ON DIFFUSIVE-DISPERSIVE TRAVELLING WAVES FOR GENERAL FLUX FUNCTIONS

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ABSTRACT. We are interested in the diffusive-dispersive travelling waves for a general flux function. First, we will construct explicitly travelling waves when the line connecting the two left-hand and right-hand states meets the graph of the flux function at least four points. Second, we will discuss about the asymptotic behavior of trajectories among those we could have travelling waves. Finally, we make some remark about the uniqueness of the Riemann problem.

CONTENTS

1. INTRODUCTION

The existence of nonclassical shocks has been studied from mainly two ways: either to construct nonclassical shocks directly or to confirm the existence of the relative travelling waves. The first trend relates

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to the selection of nonclassical shocks and the solution of the Riemann problem, proposed by follows the strategy proposed by Abeyaratne-Knowles [1, 2], and by Hayes and LeFloch [9, 10], and then developed in [14, 10, 15, 18, 19, 17, 21], etc. The second trend relates to smooth solutions of conservation laws with diffusion and dispersion, see [11, 15, 9, 8, 5, 6, 4, 16], etc. Different from classical shocks, the line connect the two states of a nonclassical shock always crosses the graph of the flux function. And in the above works, the line crosses the graph once. The existence of the nonclassical shock waves such that the line connects the two states of the shock has been widely expected. And this is the goal of this work, by means of studying the existence of travelling waves of the following conservation laws

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_{xx} u + \delta \partial_{xxx} u, \tag{1.1}$$

where the flux function is given by

$$f(u) = \big||u| - 1\big|, \qquad u \in \boldsymbol{R}. \tag{1.2}$$

We are interested in the limit of (1.1) when $\varepsilon \to 0$ and

$$\alpha = \varepsilon / \sqrt{\delta}$$

is kept constant. If limit of solutions of (1.1) when $\varepsilon \to 0$ exists, we can imply the existence of nonclassical shock waves of the conservation law

$$\partial_t u + \partial_x f(u) = 0. \tag{1.3}$$

We expect that this work will provide as an illustration to the existence of diffusive-dispersive travelling waves in the case where the flux function admits several inflection points. Furthermore, the straight line connecting between the two states of nonclassical shocks can cross the graph of the flux function at more than one point.

We note that diffusive-dispersive travelling waves were discovered by Jacobs, McKinney, and Shearer [11] for the cubic function. The developments can be found in [9, 8, 5, 6, 4], etc. Recently, non-monotone travelling waves were observed in [3]. In this work, the authors also address the existence of travelling waves when the line connecting two states of the corresponding nonclassical shock cuts the graph of the flux function at other two points in the interval between the left-hand and the right-hand states.

A geometrical distinction between the classical (see [22, 13, 20]) and nonclassical shocks is that in the case of classical shocks, the line connecting the two left-hand and right-hand states does not cross the graph of the flux function in the interval between these two states, while it is the case for nonclassical shocks. In the literature, diffusive-dispersive travelling waves were constructed explicitly. However, in most of these works, the line connecting the two states of the nonclassical shock crossing the graph of the flux function at only one point.

Our goal in this work is to obtain certain "qualitative" as well as "quantitative" properties of travelling waves of diffusive-dispersive model (1.1) for general flux functions. For this, we first construct explicitly diffusive-dispersive travelling waves where the line connecting the two states of the nonclassical shock crosses the graph of the flux function at two points. Furthermore, we will perform the construction in a direct and simple way. By this work, we want to emphasize that nonclassical shocks of "complicated" functions can exist. More precisely, the graph of the flux function between the two states of nonclassical shocks could be complicated and may admit several inflection points. Second, we will study asymptotic behavior of equilibria of differential systems related to the travelling waves of (1.1).

2. An explicit formula for travelling waves

2.1. **Preliminaries.** We will construct explicitly *travelling waves of (1.1)* i.e., smooth solution $u = u(y)$ depending on the re-scaled variable

$$y := \alpha\frac{x - \lambda t}{\varepsilon} = \frac{x - \lambda t}{\sqrt{\delta}}, \tag{2.1}$$

for some constant speed λ. Substituting $u = u(y)$, where y is given by (2.1) to (1.1), after re-scaling, the function u is a travelling wave (1.1) connecting a left-hand state u_- to a right-hand state u_+. That is $u = u(y), y \in \boldsymbol{R}$ satisfies the ordinary differential equation

$$-\lambda\frac{du}{dy} + \frac{df(u)}{dy} = \alpha\frac{d^2u}{dy^2} + \frac{d^3u}{dy^3}, \tag{2.2}$$

and the boundary conditions

$$\begin{aligned} &\lim_{y\to\pm\infty} u(y) = u_\pm, \\ &\lim_{y\to\pm\infty} \frac{du}{dy} = \lim_{y\to\pm\infty} \frac{d^2u}{dy^2} = 0. \end{aligned} \tag{2.3}$$

