ON THE ANALYTIC PICARD GROUP OF CERTAIN AFFINE ALGEBRAIC HYPERSURFACES.

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Abstract: Let $Y \subset \mathbf{P}_N$ be a non singular hypersurface with $N > 3$, let $\Gamma \subset Y$ be a transverse hyperplane section and let $\mathcal{A} := Y \backslash \Gamma$. In 1966, A. Howard established the following result:

The analytic Picard group of $\mathcal{A}$ is trivial (*)

In 1973, Gerstner and Kaup showed that (*) remained valid if Y has only isolated singularities, say $\{q_k\}$ and Γ is merely a non singular hyperplane section, provided $\{q_k\} \notin \Gamma$. The purpose of this note is to propose to the latter result, a counterexample which is a nodal and irreducible 3-dimensional hypersurface $\mathbf{Y} \subset \mathbf{P}_4$. Also a geometric characterization of a Non-Kahlerian and non singular resolution π: $\mathbf{m} \to \mathbf{Y}$ will be established.

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§ 0.- **The Motivation**

Unless the contrary is explicitly stated, all 3-dimensional (resp. 2-dimensional) connected C-analytic manifolds will be referred to simply as *threefolds* (resp. *surfaces*). For any C-algebraic variety X, let us denote by $\mathrm{Pic}(X) := H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*)$ the *analytic Picard group* of X, where $\mathcal{X}$ is the underlying C-analytic space associated to X. $\mathcal{L} \in \mathrm{Pic}(X)$ is called *numerically positive* if $\mathcal{L} . \mathcal{C} > 0$ for any compact irreducible curve $\mathcal{C} \subset X$. Also the finite sum $\Xi := \Sigma_i n_i C_i$ where $n_i \in \mathbf{Z}^+$ and $C_i \subset X$ are irreducible compact curves for any i, will be called a *1- cycle* and Ξ is said to be *numerically trivial* if $\mathcal{L} . \Xi = 0$ for any $\mathcal{L} \in \mathrm{Pic}(X)$

In 1966, A. Howard established the following result [H] (corollary 2.3)

Proposition 0.1: *Let $N > 3$, let $Y_0 \subset \boldsymbol{P}_N$ be a connected and non singular hypersurface and let $Y' \subset \boldsymbol{P}_N$ be a linear hyperplane intersecting Y_0 transversely. Let $\Gamma_0 := Y_0 \cap Y'$ and let $\mathbb{A}_0 := Y_0 \backslash \Gamma_0$ Then $\mathrm{Pic}(\mathbb{A}_0)$ is 0*

Also as noticed in [H] (p.213) the hypothesis of transversality of Y' is crucial here; in fact one has

Example 0.2:: Let $Y_2:= \{ x^2 + y^2 + z^2 + w^2 = 0 \} \subset \mathbf{P}_4$ (x:y:z:w:t) be a quadric hypersurface with a single (isolated) singular point q = (0:0:0:0:1) and let $\underline{A}_2 := Y_2 \cap \{x \neq 0\}$. Then it is clear that $\underline{A}_2 \cong \{\zeta^2 + \xi^2 + \nu^2 = -1\} \subset \mathbf{C}^4 (\zeta,\xi,\nu,\tau)$ is a non singular affine algebraic hypersurface, where $\zeta:= y/x$ $\xi := z/x$ $\nu:= w/x$ and $\tau:=t/x$ Certainly $\underline{A}_2$ is homotopically equivalent to $\underline{A}_2 \cap \{\tau = 0\}$ which has the same homotopy type as the 2-sphere S^2 ; consequently, in contrast with A_0

$$\mathrm{Pic}(\underline{A}_2) \cong H^2(\underline{A}_2, \mathbf{Z}) = \mathbf{Z}$$

By steering clear from Example 0.2, in 1973, the following generalization of Proposition 0.1 was given in [GK] (Satz 1, p.121 & Korollar 3, p.125)

Theorem 0.3: *Let $N > 3$, and let $Y \subset \mathbf{P}_N$ be an irreducible hypersurface with only isolated singularities, say $\{q_k\}$. Now, let $\Gamma \subset Y$ be a non singular hypersurface section such that $\{q_k\} \notin \Gamma$ and let $\mathcal{A} := Y \setminus \Gamma$. Then*

(a) $\pi(\mathcal{A})$, the fundamental group of $\mathcal{A}$, is finite cyclic, and

