau \in \mathbf{C}^*$

(2.5) **Lemma:**[V3] *(1) The generic affine C-bundles $\mathcal{A}_{k,\alpha,\tau}$ (i.e. $\tau \neq 0$) are free of compact curves*
(2) Meanwhile $\mathcal{A}_{k,\alpha,0}$ is the total space of a line bundle **L** *over $\mathcal{C}_\alpha$ such that $c_1(\mathbf{L}) = -k$;*

(2.6) **Definition:** [K1] (see also [V7]) Let $t \in$ C*, $\alpha \in$ C with $0 < |\alpha| < 1$, let $U := \mathbf{C}^2 \setminus \{0,0\}$ and let g: U $\to$ U be an automorphism of U defined by

$$g(z, w) := (\alpha^m z + t w^m, \ \alpha w)$$

Now one can check [K1] (p.695) that the cyclic group <g> is properly discontinuous and the quotient space $\mathbf{H}_{\alpha,t,m} := U/<g>$ is a compact surface with

$$b_1(\mathbf{H}_{\alpha,t,m}) = 1 \quad \text{and} \quad b_2(\mathbf{H}_{\alpha,t,m}) = a(\mathbf{H}_{\alpha,t,m}) = 0 \qquad (\blacklozenge)$$

where $b_i(\,)$ are the Betti numbers. Furthermore, the punctured line $U \cap \{w = 0\}$ is invariant under g, so it is mapped by the projection π: U $\to \mathbf{H}_{\alpha,t,m}$ onto a non singular elliptic curve $\Gamma_\alpha := \mathbf{C}^*/<\alpha^m>$ which is the only compact curve in $\mathbf{H}_{\alpha,t,m}$

$\mathbf{H}_{\alpha,t,m}$ is called *the non elliptic Hopf surface of Type (I)*

(2.7) It was first shown in [Ho] that $\mathbf{H}_{\alpha,t,m} \setminus \Gamma_\alpha \cong$ **Š** for any α and $t \in$ C*

(2.8) **Definition:** [H2] (V.2) For any (geometric) ruled surface π X $\to \mathcal{C}_g$ where $\mathcal{C}_g$ is a compact curve of genus $g \geq 0$, there exists a rank 2 vector bundle $\mathcal{V}_g$ on $\mathcal{C}_g$ such that $X \cong \mathbf{P}(\mathcal{V}_g)$.

Furthermore, we assume that $\mathcal{V}_g$ is *normalized* in the sense of Hartshorne [H2](V.2.8.1) and $e := -c_1(\det \mathcal{V}_g)$ will be referred to as an invariant of X

Let Ξ be the *canonical* section of X with $\mathcal{O}_X(\Xi)\cong \mathcal{O}(1)$ where $\mathcal{O}$ is the structural sheaf of $\mathbf{P}(\mathcal{V}_g)$. Then $\Xi^2= -e$. Also let $\mathbf{F}:= \pi^{-1}(x)$ for any $x\in \mathcal{C}_g$ be the fibre of X

(2.9) [V1,2] **Definition:** A surface X is said to be *strongly pseudoconvex* (or *1-convex* for short) if there exist:

(1) a 2-dimensional Stein space Y with only finitely many isolated normal singularities, say $\{p_i\}$ and

(2) a proper and surjective morphism $\pi : X\to Y$. inducing a biholomorphism

$$X \setminus S \;\cong\; Y \setminus \cup_i \{ p_i\}$$

where $S:= \cup_i\pi^{-1}(p_i)$ is called the *exceptional set* of X

(2.10) **Remark:** Obviously any Stein surface is 1-convex (with dim S = 0). So from now on 1-convex surfaces which are *not* Stein (i.e. dim S > 0) will be referred to as *proper* 1-convex surfaces

(2.11) **Lemma:** *(1) The generic principal affine C-bundles $\mathcal{A}_{k,\alpha,\tau}$ (i.e. $\tau \neq 0$) are compactifiable Stein surfaces which also admit affine structure*

(2) Meanwhile $\mathcal{A}_{k,\alpha,0}$ are compactifiable proper 1-convex surfaces with exceptional set, an elliptic curve Ξ, such that $\Xi^2 = -k < 0$

Proof: By definition, each $\mathcal{A}_{k,\alpha,\tau}$ admits an elliptic ruled surface $\pi{:}E_\alpha := \mathbf{P}(\mathcal{V}_1) \to \mathcal{C}_\alpha$ as its compactification. In particular, one can find a section $\Theta \subset E_\alpha$ such that $\mathcal{A}_{k,\alpha,\tau} \cong X:= E_\alpha\setminus\Theta$ .Now one will have the following 3 alternatives:

(a) If $\Theta^2< 0$, then $\mathcal{V}_1$ is necessarily decomposable and $\Theta = \Xi$ the canonical section; hence there exists a section at "infinity", say $\Lambda\subset E_\alpha$ such that $\Lambda^2 > 0$ and $\Theta.\Lambda = 0$, i.e. $\Lambda \subset X \cong \mathcal{A}_{k,\alpha,\tau}$ which is not possible in view of Lemma 2.5

(b) If $\Theta^2= 0$, it means that $\mathcal{A}_{k,\alpha,\tau}$ is an affine bundle of degree 0, but this is not possible, since $k \neq 0$

(c) Therefore, $\Theta^2> 0$. We infer readily that $\mathcal{A}_{k,\alpha,\tau}$ are *1-convex* see. e.g. [V5]