Integrating (2.2) and using the boundary condition (2.3), we find u such that

$$\frac{d^2u}{dy^2} + \alpha\frac{du}{dy} = -\lambda(u(y) - u_-) + f(u) - f(u_-), \quad y \in \boldsymbol{R}. \tag{2.4}$$

Using (2.3) again, we deduce from (2.4)

$$\lambda = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \tag{2.5}$$

In the follows we will fix the left-hand side u_- and find u_+ such that a smooth solution u satisfying (2.4) and (2.5). For simplicity, we assume $u_- > 0$, though the case $u_- < 0$ can be done similarly. Moreover, we are interested in the case that

(H) The solution u is constraint to the condition that the line denoted by (Δ) connecting $(u_\pm, f(u_\pm))$ cuts the graph of f at exactly four points $u_+ = u_3 < u_2 < u_1 < u_0 = u_-$. And, $u_- > 1$. Consequently, we have

$$|\lambda| < 1 \quad \text{and} \quad \lambda = \frac{f(u_i) - f(u_0)}{u_i - u_0}, \quad i = 1, 2, 3. \tag{2.6}$$

The case that the line (Δ) cuts the graph at only the two points $(u_\pm, f(u_\pm))$ will determine a classical shock, three points will determine a nonclassical shock, see [15].

The flux function f can be re-written as, see Figure 1

$$f(u) = \begin{cases} u - 1 & u \geq 1, \\ 1 - u & 0 < u < 1, \\ u + 1 & -1 \leq u < 0, \\ -u - 1 & u < -1 \end{cases} \tag{2.7}$$

2.2. **Entropy dissipation and Kinetic functions.** Let us consider the conservation laws in (1.1) with zero right-hand side

$$\partial_t u + \partial_x f(u) = 0. \tag{2.8}$$

We will study in this section the entropy dissipation of (2.9) which defines the corresponding kinetic function. And it has been known that kinetic functions define nonclassical shock waves.

Since the flux function f is Lipschitz continuous, and not smooth, we have to understand the notion of entropy in a more general sense, see [7]. Recall that a locally Lipschitz continuous function U is said to be the entropy of the conservation law (2.8) if there exists a locally Lipschitz continuous function F such that

$$F'(u) = U'(u) f'(u), \quad \text{for almost} \quad u \in \boldsymbol{R}. \tag{2.9}$$

We will constraint each discontinuity of (2.8) to satisfy a *single* entropy inequality for a given entropy pair (U, F)

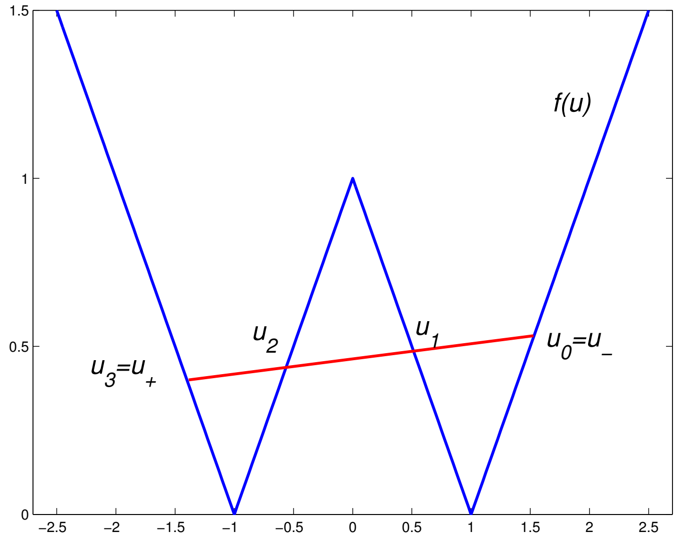

FIGURE 1. The line connecting two states of a shock crosses the graph of the flux function at exactly four points

$$\partial_t U(u) + \partial_x F(u) \leq 0. \tag{2.10}$$

Let us chose the entropy pair

$$U(u) = u^2, \quad F(u) = \begin{cases} -u^2, & u \leq -1, \\ u^2 - 2, & -1 \leq u \leq 0, \\ -u^2 - 2, & 0 \leq u \leq 1, \\ u^2 - 4, & u \geq 1. \end{cases} \tag{2.11}$$