(b) $Pic(\mathcal{A}) = 0$, provided Γ is a non singular hyperplane section

One of the main purposes of this note is to provide a simple *counterexample* to Theorem 0.3 (b) (see Counterexample1.3 below); consequently its fundamental group, in view of Theorem 0.3(a), is *non trivial* cyclic. It turns out that such a construction also provides a road map for a geometric characterization of certain Moishezon 3-folds $\mathbf{M}$ (resp. *compactifiable* 1-convex 3-folds $\mathbf{X} \subset \mathbf{M}$) which are Non-Kahlerian. That will be the goal of the last section. First of all let us introduce the following

Definition 0.4: [V1] A C-analytic manifold X is said to be *strongly pseudoconvex* (or *1-convex* for short) if X is a non singular resolution $\pi: X \longrightarrow Y$ of some *Stein* space Y which has only finitely many isolated singular points $\{q_k\}$ Henceforth, $S:= \cup_k \pi^{-1}(q_k)$ will be referred to as the *exceptional set* of X. Furthermore, X is said to be *compactifiable* if there exist a compact C-analytic manifold M and a C-analytic subvariety $\Gamma \subset M$ such that $X \cong M \setminus \Gamma$

§ 1.- **The Small Resolution**

We are now in a position to proceed to the construction of the following

Example 1.1: For any fixed integer $d \in \mathbf{Z}$ with $d \geq 3$, let Y_d be an irreducible hypersurface of degree d in $\mathbf{P}_4$ with only one isolated singular point{ p} which is an ordinary double point. Let $\Gamma_d \subset Y_d$ be a non singular hyperplane section such that $\{p\} \notin \Gamma_d$

Since {p} is an ordinary singularity, a result in [Kz] tells us that Y_d admits an *irreducible small* resolution $\pi: M_d \longrightarrow Y_d$, where M_d is a compact 3-fold with $\mathbf{Exc}(\pi)$:= the *exceptional locus* of $\pi \cong \mathbf{P}_1$. Now, Lefschetz hyperplane section theorem and Poincaré duality tell us that

$$H^1(\Gamma_d, \mathbf{Z}) \cong H_3(\Gamma_d, \mathbf{Z}) = 0$$

Certainly, $\boldsymbol{A}_d := Y_d \setminus \Gamma_d$, is Stein; hence, in view of Definition 0.4,

$X_d := M_d \setminus \Theta_d$ is a compactfiable1-convex 3-fold with exceptional set $\varepsilon \cong \mathbf{Exc}(\pi) = \mathbf{P}_1$. (^)

where $\Theta_d := \pi^{-1}(\Gamma_d)$. Since, $d \geq 3$, {p} is *factorial* [Ch]: consequently $H^2(M_d, \mathbf{Z}) \cong H_4(M_d, \mathbf{Z}) \cong \mathbf{Z}$ Thus, from the following exact sequence

$$\mathbf{Z} \cong H_4(\Theta_d, \mathbf{Z}) \xrightarrow{\iota_*} H_4(M_d, \mathbf{Z}) \cong \mathbf{Z} \to H_4(M_d, \Theta_d; \mathbf{Z}) \to H_3(\Theta_d, \mathbf{Z}) = 0$$

we infer that

$$H^2(X_d, \mathbf{Z}) \cong H_4(M_d; \Theta_d, \mathbf{Z}) \text{ is a finite group for any } d \geq 3 \qquad (\#)$$

Claim : *X_d is non Kählerian* ($)

Assume the contrary and assume that Ω is a Kähler (1,1) form on X_d upon which one has the following exact sequence

$$0 \to \mathbf{R} \xrightarrow{\iota} \mathcal{O} \to \mathcal{P} \to 0$$
$$f \mapsto \operatorname{Re} f$$

where $\mathcal{O}$ is the analytic structure sheaf of X_d , $\mathcal{P}$ is the sheaf of germs of pluriharmonic functions, see e.g. [HL] and the map ι is defined by multiplication by $\sqrt{-1}$. Hence one obtains the following exact sequence

$$H^1(X_d, \mathcal{O}) \to H^1(X_d\ \mathcal{P}) \to H^2(X_d, \mathbf{R}) \qquad (1)$$

Let us consider the following restriction morphism

$$\lambda := \pi|X_d : X_d \to \boldsymbol{A}_d \qquad (\sim)$$

Since{p}is a rational singularity, we infer from Leray spectral sequence that the left hand side group in (1) vanishes; and so does its right hand side group, in view of (#). Hence it follows readily that

$$H^1(X_d\ \mathcal{P}) = 0 \qquad (2)$$

On the other hand from the following exact sequence

$$0 \to \mathcal{P} \to \mathcal{E} \to \mathcal{K} \to 0 \qquad (\blacklozenge)$$
$$\phi \mapsto \iota\, \partial\bar\partial\phi$$

where $\mathcal{E}$ (resp $\mathcal{K}$) is the sheaf of germs of real valued differentiable functions (resp of real valued differentiable closed (1,1) forms)[HL] one gets, in view of (2), the following exact sequence

$$\Gamma(X_d, \mathcal{E}) \to \Gamma(X_d, \mathcal{K}) \to H^1(X_d, \mathcal{P}) = 0 \qquad (3)$$