(1) Now, as far as generic affine C-bundles $\mathcal{A}_{k,\alpha,\tau}$ are concerned, as previously observed (Lemma 2.5) since $\tau \neq 0$, $\mathcal{A}_{k,\alpha,\tau}$ are free of compact curves. Hence Θ is actually an ample divisor. Thus X is affine. In particular $\mathcal{A}_{k,\alpha,\tau}$ is Stein

(2) On the other hand as noticed earlier, if $\Theta := E_\alpha \setminus \mathcal{A}_{k,\alpha,0}$, then $\Theta^2 = k$; in particular $\mathcal{A}_{k,\alpha,0}$ is proper 1-convex and admits $\Xi :=$ the canonical section of $\mathbf{L}$, as exceptional set with $\Xi^2 = -k < 0$

Q.E.D.

Therefore, in view of Lemma 2.11, Problem 2.1 is completely settled by the following

(2.12) **Theorem:** [E2] [V3] *Let M non algebraic and compact surface, let $\Gamma\subset M$ be a compact analytic subvariety and let $X:= M \setminus\Gamma$. Then the following conditions are equivalent*

(i) X is Stein

(ii) $X \cong \check{\mathbf{S}}$ or $\mathcal{A}_{k,\alpha,\tau}$ for some k, α and τ as in (1.1) with $\tau\neq 0$

(iii) X admits an affine structure

(2.13) **Corollary:** *A Stein surface X is compactifiable iff X is quasi- projective*

(2.14) **Corollary:** *Any compactification M of a Stein surface X is projective algebraic provided $X \neq \check{\mathbf{S}}$ and $\mathcal{A}_{k,,\alpha,,\tau}$*

§.3. The Uniqueness issue

(3.1) As notice earlier, one has $\mathbf{H}_{\alpha,t,m} \setminus \Gamma_\alpha \cong \check{\mathbf{S}} \cong \mathbf{H}_{\beta,s,m} \setminus \Gamma_\beta$ for any α and.β. On the other hand, one can check that

$$\mathbf{H}_{\alpha,t,m} \text{ is bimeromorphic to } \mathbf{H}_{\beta,s,m} \text{ iff } \alpha = \beta \qquad (!)$$

Hence complementing (0.4), we infer from (!) that $\check{\mathbf{S}}$ admits *infinitely* many different (i.e. non bime-romorphically equivalent) non algebraic structures. Hence one would like to raise the following

(3.2) **Problem**: Up to biholomorphism, is **Š** the only compactifiable Stein surface which admits non algebraic (resp.algebraic) compactifications which, as surfaces, are not bimeromorphically equivalent ?
Our main purpose here is to provide an affirmative answer to this Problem, namely

(3.4) **Main Theorem**: *Let X be a given compactifiable Stein surface.*
Then all algebraic compactifications of X are birationally equivalent, provided X ≠ **Š**

The proof of this result will be given in the next section. We would like to exhibit here a very special but practical version of Theorem 3.4 as an illustration which has interest in its own right.. But first of all few basic ingredients are in order
(3.5) **Definition**:[I] Let $\mathcal{D}$ be a non singular algebraic curve and let $\mathcal{C}$ be its non singular compactification (which exists and is unique). Hence there exist finitely many points $\{q_i\} \in \mathcal{C}$ such that $\mathcal{D} \cong \mathcal{C} \setminus \cup_i q_i$ Now let g:= genus of $\mathcal{C}$ and n:= card $|q_i|$. Then we say that $\mathcal{D}$ is *of type (g, n)*

(3.6) **Theorem**: [I] (Theorem 5) *Let X be a Stein surface. Assume that there exist a non singular algebraic curve $\mathcal{R}$ and a surjective morphism $\pi : X \to \mathcal{R}$. Assume that*

(a) π is of maximal rank for any $x \in X$
(b) each fibre $\mathcal{D} := \pi^{-1}(t)$ for any $t \in \mathcal{R}$ is a non singular algebraic curve of type (g,n) such that

$$2g + n > 2 \qquad (\spadesuit)$$

Then X only admits algebraic compactifications which are birationally equivalent

We are now ready to state a special case of Theorem 3.4
(3.7) **Proposition**: *Theorem 3.4 holds if one assumes that $X \cong A_1 x\ A_2$, product of two non singular algebraic curves*
Proof: Notice that any non singular algebraic curve $\mathcal{D}$ does satisfy (♠) with 2 exceptions: **C** and **C***. Consequently it follows readily from Theorem 3.6 that Proposition 3.7 does hold with possibly 3 exceptions:

(i) $\mathbf{C}^2$ or
(ii) **C** x **C*** or
(iii) **C*** x **C***

However, we infer from results in [K2] (resp. in [U]) that the only compactifications of $\mathbf{C}^2$ (resp. **C** x **C***) are rational compact surfaces. Hence our proof is complete.

Q.E.D.