The entropy inequality (2.10)-(2.11) is equivalent to the constraint that any discontinuity connecting a left-hand state u_0 and a right-hand state u of the conservation law (2.8) has to satisfy the condition that the corresponding entropy dissipation is non-positive:

$$E(u, u_0) := -s(U(u) - U(u_0)) + F(u) - F(u_0) \leq 0, \tag{2.12}$$

where s is the speed of the discontinuity defined by the Rankine-Hugoniot relation

$$s = \frac{f(u) - f(u_0)}{u - u_0}. \tag{2.13}$$

Due to the symmetry, it is not restrictive to assume that $u_0 > 0$. The case $u_0 \leq 0$ will be argued similarly. Furthermore, fixing the left-hand state u_0, we will investigate the sign of $E(u, u_0)$ to determine all possible right-hand state u that can be connected to u_0 by a nonclassical shock.

Proposition 2.1. *The entropy inequality (2.12) selects the following part*

(i) *if* $u_0 \geq 2$

$$u \in (-\infty, +\infty); \tag{2.14}$$

(ii) *if* $1 \leq u_0 < 2$

$$u \in (-\infty, \frac{-2}{u_0}) \cup (1 - u_0, +\infty); \tag{2.15}$$

(iii) *if* $0 \leq u_0 \leq 1$

$$u \in (-\infty, -1 - u_0] \cup [0, 1]. \tag{2.16}$$

Proof. First, suppose $u_0 \geq 1$. Then, we have $f(u_0) = u_0 - 1, F(u_0) = u_0^2 - 4$.

If $u \geq 1$, then $f(u) = u - 1, F(u) = u^2 - 4$. And therefore,

$$\begin{aligned} E(u, u_0) &= -\frac{u - 1 - (u_0 - 1)}{u - u_0}(u^2 - u_0^2) + u^2 - 4 - (u_0^2 - 4) \\ &= 0. \end{aligned}$$

If $0 \leq u \leq 1$, then $f(u) = 1 - u, F(u) = -u^2 - 2$. And we have

$$\begin{aligned} E(u, u_0) &= -\frac{1 - u - (u_0 - 1)}{u - u_0}(u^2 - u_0^2) - u^2 - 2 - (u_0^2 - 4) \\ &= 2(u_0 - 1)(u - 1) \leq 0. \end{aligned}$$

If $-1 \leq u \leq 0$, then $f(u) = u + 1, F(u) = u^2 - 2$. Thus,

$$\begin{aligned} E(u, u_0) &= -\frac{u + 1 - (u_0 - 1)}{u - u_0}(u^2 - u_0^2) + u^2 - 2 - (u_0^2 - 4) \\ &= -2(u + u_0 - 1) \end{aligned}$$

is nonpositive if and only if

$$u \geq 1 - u_0.$$

Furthermore, the condition $1 - u_0 \geq -1$ implies $u_0 \geq 2$.

If $u \leq -1$, then $f(u) = -u - 1, F(u) = -u^2$. Then,

$$\begin{aligned} E(u, u_0) &= -\frac{-u - 1 - (u_0 - 1)}{u - u_0}(u^2 - u_0^2) - u^2 - (u_0^2 - 4) \\ &= 2(uu_0 + 2) \end{aligned}$$

is nonpositive if and only if

$$u \leq \frac{-2}{u_0}.$$

Similarly, the condition $-2/u_0 \le -1$ implies $u_0 \ge 2$.

Consider now the case $0 \le u_0 \le 1$. Then, $f(u_0) = 1 - u_0, F(u_0) = -u_0^2 - 2$.