Consequently (3) gives rise to some global differentiable function ϕ on X_d such that

$$i\, \partial\bar\partial\phi = \Omega \text{ is positive definite}$$

i.e. ϕ is *strongly plurisubharmonic*. Thus, it follows readily from the maximum principle for plurisubharmonic functions that X_d is free of compact subvarieties of positive dimension; this contradicts (^) and our claim is proved.

Remark1.2: It follows readily from ($) that M_d is not projective algebraic. Another proof of this fact can be found in [M1] (p.174) (see also [C]) where it is shown that ε is a *null homologous* 1-cycle in M_d

Counterexample 1.3 For any fixed integer $d \geq 3$, the hypersurface $Y_d \subset \mathbf{P}_4$ (x:y:z:w:t) considered in Example 1.1 can be explicitly described by the homogeneous equation, see e.g. [W1][CC],

$$Q_2(x, y, z, w)\, t^{d-2} + H_d(x, y, z, w) = 0 \qquad (\clubsuit)$$

where $Q_2(\mathbf{x})$ (resp $H_d(\mathbf{x})$) is a non degenerate quadric (resp.a homogeneous equation of degree d) in $\mathbf{P}_3(\mathbf{x})$, where $\mathbf{x}$:=(x,y,z,w). Now let

$$p := (0:0:0:0:1) \in \mathbf{P}_4(\mathbf{x}:t) \quad \text{and} \quad Y' := \{t = 0\} \cong \mathbf{P}_3(\mathbf{x})$$

Certainly

$$\{p\} \notin Y' \text{ and } \Gamma_d := Y_d \cap Y' \text{ is non singular by Bertini theorem} \qquad (4)$$

Step1: Let $\boldsymbol{m}$ be the maximal ideal sheaf of {p}.Then a main result in [HR](Theorem1) tells us that there exists an $\boldsymbol{m}$-primary ideal sheaf $\mathcal{J}(\sqrt{\boldsymbol{J}} \text{ -= } \boldsymbol{m})$ such that π: $M_d \longrightarrow Y_d$ is dominated by the monoidal transformation Π: $\boldsymbol{B} \longrightarrow Y_d$ with center $\boldsymbol{J}$ i. e. there exists a modification morphism τ: $\boldsymbol{B} \longrightarrow M_d$ such that $\pi \circ \tau = \Pi$. Since {p} is an ordinary singular point, $\boldsymbol{Q} = \mathbf{Exc}(\Pi) \cong \mathbf{P}_1 \text{ x } \mathbf{P}_1$. In fact since {p} is the only singular point in Y_d, one can assume that

$$\mathrm{B}:=\{Q_2 = H_d = 0\} \subset \mathbf{P}_3\,(\mathbf{x}) \text{ is a non singular curve.} \qquad (\heartsuit)$$

Notice that $C:= \{(\mathbf{x},t) \in Y_d | \ \mathbf{x} \in \mathrm{B}\}$ is the cone of all lines $l \in Y_d$ passing through {p}; hence we infer readily from ($\clubsuit$), ($\heartsuit$) and (4) that

$$\boldsymbol{l}._M \Theta_d = l._Y \Gamma_d = m \geq 1 \qquad (5)$$

where $\boldsymbol{l}$ is the strict transform of l by π .This situation can be summarized by the following diagram

$$\begin{array}{ccccccc}
 & & \mathcal{R}_2 & & & & \\
 & & \cap & & & & \\
\mathbf{P}_1 \text{x} \mathbf{P}_1 \cong \boldsymbol{Q} & \subset & \boldsymbol{B} & & & & \\
 & & \tau\downarrow & \Pi & & & \\
\mathbf{P}_1 \cong \varepsilon & \subset & M_d & \text{---}\pi\text{---}\longrightarrow & Y_d & \subset & \mathbf{P}_4, \\
 & & \cup & & \cup & & \\
 & \mathrm{B}_1 \subset \mathcal{R}_1 & & & \mathrm{p} \in C \supset \mathrm{B} & \subset & \mathbf{P}_3
\end{array}$$

where $\mathcal{R}_1 := \pi^*(C)$ is the compact surface (*singular* but *irreducible*) which is the union of lines $\boldsymbol{l}$ joining ε to $\mathrm{B}_1 := \pi^*(\mathrm{B})$.and $\mathcal{R}_2$ is a ruled surface; Notice that one has $\Pi^{-1}(C) \cong \boldsymbol{Q} \cup \mathcal{R}_2$, $\tau(\boldsymbol{Q}) \cong \varepsilon$ and $\mathcal{R}_1 \cong \tau(\mathcal{R}_2)$

Step2:: a) If d = 3, it follows readily from ($\clubsuit$) and (5) that m = 1; consequently (#) tells us that $H^2(X_3, \mathbf{Z}) = 0$

b) Yet this simple case shows us how to concoct a situation in which the latter group is non trivial, when d = 2N + 1 for any positive integer N.