(3.8) **Corollary:** *Let X:= $A_1 x\ A_2$, be the product of two non singular algebraic curves*
Then X only admits algebraic compactifications which are birationally equivalent with only a single exception, namely X = **Š**

§4. The generic birationality

The main purpose of this section is to devote to a complete proof of Theorem 3.4 above. But first of all few basic ingredients are in order:
(4.1) **Definition:** Let M be an analytic compactification of some Stein surface X, let Γ:= M\X and let $\mathcal{K}_M$ be the canonical bundle of M. From the vector space **V**:= $H^0(M, \mathcal{O}(m\,\mathcal{K}_M + (m-1)\Gamma)$, let us consider a basis $\{\phi_0,\ldots,\phi_N\}$ which gives rise to a well defined *meromorphic* map:

$$\mathbf{\Phi}_m : M \to \mathbf{P}_N$$
$$z \mapsto \mathbf{\Phi}_m(z) := [\phi_0 : \ldots : \phi_N]$$

where N:= dim **V** – 1
Following [Sa] (p.245), let N(X):= {m >0| dim **V** > 0} and let us define

$$k_a(X) := \begin{cases} \max_m \{\dim \Phi_m(M)\} & \text{if } N(X) \neq \varnothing \\ -\infty & \text{if } N(X) = \varnothing. \end{cases} \qquad (0)$$

Notice that $k_a(X)$ which will be referred to as the *analytic Kodaira dimension* of X, is a bimeromorphically invariant. On the other hand, we have

(4.2) **Definition**: [Ii.2,3] In formula (0) of the definition (4.1) , if one replaces the vector space **V** by **W**:= $H^0(M, \mathcal{O}(m(\mathcal{K}_M + \Gamma))$, then one obtains the so called *logarithmic Kodaira dimension* of X which will be denoted from now on by $k_l(X)$
Notice that in contrast with $k_a(X)$, $k_l(X)$ which is a birational invariant, is not biholomorphically invariant , an aspect which will be fully exploited later on in our strategy
Also, it is obvious from the definition, that in the special case where $\Gamma = \varnothing$, $k_l(M)$ and $k_a(M)$ will coincide with the standard notion of Kodaira dimension $k(M)$ for compact surfaces M. Now in general, if M is a compactification of some Stein surface X, then one has [Ii(2)][Sa]

$$-\infty \leq k(M) \leq k_a(X) \leq k_l(X) \leq 2 \qquad (1)$$

(4.3)**Example:** Let $X_1 := \mathbf{P}_2 \setminus \Gamma$ where Γ consists of 3 lines in general position. Let $\mathcal{V}$ be an indecomposable rank 2 bundle over an elliptic curve with invariant e = 0. From [H2](V.2) we know that there exists a unique section $\Gamma \subset \mathbf{P}(\mathcal{V})$ such that $\Gamma^2 = 0$. Now let $X_2 := \mathbf{P}(\mathcal{V}) \setminus \Gamma$ Then one can check that:
(1) $X_1 \cong X_2 \cong \check{\boldsymbol{S}}$
(2) $k_l(X_1) = 0$ and $k_l(X_2) = -\infty$
(3) $k_a(X_1) = k_a(X_2) = -\infty$

(4.4) **Proposition**:[Sa] $k_a(X) = 2$ *iff* $k_l(X) = 2$
(4.5) **Proposition**: [Sa] *Let Γ be a compact curve of degree d in* $\mathbf{P}_3$ *and let* $X := \mathbf{P}_3 \setminus \Gamma$. *Then*

$$k_a(X) = \begin{cases} 2 & \text{if } d > 3 \\ -\infty & \text{otherwise} \end{cases}$$

As far as Stein surfaces with $k_a(X) < 2$ are concerned, we have the following crucial results
(4.6) **Theorem**: [V2] (Lemma 2) *Let M be an algebraic compactification of a Stein surface X. Assume that* $k_a(X) < 2$. *Then M is a ruled surface*
Combining Theorem 4.6 with a main result in [Sr] (Theorem 3.4) one obtains the following

(4.7) **Theorem**: *Let* $\pi: \mathcal{R}_g \to \mathcal{C}_g$ *with* $g \geq 1$, *be an irrational ruled surface and let* $\Gamma \subset \mathcal{R}_g$ *be a compact curve. Assume that* $X := \mathcal{R}_g \setminus \Gamma$ *is Stein and admits non birationally algebraic compactifications. Then*

(a) $g = 1$ *i.e.* $\pi: \mathcal{R}_1 \to \mathcal{C}_1$ *is necessarily an elliptic ruled surface; furthermore*
(b) Γ is either
(i) a section or
(ii) an irreducible 2-section, or
(iii) a reducible 2-section, $\mathcal{C} \cup \mathcal{D}$ *where* $\mathcal{C}$ *(resp.* $\mathcal{D}$*) is a section*

(4.8) **Notations:** From now on, a surface X:= $\mathcal{R}_1\backslash\Gamma$ in Theorem 4.7 will be denoted by $\mathcal{X}_{I}$ (resp $\mathcal{X}_{II}$, resp $\mathcal{X}_{III}$) if Γ is of type (b) (i) (resp.(ii), resp.(iii)) of Theorem 4.7. Also from now on let us adhere to the following convention $\mathcal{R}_1$= **P**($\mathcal{V}$) for some rank 2 bundle $\mathcal{V}$ over C_1 with invariant e

(4.9) **Definition**:[F] Let π:M→ C be a ruled surface and let Γ⊂ M be a compact curve . Then X:= M\ Γ is said to admit a C*-*fibration* if Γ. **F** = 2 for generic fibre **F** in M. From now on a C*-fibration structure on X will be denoted by f:X → C where f:=π|X

Now let us look at the following alternatives:

(4.10) By taking Proposition 4.4 into account, we have
Theorem: [Sa] *Assume that* $\kappa_l(X) = 2$ *and X is quasi projective. Then all algebraic compactifications of X are bimeromorphically equivalent*