If $u > 1$, then

$$\begin{aligned} E(u, u_0) &= -\frac{u - 1 - (1 - u_0)}{u - u_0}(u^2 - u_0^2) + u^2 - 4 - (-u_0^2 - 2) \\ &= 2(1 - u_0)(u - 1) > 0. \end{aligned}$$

If $0 \le u \le 1$, then

$$E(u, u_0) \equiv 0.$$

If $-1 < u < 0$, then

$$\begin{aligned} E(u, u_0) &= -\frac{u + 1 - (1 - u_0)}{u - u_0}(u^2 - u_0^2) + u^2 - 2 - (-u_0^2 - 2) \\ &= -2uu_0 > 0. \end{aligned}$$

Finally, if $u \le -1$, then

$$\begin{aligned} E(u, u_0) &= -\frac{-u - 1 - (1 - u_0)}{u - u_0}(u^2 - u_0^2) - u^2 - (-u_0^2 - 2) \\ &= 2(u + u_0 + 1) \end{aligned}$$

is nonpositive if and only if

$$u \le -1 - u_0.$$

This terminates the proof.

□

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Therefore, we can define kinetic functions associate with the conservation law (2.8) as follows.

Definition 2.2. *A kinetic function associate with the conservation law (2.8) is any locally Lipschitz continuous function φ so that*

$$u \cdot \varphi(u) < 0 \quad \text{and} \quad E(u, \varphi(u)) \le 0. \tag{2.17}$$

2.3. **Construction of the travelling waves.** Under the hypothesis (H), we will find the diffusive-dispersive travelling waves of (1.1).

Theorem 2.3. *Given a left-hand state $u_- > 1$ (for definitiveness). For a range of right-hand states $u_+ < -1$ the travelling wave connecting u_- and u_+ can be constructed explicitly.*

Proof. To find u satisfy (2.3) and (2.4), we first solve

$$\begin{aligned} \frac{d^2u}{dy^2} + \alpha\frac{du}{dy} &= -\lambda(u(y) - u_0) + f(u) - f(u_0), \quad y \in \boldsymbol{R}, \\ u(-\infty) &= u_0, \\ \frac{du}{dy} &= 0. \end{aligned} \tag{2.18}$$

Since $u_0 > 1$, we first restrict our consideration in the range $u \geq 1$. In this case $f(u_0) = u_0 - 1$ and $f(u) = u - 1$. It is therefore derived from (2.18) that

$$\begin{aligned} \frac{d^2u}{dy^2} + \alpha\frac{du}{dy} &= -\lambda(u(y) - u_0) + (1-\lambda)(u - u_0), \quad y \in \boldsymbol{R}, u \geq 1, \\ u(-\infty) &= u_0, \\ \frac{du}{dy} &= 0. \end{aligned} \tag{2.19}$$

Setting $v = u - u_0$, we can re-write the differential equation in (2.19) as

$$v'' + \alpha v' + (\lambda - 1)v = 0. \tag{2.20}$$

Setting

$$\beta_1 := \frac{-\alpha - \sqrt{\alpha^2 + 4(1-\lambda)}}{2} < 0 < \beta_2 := \frac{-\alpha - \sqrt{\alpha^2 + 4(1-\lambda)}}{2}, \tag{2.21}$$

we obtain the solution of the equation (2.20) as

$$v(y) = Ae^{\beta_1 y} + Be^{\beta_2 y},$$

where A and B are constant. On substituting $u = v + u_0$,

$$u(y) = Ae^{\beta_1 y} + Be^{\beta_2 y} + u_0.$$

Now, using the limit condition in (2.18)

$$u(y) \to u_0 \quad y \to -\infty$$

we deduce $A = 0$. Thus,

$$u(y) = Be^{\beta_2 y} + u_0. \tag{2.22}$$

Taking the derivative of u, we get

$$u'(y) = B\beta_2 e^{\beta_2 y}.$$

Since we want the solution u to attain values smaller than u_0, we have to take $B < 0$. Thus, the function u will be decreasing from u_0 and it reaches the value $u = 1$ when

$$u(y) = Be^{\beta_2 y} + u_0 = 1, \tag{2.23}$$

or

$$y = y_0 := \frac{1}{\beta_2} \ln \frac{1 - u_0}{B}.$$

Next, we consider the problem (2.18) for $y > y_0$ (thus, $u < 1$) and we restrict attention to the range $0 \le u \le 1$. Since λ, u_0, u_1 satisfy (2.6), we can re-write the differential equation of (2.18) as

$$\begin{aligned} \frac{d^2u}{dy^2} + \alpha \frac{du}{dy} &= -\lambda(u(y) - u_0) + f(u) - f(u_0) \\ &= -\lambda(u(y) - u_1) + f(u) - f(u_1) \end{aligned}$$

and so, for $0 \le u, u_1 \le 1$ the last differential equation becomes

$$\begin{aligned} \frac{d^2u}{dy^2} + \alpha \frac{du}{dy} &= -\lambda(u(y) - u_1) - u(y) + u_1 \\ &= -(\lambda + 1)(u(y) - u_1). \end{aligned} \tag{2.24}$$