Indeed, instead of the defining equation ($\clubsuit$), let us consider, the nodal hypersurface $\ddot{Y}_{2N+1}$, defined by

$$t^{2N+1}\, Q_2\,(x, y, z, w) + t\, Q_2^N(x, y, z, w) + H_{2N+1}\,(x, y, z, w) = 0 \qquad (\clubsuit\clubsuit)$$

where $Q_2(x, y, z, w)$ (resp. $H_{2N+1}(x, y, z, w)$) is a non degenerate quadric (resp. a generic homogeneous polynomial of degree 2N + 1) in $\mathbf{P}_3$(x:y:z:w). Then in view of [CC] (section II, p.288) $\ddot{Y}_{2N+1}$carries the same cone of lines, which in view of ($\clubsuit\clubsuit$), meets the hyperplane section Γ_{2N+1} along the *non reduced N-uple* curve

$$\check{\boldsymbol{c}} := \{Q_2^N = H_{2N+1} = 0\} \subset \mathbf{P}_3\,(\mathbf{x}) \qquad (\heartsuit\heartsuit)$$

From,($\heartsuit\heartsuit$) and (5) one can check that m = N. We infer from (#) that

$$H^2(X_{2N+1}, \mathbf{Z}) = \mathbf{Z}/N\mathbf{Z} \qquad (\%)$$

Notice that, although Γ_{2N+1} is non singular, it is not in general a transverse hyperplane section

Step3: From the morphism (~), let us consider the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^1(\varepsilon, \mathbf{Z}) & \rightarrow & H^2(X_d, \varepsilon; \mathbf{Z}) & \rightarrow & H^2(X_d, \mathbf{Z}) & \rightarrow & H^2(\varepsilon, \mathbf{Z}) \\
\uparrow\lambda_1 & & \uparrow\lambda_2 & & \uparrow\lambda_3 & & \uparrow\lambda_4 \\
0 \cong H^1(\{p\}, \mathbf{Z}) & \rightarrow & H^2(\boldsymbol{A}_d, \{p\}; \mathbf{Z}) & \rightarrow & H^2(\boldsymbol{A}_d, \mathbf{Z}) & \rightarrow & H^2(\{p\}, \mathbf{Z}) \cong 0
\end{array} \qquad (\spadesuit)$$

By construction $X_d \setminus \varepsilon \cong \boldsymbol{A}_d \setminus \{p\}$, it follows readily that λ_2 is an isomorphism. Therefore, from the above diagram one obtains the following exact sequence

$$0 \cong H^1(\varepsilon, \mathbf{Z}) \rightarrow H^2(\boldsymbol{A}_d, \mathbf{Z}) \longrightarrow H^2(X_d, \mathbf{Z}) - \rho^* \rightarrow H^2(\varepsilon, \mathbf{Z}) \cong \mathbf{Z}$$

:

In view of ($), X_d is Non Kählerian; hence we infer from a main result in [V1] (Theorem III) that the restriction application ρ^* is the zero map. Hence our desired construction is complete, provided one takes (%) into account.

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§.2. The Borel- Moore Homology

The major shortcoming in the proof of Theorem 0.3(b) stems from the *erroneous* assertion that the affine algebraic hypersurface $\mathcal{A}$ is *simply connected* which we shall discuss next. But first of all let us introduce the following:

Definition 2.1: ([K1] p.3) Let Y, $\mathcal{A}$ and Γ be as in Theorem 0.3.. Then, for any $k \geq 0$, let

$$\mathcal{H}_k(\mathcal{A}) := H_k(Y, \Gamma; \mathbf{Z})$$

where $H_k(Y, \Gamma; \mathbf{Z})$ stands for the *relative singular homology* of the pair (Y, Γ)

$\mathcal{H}_k(\mathcal{A})$ is the so called k^{th} *Borel-Moore integral homology of* $\mathcal{A}$.