(4.11) **Theorem** : (see e.g.Remark of Lemma 2.4 in [GS] (p.120))
Assume that $\kappa_l(X) =1$ *and furthermore assume that X is affine*
Then X admits a structure of a C-fibration f: X → C which is uniquely determined Precisely if g:=X' → C' is another C*-fibration, such that κ: X ≅ X' , then there exists an isomorphism σ: C ≅ C' such that* $g = \sigma \circ f \circ \kappa$

(4.12) **Theorem**: *Let M be a compactification of X. Assume that* $\kappa_l(X) = 0$.
Then M is a rational surface provided X ≠ **Š**.
Proof: Assume that M = $\mathcal{R}_g$ is an irrational surface with g ≥ 1. In view of the hypothesis,

$$\mathcal{K}+\Gamma \equiv 0 \qquad (\wedge)$$

where $\mathcal{K}$ is the canonical bundle of $\mathcal{R}_g$, ≡ stands for numerical equivalence, and Γ := $\mathcal{R}_g\backslash$X. In view of Theorem 4.7, one can assume that Γ is free of fibre components.
(1) Assume that Γ is irreducible. Hence, from (^) , we have that

$$\Gamma^2 = \mathcal{K}^2$$
$$= - 8(1 - g) \leq 0$$

Since X is Stein, so necessarily

$$\Gamma^2 = 0 \text{ i.e. } g = 1. \qquad (\%)$$

In this situation, one has 3 alternatives for the elliptic ruled surface $\mathcal{R}_1$:= **P**($\mathcal{V}$)
(i) $\mathcal{V}$ is a decomposable rank 2 bundle with e = 0. In this case its canonical section Γ will satisfy (%). From the decomposability assumption of $\mathcal{V}$, it follows that $\mathcal{R}_1$ also contains compact curves Θ with Θ.Γ = 0, i.e. Θ ⊂ X, contradicting the fact that X is Stein
(ii) $\mathcal{V}$ is an indecomposable rank 2 bundle with invariant e = -1. Then

$$\Gamma \equiv 2\Xi - \mathbf{F} \qquad (\bullet)$$

which is an elliptic curve will satisfy (%). But it was shown [N] (Lemma 6.8) that in this situation, X also contains compact curves. That will contradict the Steiness of X. In fact, in [Su] by realizing $\mathcal{R}_1$ as an hyperelliptic surface over $\mathbf{P}_1$, (at least) 3 disjoint curves of type (•) in X, were explicitly exhibited. Consequently, this case cannot happen, as long as X is required to be Stein.
(iii) $\mathcal{V}$ is an indecomposable rank 2 bundle with invariant e = 0.
In this case, there exist a unique section Γ ⊂ **P**($\mathcal{V}$) such that Γ^2 = 0. Here one can check that X ≅ **Š**

(2) Assume that Γ:= $\mathcal{R}_1\backslash$X is reducible. Then let Λ ⊂ Γ be an irreducible component. As mentioned earlier, one can assume that Γ is free of fibre components; hence Λ .**F** > 0 for any fibre **F** of $\mathcal{R}_1$ Hence

$$0 = (\Lambda.\mathcal{K}+\Gamma) = \Lambda^2 + \mathcal{K}.\Lambda + \Lambda. (\Gamma \backslash \Lambda)$$
$$= 2\, g(\Lambda) - 2 + \Lambda .(\Gamma \backslash \Lambda)$$

Hence $g(\Lambda) = 1$ and $\Lambda. (\Gamma \setminus \Lambda) = 0$ i.e. Λ is isolated in Γ. But X is Stein so $\Lambda = \Gamma$ and the same argument as above will apply **Q.E.D**

(4.13) ***Remark****:* (a) It follows from the arguments in (1) (ii) we infer that the surfaces $X \cong \mathcal{X}_m$ where m = II or III are *affine*
(b) A complete list of rational surfaces satisfying Theorem 4.12 can be found in [Ii1] Prop.6 and 16

We are in now in a position to provide a complete proof of the Main Theorem.
Proof: We are going to show that all algebraic compactifications of $X = \mathcal{X}_m$ where m = I or II or III are birationally equivalent, unless $X \cong \check{\mathbf{S}}$

Step 1: Assume that

$$X = \mathcal{X}_m \text{ with } m = I \text{ or } II . \qquad (\P)$$

Let us assume that X admits a rational compactification say M. So let us consider the following exact sequence of homology groups with **C** coefficients

$$0 = H_3(M) \rightarrow H_3(M,\Lambda) \rightarrow H_2(\Lambda) \rightarrow H_2(M) \rightarrow H_2(M,\Lambda) \rightarrow H_1(\Lambda) \rightarrow H_1(M) = 0 \qquad (2)$$

where $\Lambda := M \setminus X$
On the one hand, in view of the hypothesis (¶), one can check that

$$b_1(X) = 2 \quad \text{and} \quad b_2(X) = 1. \qquad (3)$$

On the other hand, by duality, we have

$$H_3(M, \Lambda) = H^1(X) \text{ and } H_2(M, \Lambda) = H^2(X) \qquad (4)$$

Since C-dim.$H_2(\Lambda) =: \mu(\Lambda)$ is equal to the number of irreducible components of Λ, by combining (2), (3) and (4), one can check that:
(a) If $M = \mathbf{F}_n$, then $\mu(\Lambda) = 4$, Λ consists of 2 sections and 2 fibres and $X \cong \check{\mathbf{S}}$
(b) If $M = \mathbf{P}_3$, then $\mu(\Lambda) = 3$, Λ consists of 3 lines in general position and $X \cong \check{\mathbf{S}}$