Setting $v = u - u_1$, we have

$$v'' + \alpha v' + (\lambda + 1)v = 0, \quad y > y_0. \tag{2.25}$$

Later, we will see that the case $\alpha^2 - 4(\lambda + 1) < 0$ does not imply the existence of travelling waves. Let us assume now $\alpha^2 - 4(\lambda + 1) > 0$. Set

$$\beta_3 := \frac{-\alpha - \sqrt{\alpha^2 - 4(\lambda + 1)}}{2} < \beta_4 := \frac{-\alpha - \sqrt{\alpha^2 - 4(\lambda + 1)}}{2} < 0. \tag{2.26}$$

Then, the solution of (2.25) is

$$v(y) = Ce^{\beta_3 y} + De^{\beta_4 y},$$

where C, D are constants. On substituting $u = v + u_1$ we get

$$u(y) = Ce^{\beta_3 y} + De^{\beta_4 y} + u_1.$$

Using the initial conditions at y_0, we will solve for the constants C, D

$$\begin{aligned} u(y_0) &= Ce^{\beta_3 y_0} + De^{\beta_4 y_0} + u_1 = 1, \\ u'(y_0) &= C\beta_3 e^{\beta_3 y_0} + D\beta_4 e^{\beta_4 y_0} = \beta_2(1 - u_0). \end{aligned} \tag{2.27}$$

These two simultaneous linear equations give the constants C, D:

$$C = \frac{\begin{vmatrix} 1-u_1 & e^{\beta_4 y_0} \\ \beta_2(1-u_0) & \beta_4 e^{\beta_4 y_0} \end{vmatrix}}{\begin{vmatrix} e^{\beta_3 y_0} & e^{\beta_4 y_0} \\ \beta_3 e^{\beta_3 y_0} & \beta_4 e^{\beta_4 y_0} \end{vmatrix}} = \frac{\beta_4(1-u_1) - \beta_2(1-u_0)}{e^{\beta_3 y_0}(\beta_4 - \beta_3)}, \tag{2.28}$$

and

$$D = \frac{\begin{vmatrix} e^{\beta_3 y_0} & 1-u_1 \\ \beta_3 e^{\beta_3 y_0} & \beta_2(1-u_0) \end{vmatrix}}{\begin{vmatrix} e^{\beta_3 y_0} & e^{\beta_4 y_0} \\ \beta_3 e^{\beta_3 y_0} & \beta_4 e^{\beta_4 y_0} \end{vmatrix}} = -\frac{\beta_3(1-u_1) - \beta_2(1-u_0)}{e^{\beta_4 y_0}(\beta_4 - \beta_3)}. \tag{2.29}$$

Substituting C and D from (2.28) and (2.29) to (2.26), and after simplifying terms, we find the solution $u = u(y)$ for $y > y_0$ to be

$$u(y) = \frac{u_0 - 1}{\beta_4 - \beta_3}\Big[(\beta_2 + \frac{1-\lambda}{1+\lambda}\beta_4)e^{\beta_3(y-y_0)} - (\beta_2 + \frac{1-\lambda}{1+\lambda}\beta_3)e^{\beta_4(y-y_0)}\Big] + u_1. \tag{2.30}$$

Differentiating u from (2.30) we get

$$u'(y) = \frac{u_0 - 1}{\beta_4 - \beta_3}\Big[(\beta_2 + \frac{1-\lambda}{1+\lambda}\beta_4)\beta_3 - (\beta_2 + \frac{1-\lambda}{1+\lambda}\beta_3)\beta_4 e^{(\beta_4-\beta_3)(y-y_0)}\Big]e^{\beta_3(y-y_0)} + u_1. \tag{2.31}$$