In [GK] (Korollar 3, p.124), a strategy to prove Theorem 0.3 (b) followed exactly the same pattern as the one in [H]. In particular, it was established, by using the universal coefficient theorem

$$H^2(\mathcal{A}; \mathbf{Z}) \cong \mathrm{Hom}(H_2(\mathcal{A}, \mathbf{Z}); \mathbf{Z}) \oplus \mathrm{Tor}\, H_1(\mathcal{A}, \mathbf{Z})$$

and, by quoting (as explicitly mentioned in Korollar 2, p123 in [GK]) the fact that

$$\mathcal{H}_1(\mathcal{A}) = 0$$

a proof of which was given in [K2] (Korollar 2.8 (ii)); however, in general

$$\mathcal{H}_1(\mathcal{A}) \neq H_1(\mathcal{A}, \mathbf{Z})$$

namely the Borel-Moore homology is not necessarily the same as the singular one..
In fact, this is true if Y is non singular. Indeed, if Y is non singular, we have

$$\mathcal{H}_1(\mathcal{A}) := H_1(Y, \Gamma : \mathbf{Z}) \cong H^{2N-3}(Y, \Gamma; \mathbf{Z}) \cong H_1(\mathcal{A}, \mathbf{Z})$$

in view of the Poincaré (resp. Alexander-Lefschetz) duality., since **R**–dimension Y = 2N – 2. On the other hand, when the ambient space Y is singular, this is not always the case;

Remark 2.2: Counterexample 1.3 is , in some extent rare, as shown by the following result which was kindly communicated to us by Prof. P. Deligne.

Theorem 2.3*: Let $Y_d \subset \boldsymbol{P}_4(x{:}y{:}z{:}w{:}t)$ be the nodal hypersurface defined by equation (♣) in section 1.3 above. For generic $H_d(x, y, z, w)$, let $\Gamma_d \subset Y_d$ be a transverse hyperplane section and let $\boldsymbol{A}_d := Y_d \setminus \Gamma_d$. Then*

$$Pic(\mathbf{A}_d) \cong H^2(\mathbf{A}_d, \mathbf{Z}) = 0 \qquad \text{for any } d.$$

Proof: Since $H_d(x, y, z, w)$ is generic, one gets, by perturbation a differentiable 3-fold $\boldsymbol{M}$ such that

$$\boldsymbol{M} \approx \boldsymbol{Y} \qquad (6)$$

where $\boldsymbol{Y}$ is some non singular hypersurface of degree d in $\mathbf{P}_4$ and $\approx$ stands for diffeomorphism, and that, $\boldsymbol{M}$ carries a vanishing cycle $\delta \cong S^3$ the 3-sphere, which can be contracted topologically to $\{p\}$. Precisely there exists a topological morphism f: $\boldsymbol{M} \longrightarrow Y_d$ such that

$$\boldsymbol{M} \setminus S^3 \approx Y_d \setminus \{p\}. \qquad (!)$$

Now, let g: $\boldsymbol{M} \setminus f^{-1}(\Gamma_d) =: \boldsymbol{A} \longrightarrow A_d$ be the restriction topological modification morphism. We infer from (!) that

$$\boldsymbol{A} \setminus S^3 \approx A_d \setminus \{p\} \qquad (!!)$$

Consequently, by taking (!!) into account, similar commutative diagram as the one of (♠), in section 1.3, applies to the morphism g in this situation and one gets the following exact sequence

$$0 \cong H^1(\delta, \mathbf{Z}) \longrightarrow H^2(A_d \mathbf{Z}) - g^* \longrightarrow H^2(\boldsymbol{A}, \mathbf{Z}) \qquad (7)$$

In view of a result in [H](corollary 2.3) and (6) it follows readily that $H^2(\boldsymbol{A}, \mathbf{Z}) = 0$; hence our conclusion will follow from (7). **Q.E.D.**

§3. The Threefold Paradigm.

Notice that the previous examples only occurred in dimension 3 and this is by no means accidental as shown by the following

Proposition 3.1: *Let $Y \subset \boldsymbol{P}_V$ be a (strict) complete intersection with only isolated singularities, say$\{p_i\}$ and let $\pi : M \to Y$ be a non singular resolution. Assume that*

*(1) **Exc**(π) is of pure dimension one and*

(2) dim.$Y > 3$.

Then such a resolution does not exist

Proof: Assume the contrary that such a resolution does exist.
Let $\cup_i C_i$ be an irreducible decomposition of **Exc**(π), and let $\mathcal{K}$ be the canonical bundle of M. Let $\Gamma \subset Y$ be a non singular hyperplane section such that $\{p_i\} \notin \Gamma$ and let $X := M \setminus \Theta$ where $\Theta := \pi^{-1}(\Gamma)$. Since X is 1-convex with exceptional set $S \cong$ **Exc**(π) and since with C-dim $X \geq 4$ and dim $S = 1$, it follows readily that

(a) $\mathcal{K}|S$ is ample (Theorem 1.5 in [V2]) and

(b) for any irreducible and positive dimensional compact subvariety $Z \subset M$ such that $Z \neq C_i$, for any i, necessarily $Z \cap \Theta \neq \varnothing$. Hence we infer from Moishezon-Nakai criterion that for some N>>0, $\mathcal{K} \otimes L^N$ is ample where L is the line bundle on M determined by Θ.