Step 2: As notice earlier, (0.8), $\check{\mathbf{S}}$ has a structure of an affine C-bundle of degree 0 over an elliptic curve; So let $\mathcal{R}_1$ be a compactification of $\check{\mathbf{S}}$. Then it follows readily that $\Gamma := \mathcal{R}_1 \setminus \check{\mathbf{S}}$ is a section with $\Gamma^2 = 0$. We infer readily that $\mathcal{X}_{II}$ can not admit a rational compactification

Step 3: In step 1, assume that

$$X = \mathcal{X}_{III} \qquad (\P\P)$$

Since $\mathcal{C}$ and $\mathcal{D}$ are sections and since $b_i(X) = 0$ for $i \geq 3$, the topological Euler number $\chi(X)$ of X can be expressed as follows:

$$\begin{aligned} \chi(X) &= b_0(X) - b_1(X) + b_2(X) \\ &= \chi(\mathcal{R}_1) - \chi(\Lambda) \\ &= \nu > 0 \end{aligned} \qquad (5)$$

where $\nu :=$ Card$(\mathcal{C} \cap \mathcal{D})$. In view of Lemma (7.10)[F]

$$b_1(X) = 2 \text{ (resp.= } 3) . \qquad (6)$$

Now let us assume that

$$\bar{k}(X) = -\infty . \qquad (7)$$

Then we shall use the following informations to rule out any rational compactfication for (¶¶)

(α) Assume that all the components of Λ are projective lines. By excluding the case where $X \cong \check{\mathbf{S}}$, one has the following alternatives:
(i) If $M = \mathbf{P}_3$, then one can check that M is the product of 2 affine curves. Therefore Theorem (3.6) will exclude this possibility for X

(ii) On the other hand, let us assume that $M = \mathbf{F}_n$. Then some careful calculations (see e.g. [F] Lemma 7.9) show us that, in view of (2) and under the constraints (6), Λmust consist of 1 section and 3 fibres (resp. 1 section and 4 fibres). Now one has

$$\begin{aligned}\chi(X) &= \chi(\mathbf{F}_n) - \chi(\Lambda) \\ &= 4 - (1 - 0 + \mu(\Lambda)) \\ &= -1 \text{ (resp. } -2)\end{aligned}$$

which in either case will contradict (5) Hence a rational compactification M of $\mathcal{X}_{III}$ is not possible:

(β) Otherwise, at least one component of Λ say Ξ is either

(a) an irrational curve (with possibly singularities) or

(b) a rational curve with one node

This certainly will be the case when ν is large. Indeed, since $1 \leq b_2(M) \leq 2$, ie. $\mu(\Lambda) \leq 5$, then (5) will tell us that $b_2(M)$ is big and so does $b_1(\Lambda)$in view of (2).

(i) If $M = \mathbf{P}_3$, Proposition 4.5 tells us that $\kappa_a(X) = 2$ contradicting (7). Hence M cannot occur

(ii) On the other hand if $M = \mathbf{F}_n$, we infer from Hurwitz's Formula that Ξ is a mutisection. Now in this situation one can assume that $M = \mathbf{F}_0 = \mathbf{P}_1 \times \mathbf{P}_1$, [Ve](Theorem4.1) Then from definition 4.2 , one can check that $\kappa_1(X) \geq 0$.

Now in view of the presence of the elliptic ruled surface $\mathcal{R}_1$ which is a compactification of $\mathcal{X}_{III}$, we infer from Theorem 4.12 (resp. Theorem 4.11, resp. Theorem 4.10) that such a rational compactification M for (¶¶) cannot occur.

Step 4: Let us use the same conventions as in (4.8). Assume that $X = \mathcal{X}_I$ admits another elliptic ruled surface, say $\mathcal{E}_1 \rightarrow \mathcal{D}_1$as its compactification, i.e.there exists a compact curve $\Theta \subset \mathcal{E}_1$ such that

$$\sigma : X \cong X.' := \mathcal{E}_1 \setminus \Theta \qquad (8)$$

where $\mathcal{E}_1 = \mathbf{P}(\mathcal{W})$ for some rank 2 bundle $\mathcal{W}$over $\mathcal{D}_1$with invariant e'.

Since one excludes the case where $X \cong \check{\mathbf{S}}$, one can assume that

$$\Gamma^2 = k > 0, \qquad (9)$$

i.e. X is an affine surface where $\Gamma := \mathcal{R}_1 \setminus X$. In view of Theorem 4.7, one can assume that Θ has no fibre components. Now by using Mumford-Ramanujam theory [E2] [Sh] [U] one can show, on the basis of (8), that ∂U is *homotopically equivalent* to ∂V where U (resp.V) is some tubular neighborhood of Γ (resp.Θ); we infer from (9) that $\Theta^2 = k$