In the range of λ such that $u'(y) < -m < 0$ for some constant m (with appropriate values of $\beta_2, \beta_3, \beta_4$), the function u will reach the value $u = 0$ at some value $y = y_1 > y_0$. Thus,

$$u(y_1) = 0, \quad u'(y_1) = u^*. \tag{2.32}$$

Third, we consider the problem (2.18) for $y > y_1$ and $-1 \le u \le 0$. Since λ, u_0, u_2 satisfy (2.6), we can re-write the differential equation of (2.18) as

$$\begin{aligned} \frac{d^2u}{dy^2} + \alpha\frac{du}{dy} &= -\lambda(u(y) - u_0) + f(u) - f(u_0) \\ &= -\lambda(u(y) - u_2) + f(u) - f(u_2) \end{aligned}$$

and so, for $0 \ge u, u_2 \ge -1$ the last differential equation becomes

$$\begin{aligned} \frac{d^2u}{dy^2} + \alpha\frac{du}{dy} &= -\lambda(u(y) - u_2) + u(y) + 1 - (u_2 + 1) \\ &= (1-\lambda)(u(y) - u_2). \end{aligned} \tag{2.33}$$

Setting $v = u - u_2$, we have

$$v'' + \alpha v' + (\lambda - 1)v = 0, \quad y > y_1. \tag{2.34}$$

Therefore, the solution of (2.33) is given by

$$v(y) = Ee^{\beta_1 y} + Fe^{\beta_2 y},$$

where E and F are constant, and $\beta_{1,2}$ are given by (2.21). Once substituting $u = v + u_2$, we get

$$u(y) = Ee^{\beta_1 y} + Fe^{\beta_2 y} + u_2.$$

Using the initial conditions (2.32) we can determine the coefficients E, F as follows

$$\begin{aligned} u(y_1) &= Ee^{\beta_1 y_1} + Fe^{\beta_2 y_1} = -u_2, \\ u'(y_1) &= E\beta_1 e^{\beta_1 y_1} + F\beta_2 e^{\beta_2 y_1} = u^*. \end{aligned} \tag{2.35}$$

It is therefore derived from (2.35) that

$$E = \frac{\begin{vmatrix} -u_2 & e^{\beta_2 y_1} \\ u^* & \beta_2 e^{\beta_2 y_1} \end{vmatrix}}{\begin{vmatrix} e^{\beta_1 y_1} & e^{\beta_2 y_1} \\ \beta_1 e^{\beta_1 y_1} & \beta_2 e^{\beta_2 y_1} \end{vmatrix}} = -\frac{u_2\beta_2 + u^*}{e^{\beta_1 y_1}(\beta_2 - \beta_1)}, \tag{2.36}$$

and

$$F = \frac{\begin{vmatrix} e^{\beta_1 y_1} & -u_2 \\ \beta_1 e^{\beta_1 y_1} & u^* \end{vmatrix}}{\begin{vmatrix} e^{\beta_1 y_1} & e^{\beta_2 y_1} \\ \beta_1 e^{\beta_1 y_1} & \beta_2 e^{\beta_2 y_1} \end{vmatrix}} = \frac{u^* + u_2\beta_1}{e^{\beta_2 y_1}(\beta_2 - \beta_1)}. \tag{2.37}$$

Substituting E, F, we obtain

$$\begin{aligned} u(y) &= -\frac{u_2\beta_2 + u^*}{e^{\beta_1 y_1}(\beta_2 - \beta_1)} e^{\beta_1 y} + \frac{u^* + u_2\beta_1}{e^{\beta_2 y_1}(\beta_2 - \beta_1)} e^{\beta_2 y} + u_2 \\ &= -\frac{u_2\beta_2 + u^*}{\beta_2 - \beta_1} e^{\beta_1 (y - y_1)} + \frac{u^* + u_2\beta_1}{\beta_2 - \beta_1} e^{\beta_2 (y - y_1)} + u_2. \end{aligned} \tag{2.38}$$

We need this function to be decreasing, and until a certain time, it reaches $u = -1$ at some value $y = y_2$, so that

$$u(y_2) = -1, \quad u'(y_2) = u_*. \tag{2.39}$$

Finally, we consider the problem (2.18) for $y > y_1$ and $u \le -1$. Since λ, u_0, u_3 satisfy (2.6), we can re-write the differential equation of (2.18) as

$$\begin{aligned} \frac{d^2u}{dy^2} + \alpha\frac{du}{dy} &= -\lambda(u(y) - u_0) + f(u) - f(u_0) \\ &= -\lambda(u(y) - u_3) + f(u) - f(u_3) \end{aligned}$$

and so, for $u, u_3 \leq -1$ the last differential equation becomes

$$\begin{aligned}\frac{d^2u}{dy^2} + \alpha\frac{du}{dy} &= -\lambda(u(y) - u_3) - 1 - u(y) - (-1 - u_3) \\ &= -(1+\lambda)(u(y) - u_3).\end{aligned} \tag{2.40}$$