Consequently, M is projective algebraic .Therefore GAGA type theorem implies that π is actually a projective morphism. By virtue of our hypothesis (2), $\{p_i\}$ are algebraically factorial [G] (XI 3.1.4); hence van der Waerden theorem tells us that S is of pure codimension one. Contradiction ! **Q.E.D.**

In sharp contrast to this situation, its 3-fold analogue is completely characterized by the following

Theorem 3.2: *Let $\pi : M \to Y$ and X be as in Proposition 3.1 with C-dim. Y = 3.*
Assume that

$$S := \mathbf{Exc}(\pi) \text{ is of pure dimension one} \qquad (\dagger)$$

Then the following conditions are equivalent:

(α) M is non-Kählerian

(β) M carries a null homologous 1-cycle Ξ

(γ) X carries a null homologous 1- cyle $\boldsymbol{\Lambda} \cong \Xi$

Proof: (α) ===> (β)

Step 1: We shall follow here closely an idea in [HL]. Let us consider the following homomorphism of Frechet spaces :

$$d: \mathcal{E}^{1,1}(M) \quad \to \quad \mathcal{E}^{2,1}(M) \oplus \mathcal{E}^{1,2}(M) \qquad (8)$$

where for any non negative pair of integers p and q, $\mathcal{E}^{p,q}(M)$ denotes the Frechet space of global, real-valued, smooth (p,q) forms on M and let $\mathcal{D}^{p,q}(M)$, be its dual space,. Notice that (8) induces a dual homomorphism

$$\partial.+ \partial : \mathcal{D}^{1,2}(M) \oplus \mathcal{D}^{2,1}(M) \to \mathcal{D}^{1,1}(M) \qquad (9)$$

Now let

$$\mathcal{P} := \{\phi \in \mathcal{E}^{1,1}(M) \mid \phi > 0\} \qquad \text{and}$$
$$\mathcal{K} := \{ \phi \in \mathcal{E}^{1,1}(M) \mid d\phi = 0\} \qquad (\blacklozenge\blacklozenge)$$

By hypothesis $\mathcal{P} \cap \mathcal{K} = \varnothing$. Following Hahn- Banach Theorem, one can find a continuous linear form say T, on $\mathcal{E}^{1,1}(M)$ separating $\mathcal{P}$ from $\mathcal{K}$, i.e. a *current* $T \in \mathcal{D}^{1,1}(M)$ such that

(a) $T(\phi) > 0$ for any $\phi \in \mathcal{P}$ i.e. T is a *positive* current and
(b) $T(\phi) = 0$ for every $\phi \in \mathcal{K}$ i.e. $\iota\partial\partial\, T = 0$

Since $M \backslash S \cong Y \backslash (\cup_k p_k) \}$ is quasi projective, (a) implies that $\mathrm{Supp}(T) \subset S$. In view of (b), a result in [HL](Lemma 32) tells us that $T = \Sigma_i r_i\ C_i$ where C_i are irreducible components of S and $r_i \in \mathbf{R}^+$.

Step 2: On the other hand from the Leray spectral sequence associated to the morphism π , one obtains the following exact sequence

$$H^2(Y, O_Y) \to H^2(M, O_M) \to \Gamma(Y, \mathcal{R}^2(\pi_*(O_M))) \qquad (10)$$

Notice that in (10) the first homology group vanishes in view of [S] (Prop.5(b), p.273); so does the third one, in view of the hypothesis (†). Consequently $H^2(M, O_M) = 0$ and an extension of the exact sequence (1) will give rise to the following one

$$H^1(M, \boldsymbol{P}) \xrightarrow{\ \rho\ } H^2(M, \mathbf{R}) \to H^2(M, O_M) \cong 0$$

On the other hand, from the exact sequence (♦) in section 1 and the definition (♦♦) above, one has $H^1(M, \boldsymbol{P}) \cong \mathcal{K}/\iota\, \partial\partial \mathcal{E}^{0,0}(M)$; therefore, one deduces from (b) and the surjectivity of ρ, that $T(\phi) = 0$ for any $\phi \in H^2(M, \mathbf{R})$. Hence, by duality, T is a null-homologous 1-cycle in $H_2(M, \mathbf{R}) := H_2(M, \mathbf{Z}) \otimes \mathbf{R}$ which is finitely generated. Thus, there exist integers $n_i \in \mathbf{Z}^+$ such that $\Xi := \Sigma_i n_i C_i$ is the desired null homologous 1-cycle