Case 1: Assume that $\mathcal{V}$is a decomposable vector bundle. Hence $\mathcal{R}_1$ admits a canonical section Ξ such that $\Xi^2 = -\varepsilon < 0$. From (8), it follows that $\sigma(\Xi \setminus (\Xi \cap \Gamma)) =: € \subset \mathcal{E}_1$is an algebraic curve. Since dim. € = 1, it admits a unique algebraic structure[Si] , so let Ξ'be the compactification of € . Then one can deduce from (8) that $\Xi \cong \Xi'$ and $(\Xi')^2 = \Xi^2 = -\varepsilon < 0$. Since $\mathcal{E}_1$ is an elliptic ruled surface, this will imply [H2] (V.2) that $\mathcal{W}$ is also a decomposable rank 2 bundle and that $\mathcal{E}_1$will admit Ξ' as its canonical section. Now by identifying the base curve of the elliptic ruled surface with its canonical section, we infer readily that $\mathcal{C}_1 \cong \mathcal{D}_1$ and $e = e' = \varepsilon > 0$ Consequently $\mathcal{R}_1 \cong \mathcal{E}_1$

Case 2: Assume that $\mathcal{V}$is indecomposable. Then the same argument as above shows that $\mathcal{W}$must be also indecomposable. Hence (8) is actually an isomorphism of affine C-bundles of degree $\neq 0$. As note earlier , Theorem 2.4, (see also [Ka]), each affine C-bundle **A** of degree $-k < 0$, over an elliptic curve **e**, is completely determined by its linear part **L** which is a line bundle over **e** such that $c_1(\mathbf{L}) = -k$; however , from Lemma 2.10, the total space of **L** is in fact a proper 1-convex surface which is determined by its exceptional set S which in turn is the canonical section of the elliptic ruled surface. By identifying the base curve of the elliptic ruled surface with its canonical section, again we infer readily that $\mathcal{C}_1 \cong \mathcal{D}_1$; hence $\mathcal{R}_1 \cong \mathcal{E}_1$

(4.13) **Remark:** This result is an *analogue* of Theorem (4.11) for affine surfaces X which admit a **C**-fibration structure over some elliptic curve $\mathfrak{e}$ and which satisfy the condition $k_l(X) = -\infty$

Step 5: With the same situation in Step 4 with $X = \mathcal{X}_{II}$ or $\mathcal{X}_{III}$

Assume that $k_1(X) = -\infty$. Since X is Stein, then Θ got to be a section without any fibre components.. Thus it brings us back to case Step 4. Otherwise, one can check that $k_1(X) \geq 0$, Then similar arguments as above, will show that $k_1(X) = 0$ is not possible. Now if $k_1(X) = 1$ (resp = 2) we infer from Theorem 4.11 (resp. Theorem 4.10) that all algebraic compactifications of X are birationally equivalent

Step 6: Assume that $\mathcal{X}_m$ with m = I or II or III, admits an irrational ruled surface $\mathcal{R}_g$ with g > 1 as its compactification. Clearly this case can not occur in view of Theorem 4.7

Q.E.D.

§ 5.- **The Affine structures**

As far as affine surfaces are concerned, we have the following:

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(5.1) **Proposition :** [V3,4] *Let X be an affine surface. Then* $k_a(X) = -\infty$ *or 2*

In parallel to 5.1, we have

(5.2) **Theorem:** *Let X be a compactifiable Stein surface. Then* $k_a(X) = -\infty$ *or 2*

Proof: Let M be a compactification of X and let $\Gamma := M \setminus X$

Case1: Assume that $M = \mathbf{P}_2$. Then our conclusion will follow from Prop. 4.5

Case 2: Assume that M is a ruled surface. Then a main result in [V5,6] tells us that X is biholomorphic to, either

(a) an affine surface, or
(b) $\mathbf{P}(\mathcal{V}) \setminus \Gamma$ where $\mathcal{V}$ is an indecomposable rank 2 bundle over a compact curve $\mathcal{C}_g$ with g > 0 with invariant e < 0 and Γ is a section with $\Gamma^2 = 0$

For (a) Proposition 5.1 will apply . As far as (b) is concerned, one has

Claim: If $\Gamma \subset \mathbf{P}(\mathcal{V})$ is a section then $k_a(X) = -\infty$ where $X := \mathbf{P}(\mathcal{V}) \setminus \Gamma$

Proof of the Claim: Assume such that $k_1(X) \geq 0$. Let us consider the linear system $|m(\mathcal{K}_M + \Gamma)|$ for any integer m > 0 Since Γ is a section, $\mathcal{K}_M + \Gamma \equiv -\Xi + k\mathbf{F}$ for some integer k. Hence, one can find at least one effective element $D \in |m(\mathcal{K}_M + \Gamma)|$. But $D.\mathbf{F} = -m < 0$. Contradiction.

Consequently $k_l(X)$ (and a fortiori, $k_a(X)$) is equal to $-\infty$

Case3: Assume that M is algebraic and $k(M) \geq 0$. Then it follows from [V2] that $k_a(X) = 2$

Case 4: Assume that M is a non algebraic surface. Then it follows from Theorem 2.12 that X is biholomorphic to, either

(a) $\check{\mathbf{S}}$ which as affine line bundle of degree 0, admits an elliptic ruled surface as compactification such that Γ is a section with $\Gamma^2 = 0$. Thus from the Claim above, $k_a(X) = -\infty$ or
(b) $\mathcal{A}_{k,\alpha,\tau}$ which as affine C bundle of degree -k, will admit an elliptic ruled surface as compactification such that Γ is a section with $\Gamma^2 > 0$. Hence again $k_a(X) = -\infty$

Q.E.D.