Setting $v = u - u_3$, we have

$$v'' + \alpha v' + (\lambda + 1)v = 0, \quad y > y_2. \tag{2.41}$$

Using the definition of $\beta_{3,4}$ from (2.26), we obtain the solution of (2.41) as

$$v(y) = Ge^{\beta_3 y} + He^{\beta_4 y},$$

where G, H are constants. On substituting $u = v + u_3$ we get

$$u(y) = Ge^{\beta_3 y} + He^{\beta_4 y} + u_4.$$

Using the initial conditions at y_2, we will solve for the constants G, H

$$\begin{aligned}u(y_2) &= Ge^{\beta_3 y_2} + He^{\beta_4 y_2} + u_3 = -1, \\ u'(y_2) &= G\beta_3 e^{\beta_3 y_2} + H\beta_4 e^{\beta_4 y_2} = u_*.\end{aligned} \tag{2.42}$$

These two simultaneous linear equations give the constants G, H:

$$G = \frac{\begin{vmatrix} -1 - u_3 & e^{\beta_4 y_2} \\ u_* & \beta_4 e^{\beta_4 y_2}\end{vmatrix}}{\begin{vmatrix} e^{\beta_3 y_2} & e^{\beta_4 y_2} \\ \beta_3 e^{\beta_3 y_2} & \beta_4 e^{\beta_4 y_2}\end{vmatrix}} = -\frac{\beta_4(1 + u_3) + u_*}{e^{\beta_3 y_2}(\beta_4 - \beta_3)}, \tag{2.43}$$

and

$$H = \frac{\begin{vmatrix} e^{\beta_3 y_2} & -1 - u_3 \\ \beta_3 e^{\beta_3 y_2} & u_*\end{vmatrix}}{\begin{vmatrix} e^{\beta_3 y_2} & e^{\beta_4 y_2} \\ \beta_3 e^{\beta_3 y_2} & \beta_4 e^{\beta_4 y_2}\end{vmatrix}} = \frac{\beta_3(1 + u_1) + u_*}{e^{\beta_4 y_2}(\beta_4 - \beta_3)}. \tag{2.44}$$

Substituting G, H, we obtain the solution u satisfying the limit conditions (2.3)

$$\begin{aligned}u(y) &= -\frac{\beta_4(1 + u_3) + u_*}{e^{\beta_3 y_2}(\beta_4 - \beta_3)}e^{\beta_3 y} + \frac{\beta_3(1 + u_1) + u_*}{e^{\beta_4 y_2}(\beta_4 - \beta_3)}e^{\beta_4 y} + u_3 \\ &= -\frac{\beta_4(1 + u_3) + u_*}{\beta_4 - \beta_3}e^{\beta_3(y - y_2)} + \frac{\beta_3(1 + u_1) + u_*}{\beta_4 - \beta_3}e^{\beta_4(y - y_2)} + u_+.\end{aligned} \tag{2.45}$$

□

3. Travelling waves with general fluxes

In this section, we will discuss about the existence of travelling waves to the general conservation law with diffusion and dispersion

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_{xx} u + \delta \partial_{xxx} u, \tag{3.1}$$

where f is any smooth function and we require $\alpha = \varepsilon/\sqrt{\delta}$ to be constant. Since this is very related to the stability of equilibrium points of differential equations. Therefore, we need to state the concept of equilibria of differential equations first.

3.1. **Equilibria of differential equations.** We need to derive from the original problem a differential equation in $\boldsymbol{R}^2$. Arguing as in Section 2, we obtain the travelling waves $u = u(y)$, where $y = \alpha \frac{x-\lambda t}{\varepsilon}$, connecting a left-hand state u_- to a right-hand state u_+ satisfy the quasi-linear differential equation

$$\frac{d^2u}{dy^2} + \alpha \frac{du}{dy} = -\lambda(u(y) - u_-) + f(u) - f(u_-), \quad y \in \boldsymbol{R}, \tag{3.2}$$

and the boundary conditions

$$\begin{aligned} &\lim_{y\to\pm\infty} u(y) = u_\pm, \\ &\lim_{y\to\pm\infty} \frac{du}{dy} = \lim_{y\to\pm\infty} \frac{d^2u}{dy^2} = 0. \end{aligned} \tag{3.3}$$