(β) ===> (γ)

In view of Lefschetz hyperplane section theorem, $H_3(\Gamma, \mathbf{R}) = 0$. Consequently, one obtains the following natural map of vector spaces

$$H_4(M, \mathbf{R}) \to H_4(M, \Gamma; \mathbf{R}) \to H_3(\Gamma, \mathbf{R}) = 0 \qquad (11)$$

By duality

$$H^2(M, \mathbf{R}) \cong H_4(M; \mathbf{R}) \quad \text{and} \quad H^2(X, \mathbf{R}) \cong H_4(M, \Gamma; \mathbf{R}) \qquad (12)$$

From (12) and the dual of finite dimensional vector spaces, the surjectivity of (11) is equivalent to the injectivity of

$$H_2(X, \mathbf{R}) \to H_2(M; \mathbf{R})$$

Consequently Ξ is homologous to zero in $H_2(X, \mathbf{R})$

$(\gamma) \Longrightarrow (\alpha)$ is trivial **Q.E.D.**

However, when dim.**Exc**$(\pi) = 2$, the situation is quite delicate, since Pic(M) (resp. Pic(X)) would be, in general no longer isomorphic to $H^2(M, \mathbf{R})$ (resp. $H^2(X, \mathbf{R})$) unless $\{p_i\}$are rational singularities However, we have the following result

Proposition 3.3: *Let $\pi: M \to Y$ be as in Theorem 3.2.*
Assume that, for any $r \neq s$,

$$E_r \cap E_s = \varnothing \qquad (\dagger\dagger)$$

*where $E_r \in$ **S:= Exc(π)** with $1 \leq r,s \leq p$, are irreducible 2-dimensional components*
Then M is Kahlerian iff M is free of numerically trivial 1-cycles

Proof: *Step1:* Assume that M is non-Kahlerian and that C-dim.Exc$(\pi) = 2$. (such a non Kahlerian 3-fold does exist, see Examples below). Certainly $M \setminus \mathrm{Exc}(\pi) \cong Y \setminus (\cup_k p_k)$ is quasi projective. Hence a main result in [M2] (see also [Hi]) tells us that there exist an ideal sheaf $\mathcal{J} \subset \mathcal{O}_M$ and a monoidal transformation $\mu: W \to M$ with center $\mathcal{J}$ such that

(a) support of $\mathcal{J} =: \mathcal{C} = \cup_j C_j$, where each C_j is a compact non singular curve $\subset S$ with $1 \leq j \leq q$
(b) W is projective algebraic

Consequently, in view of (b),

$$M \setminus \mathcal{C} \cong W \setminus \mu^{-1}(\mathcal{C}) \text{ is quasi-projective.} \qquad (13)$$

Step 2: In view of (b), let $H \in \mathrm{Pic}(W)$ be a very ample divisor and let $\mu_*(H) =: \mathbb{H} \in \mathrm{Pic}(M)$. Now let $\mathcal{L}_r \in$ Pic(M) be determined by E_r . Then we have the following alternatives:
(i) Assume that there exists a component $\mathcal{C}' \subset \mathcal{C}$ such that for some r, $\mathcal{C}' \subset E_r$ and that $\mathcal{L}_r.\mathcal{C}' = 0$,.
Then certainly $\mathcal{C}'$ is a null homologous 1-cycle in view of $(\dagger\dagger)$ and the fact that Pic(Y) = **Z**.
(ii) Assume that there exists an $\mathcal{L} \in$ Pic(M) such that $\mathcal{L} \mid C_j > 0$ (resp. $\mathcal{L} \mid C_j < 0$) for any j. Then, in view of (13), for some integer N >> 0 , $\mathcal{E} := \mathcal{L} \otimes \mathbb{H}^N$ (resp. $\mathcal{E}' := \mathcal{L}^* \otimes \mathbb{H}^N$) is numerically positive . Therefore, we infer from a result in [Ko] (Corollary 5.1.5) that M is projective algebraic; contradiction to the hypothesis that M is Non-Kahlerian
(iii) Consequently it remains the mixed case, and we can consider only the components E_ν with $\nu \in \{1,..,p\}$ such that each such E_ν contains or meets at least two of the C_j
In view of (i) and (ii), one can find positive integers ν_m such that $\mathcal{L}_\nu.Z_\nu = 0$ where $Z_\nu := \Sigma_m \nu_m C_m$ .Since there are only finitely many such E_ν, from the fact that rank Pic(Y) = 1 and the hypothesis $(\dagger\dagger)$, it follows that $\Xi := \Sigma_\nu Z_\nu$ is the desired numerically trivial 1-cycle **Q.E.D.**

In order to illustrate the above results, we would like to exhibit the following concrete examples which mirror certain peculiar aspects of Moishezon 3-folds.