(5.3) **Remark:** (1) In [V3] a proof of Theorem 5.2 was also given. However it was incomplete
(2) As notice in (0.3) (b) (resp. Lemma 2.11), the toric surface $\check{\mathbf{S}}$ (resp. the generic affine C-bundles $\mathcal{A}_{k,\alpha,\tau}$) admit affine structures. On the other hand, as previously noticed, the analytic Kodaira dimension is bimeromorphically invariant; hence the above results naturally lead us to the following

(5.4) **Problem**: Do compactifiable Stein surfaces X always admit some affine structure ?.
Our main goal here is to provide a negative answer to this Problem.

(5.5) **Counterexample**: Let $\mathcal{V}_g$ be an indecomposable rank 2 vector bundle over some non singular compact curve of genus g ≥1 with invariant e = 0. Then one can check that the ruled surface $\mathcal{R}_g :=$ $\mathbf{P}(\mathcal{V}_g)$ carries a section Γ such that

$$\Gamma^2 = -e = 0. \qquad (\clubsuit)$$

On the other hand it was shown [V5,6] that $\mathcal{X}_g := \mathcal{R}_g \setminus \Gamma$ is indeed Stein. Notice that $\mathcal{X}_1 \cong \check{\mathbf{S}}$. We deduce from (♣) that $\mathcal{X}_g$ are not affine.. Then it follows readily from Theorem 3.4 , that, in contrast with $\mathcal{X}_1$, all compactifications of $\mathcal{X}_g$ for g > 1, are birationally equivalent; therefore $\mathcal{X}_g$ do not admit any affine structure if g > 1.

(5.6) Notice that $\kappa_a(\mathcal{X}_g) = -\infty$. In [V4,] it was anticipated that Problem (5.4) might have an affirma-tive answer provided κ_a (X) = 2, which is not always the case as shown by the following

(5.7) **Counterexample**: *Step1:* Here we follow closely an idea in [B]. From now on let us denote $\mathcal{R}_g$ simply by $\mathcal{R}$ and let us select a sufficiently ample divisor, say δ on $\mathcal{R}$ such that

(a) $\mathbf{K}_{\mathcal{R}} + \delta$ is very ample and
(b) |2δ| contains an effective smooth divisor, say Δ.

Let $\pi: \mathcal{M} \to \mathcal{R}$ be a double cover of $\mathcal{R}$, ramified along Δ.Then from the Leray spectral sequence one has

$$H^0(\mathcal{M}, \mathbf{K}_{\mathcal{M}}) \cong H^0(\mathcal{R}, \mathbf{K}_{\mathcal{R}}) \oplus H^0(\mathcal{R}, \mathbf{K}_{\mathcal{R}} + \delta) \qquad (\heartsuit)$$

Since $\mathcal{R}$ is a ruled surface the first summand in (♥) is zero. Furthermore (a) and (b) will guarantee that $\mathcal{M}$ is a surface of general type

Step2. Let $\Theta := \pi^*(\Gamma)$. Hence in view of (♣),

$$\Theta^2 = 0 \qquad (\clubsuit\clubsuit)$$

Furthermore $\mathcal{Z} := \mathcal{M} \setminus \Theta$, being a finite cover of the Stein surface $\mathcal{R}$, is itself Stein. Since $\mathcal{M}$ is of general type it follows readily that $\kappa_a(\mathcal{Z}) = 2$. In view of (♣♣), $\mathcal{Z}$ is not affine. Since $\kappa_a(\mathcal{Z}) = 2$, Theorem 4.10 tells us that analytic compactifications of $\mathcal{Z}$ are biholomorphic. We infer readily that $\mathcal{Z}$ does not admit any affine structure.

§6. Some further prospects.

(6.1) The above constructions provide us concrete examples of compact algebraic surfaces M which are ruled (resp. of general type) namely $\mathcal{R}_g$ (resp. $\mathcal{M}_g$), for any g > 1 (resp. any g> 0), such that M is a compactification of a Stein surface X, namely $\mathcal{X}_g$ (resp. $\mathcal{Z}_g$) which does not admit any affine structure Such construction was motivated by a question raised by Hartshorne. First of all let us introduce the following:

(6.2)**Definition**: Let Γ be a connected compact curve in a given compact surface M. Then Γ is said to be *topologically of positive* (resp. *null*, resp. *negative*) *type* if $\Gamma^2 > 0$ (resp. = 0, resp. < 0)

(6.3) **Problem:** ([H1] Problem 3.4,p.235) Let M be a compact surface, let $\Gamma \subset$ M be an irreducible compact curve and let X:= M\\.Γ
Assume that X is free of compact curves.
Is X always Stein, provided Γ is topologically of *non negative type* ?
In [V5,6] an affirmative answer to Problem 6.3 was given, provided κ (M) = $-\infty$; concisely one has

(6.4) **Theorem:** [V5,6] *Let M be a compactification of some surface X.*
Assume that κ (M) = $-\infty$.
*(I) Assume that M is non algebraic. Then X is Stein iff $X \cong$ **Š** or $\mathcal{A}_{k,\alpha,\tau}$*
(2) Assume that M is algebraic. Then X is Stein iff X is either affine or X = $\mathcal{X}_g$ for any g > 0
Complementing this result, it is not very hard to prove the following, see also [V1]

(6.5) **Theorem:** *Let M be a compact algebraic surface, let $\Gamma \subset M$ be a compact analytic subvariety and let X:= M \\Γ. Assume that κ (M) = 0 or 1*
Then X is Stein iff X is affine

In order to complete this picture, one naturally would like to raise the following

(6.6) **Problem**: Are $\mathcal{M}_g$, for any g > 0, up to biholomorphism, the only compact surfaces of general type which compactify Stein surfaces which are not affine, namely $\mathcal{Z}_g$?
To round off this discussion, we would like to provide some applications of Theorem 3.4 to another aspect of Problem 0.4 above, namely

(6.7) **Problem**: Let M_i be given compact surfaces and let $\Gamma_i \subset M_i$ be compact connected curves, with i = 1 or 2. Assume that $X_i := M_i \setminus \Gamma_i$ are biholomorphic.
Are M_i bimeromorphically equivalent if Γ_i are topologically of the same type ?
In this direction, we have the following:

(6.8) **Theorem**: *Problem 6.7 admits an affirmative answer, provided*
both Γ_i are topologically of negative (resp. positive) type. ($)

Proof: a) Assume that both Γ_i are topologically of negative type. Grauert's criterion tells us that there exist, normal 2 dimensional compact normal C-analytic spaces, say Y_i with one isolated singular point $\{\gamma_i\}$ and morphisms π_i: $M_i \to Y_i$ inducing biholomorphisms

$$X_i = M_i \setminus \Gamma_i \cong Y_i \setminus \{p_i\}. \qquad (\sim)$$

Then Hartogs extension Theorem tells us that the isomorphism $X_1 \cong X_2$ will extend to a biholomorphism $M_1 \cong M_2$

b) Assume that Γ_i are both topologically of positive type. Then it follows readily from Chow-Kodaira Theorem that M_i are projective algebraic. Also we infer from [V2] that X_i are 1-convex with exceptional set S_i

(1) Assume that dim. $S_i > 0$. Then a main result in [V2] tells us that M_i are biholomorphic
(2) Assume that dim $S_i = 0$, i.e. X_i are Stein.

(a) Assume that X_i = **Š.** Since Γ_i are topologically of positive type, M_i are necessarily rational surfaces. Thus we are done.
(b) Assume that $X_i \neq$ **Š** Then Theorem 3.4 tells us that M_i are birationally equivalent
Q.E.D.
(6.9) **Corollary:** *Theorem 6.8 will hold if one replaces the hypothesis ($) by*

none of the Γ_i are topologically of zero type ($$)

Proof: Assume that Γ_1 is topologically of negative type, it follows readily from (~) and Hartogs extension Theorem that

$$\Gamma(X_1, \mathcal{O}_1) = \mathbf{C}. \quad (10)$$

Now if Γ_2 is topologically of positive type, then X_2 is 1-convex, i.e.

$$\dim \Gamma(X_2, \mathcal{O}_2) = \infty \quad (11)$$

in view of the definition 2.9. Therefore (10) and (11) contradict the hypothesis that $X_1 \cong X_2$. Thus Γ_2 must be also topologically of negative type. Hence our conclusion will follow from Theorem 6.8

(b) Assume that Γ_1 is topologically of positive type. Then the same argument as above will tell us that Γ_2 must be also topologically of positive type. Again our conclusion will follow from Theorem 6.8 **Q.E.D.**

(6.10) **Question**: Does Problem 6.7 admits an affirmative answer if both Γ_i are topologically of null type ?

Obviously the answer is No. However, our current study shows that, drastically, an answer to Question 6.10 is still negative, even if one assumes, furthermore that

(a) M_i are both algebraic (resp. both non algebraic) and
(b) X_i are Stein

However on the positive side, we have

(6.11) **Proposition:** *Question 6.10 admits a positive answer, provided, either*
(a) $X_1 \cong X_2 =: X$ is Stein and $X \neq \check{\mathbf{S}}$ or
(b) $X_1 \cong X_2 =: X$ is proper 1-convex

Proof: (a) *Step 1*: Assume that M_1 is non algebraic. Since $(\Gamma_1)^2 = 0$, it follows from Theorem 2.12 that $X \cong \mathcal{A}_{k,\alpha,\tau}$. Assume that M_2 is algebraic. Since $X_2 \cong \mathcal{A}_{k,\alpha,\tau}$ and $(\Gamma_2)^2 = 0$, Lemma 2.11 tells us that this not possible. Hence M_2 got to be non algebraic. We infer from Theorem 0.3 that $M_1 \cong M_2$

Step 2: Assume that M_1 is algebraic and since $(\Gamma_1)^2 = 0$ it follows from Theorem 2.12 that $X \neq \mathcal{A}_{k,\alpha,\tau}$ (and , by hypothesis, $\neq \check{\mathbf{S}}$). Assume that M_2 is non algebraic. Then Corollary 2.12 tells us that this is not possible . Consequently M_2 is also algebraic. Since $X_i \neq \check{\mathbf{S}}$ the main Theorem will apply and our conclusion will follow

(b) Assume that $M_1$is non algebraic. We infer from a main result in [V2] that

$$X \cong \mathcal{A}_{k,\alpha,0} \quad (\#)$$

Now if M_2 is algebraic, then in view of (#) and $(\Gamma_2)^2 = 0$, Lemma 2.11 tells us that this not possible. Hence M_2 is also non algebraic and our conclusion will follow from [V2]

Now assume that $M_1$is algebraic. Then, following [V1]

$$X \neq \mathcal{A}_{k,\alpha,0} \quad (\#\#)$$

Now if M_2 is non algebraic, we infer from [V1], $(\Gamma_2)^2 = 0$ and (##) that this is not possible. Hence M_2 is also algebraic and our conlusion will follow from a main result in [V2]

Q.E.D

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