Example 3.4: For any fixed integer $d \geq 3$, let $\pi: M_d \longrightarrow Y_d$ be an irreducible small resolution as in Example 1.1 . Let $\lambda: M_0 \longrightarrow M_d$ be the blow up of M_d along $\varepsilon \cong \mathbf{P}_1$. Then one can check that M_0 is projective algebraic with **Exc**$(\lambda) \cong \mathbf{P}_1 \mathrm{x}\, \mathbf{P}_1$. Notice that rank Pic$(M_0)$= 2

On the other hand let $\kappa: M_1 \longrightarrow M_d$ be the blow up of M_d along $\cup_j a_j \in \varepsilon$ where $\{a_j\}$, with $1 \leq j \leq k$, are k distinct points. Let σ (resp.$E_j \cong \mathbf{P}_2$) be the strict transform of ε (resp. a_j) by κ Then one can check that $\Lambda := \sigma + \Sigma_j C_j$ is a null-homologous 1-cycle, where $\mathbf{P}_1 \cong C_j \subset E_j$. Notice that $E_i \cap E_j = \varnothing$ and rank Pic(M_1)= k + 1

Example 3.5: Let $\hat{Y} := \mathbf{H}_1 \cap \mathbf{H}_2$, where $\mathbf{H}_i$, (i = 1 or 2) are quadric hypersurfaces in $\mathbf{P}_5$, such that $\hat{Y}$ contains exactly 1 singular point {p} [F] which is locally defined by$\{ x^2 + y^2 + z^2 + w^4 = 0\}$ in $\mathbf{C}^4(x, y, z, w)$.Then one can check that $\hat{Y}$ admits a non singular resolution $\pi: M_2 \to \hat{Y}$ such that

(a) $\mathcal{R}:=\mathbf{Exc}(\pi) \cong \mathbf{F}_2$, a Hirzebruch ruled surface

(b) $\mathcal{R}$ carries a rational 1-cycle $\Xi := \zeta + \xi$ which is null homologous in M_2 , where $\zeta \cong \mathbf{P}_1$ and $\xi \cong \mathbf{P}_1$ are 2 disjoint sections in $\mathbf{F}_2$, such that $\xi^2 = - \xi^2 = 2$, and

(c) rank $\mathrm{Pic}(M_2) = 2$

Example 3.6: Let $\ddot{Y} := \boldsymbol{H}_1 \cap \boldsymbol{H}_2$, where $\boldsymbol{H}_i$, (i = 1 or 2) are quadric hypersurfaces in $\mathbf{P}_5$, such that $\ddot{Y}$ contains exactly 1 singular point {q} [W2](Beispiel 1, p.25) which is locally defined by$\{ x^2 + y^2 + z^2 + w^6 = 0\}$ in $\mathbf{C}^4$ but which is *not* factorial. Then one can check that $\ddot{Y}$ admits a non singular *projective* resolution $\tau: M_3 \to \ddot{Y}$ such that $\mathbf{Exc}(\tau) \cong \mathbf{P}_1$ and rank $\mathrm{Pic}(M_3)= 2$

Example 3.7: In [Ko] (Example 4.3.1) was exhibited a normal 3 dimensional Moishezon space $\mathbf{M}$ with only isolated rational singularities$\{q_i\}$. Furthermore, $\mathbf{M}$ contains 2 compact non singular curves, say, C and D such that

(i) C is rational , D is of arbitrary genus and $C \cap D = \varnothing$

(ii) The 1-cycle C + D is numerically trivial, and

(iii) $\cup_i q_i \subset C$

Now let $\pi : M_4 \to \mathbf{M}$ be the blow-up of $\mathbf{M}$ with center $\cup_i q_i$, let $\mathcal{C}$ (resp. $\mathcal{D}$) be the strict transform of C (resp.D) by π and let $E_i:= \pi^{-1}\{q_i\}\cong \mathbf{P}_2$. Notice that $E_i \cap E_j =\varnothing$. Then certainly

(a) M_4 is a Moishezon 3-fold

(b) M_4 carries a null-homologous 1-cycle $\Lambda:= \mathcal{C} + \Sigma_i m_i C_i + \mathcal{D}$

where m_i := multiplicity of q_i, and $\mathbf{P}_1 \cong C_i \subset E_i$

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