The shock speed λ also satisfies

$$\lambda = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \tag{3.4}$$

Setting

$$v = \frac{du}{dy} \tag{3.5}$$

we can re-write the second-order differential equation (3.2) to the following 2×2- system of first-order differential equations

$$\begin{aligned} u' &= v, \\ v' &= -\alpha v - \lambda(u - u_-) + f(u) - f(u_-), \end{aligned} \tag{3.6}$$

where the prime " $'$ " stands for the derivative with respect to y. Moreover, by setting

$$w = (u, v)^T, \quad g(w) = \big(v, -\alpha v - \lambda(u - u_-) + f(u) - f(u_-)\big)^T, \tag{3.7}$$

we can re-write the system (3.6) in a more compact of autonomous differential equations

$$w' = g(w), \quad w \in \boldsymbol{R}^2. \tag{3.8}$$

Suppose the line connecting u_- and u_+ cuts the graph of f at the points $u_i, i \in I$ in the interval between u_- and u_+, ($u_\pm$ inclusive). Then, these points satisfy

$$\lambda = \frac{f(u_i) - f(u_-)}{u_i - u_-}, \quad u_i \neq u_-. \tag{3.9}$$

Accordingly, the differential equation (3.8) admits $(u_i, 0), i \in I$ as its equilibrium points, i.e., the points at which the right-hand side vanishes:

$$g(u_i, 0) = 0, \quad i \in I. \tag{3.10}$$

More generally, we introduce the notion of equilibrium points of differential equations. Let us consider the autonomous differential equation

$$\frac{dX(t)}{dt} = F(X(t)), \quad t > 0, X \in D \subset \boldsymbol{R}^n. \tag{3.11}$$

Definition 3.1. *A point $X_0 \in D$ is called an equilibrium point of (3.11) if*

$$F(X_0) = 0.$$

3.2. **LaSalle's Invariance Principle.** From the above analysis, we will study the asymptotic behavior of equilibria of differential equations. Here, we will invoke LaSalle's *invariance principle*, [12]. First, we need to introduce a few definitions.

Consider the differential equation (3.11). A set $M \subset D$ is said to be an *invariant set* with respect to (3.11) if

$$X(0) \in M \quad \Rightarrow \quad X(t) \in M, \quad \forall t \in \boldsymbol{R}. \tag{3.12}$$

A set $M \subset D$ is said to be a *positively invariant set* with respect to (3.11) if

$$X(0) \in M \quad \Rightarrow \quad X(t) \in M, \quad \forall t \geq 0. \tag{3.13}$$

And similarly for a *negatively invariant set.*

Therefore, a set M is invariant if and only if it is both positively and negatively invariant.

We also say that $X(t)$ approaches a set M as t approaches infinity, if for every $\varepsilon > 0$, there is $T > 0$ such that the distance from a point p to a set M is less than ε:

$$\operatorname{dist}(X(t), M) := \inf_{X \in M} ||p - X|| < \varepsilon, \quad \forall t > T. \tag{3.14}$$

Suppose $V : D \to \boldsymbol{R}$ be a continuous differentiable function such that

$$\dot{V}(X) := \nabla V(X) \cdot F(X) \leq 0 \quad X \in \Omega. \tag{3.15}$$

We defined

$$E = \{X \in \Omega \mid \quad \dot{V}(X) = 0\}. \tag{3.16}$$

The LaSalle's invariance principle states that *if Ω is a compact set that is positively invariant with respect to (3.11), and M is the largest invariant set in E, then every solution starting in Ω approaches M as $t \to \infty$.*

3.3. **Existence of travelling waves.** Our interest is the existence of a solution of (3.8) satisfying the boundary conditions by applying LaSalle's invariance principle twice. This can be done by showing that E contains equilibria of (3.8) and that no solution can stay identically in E, except trivial (constant) solutions.

Setting

$$h(u) := -\lambda(u - u_-) + f(u) - f(u_-). \tag{3.17}$$

By applying the LaSalle invariance principle, we can prove a theorem in which u is required not to be a contact discontinuity.

Theorem 3.2. *Suppose from the condition $h(u(y)) \equiv 0$ we must imply that there exists $i \in I$ such that $u(y) \equiv u_i, i \in I$. Then, every trajectory of (3.6) approaches the set of equilibria of (3.6) in both directions $y \to \pm\infty$.*

Proof. A Lyapunov function candidate can be taken as

$$L(w) = L(u, v) = \int_0^u h(y)dy + \frac{1}{2}v^2.$$

Let c be any constant such that

$$\Omega_c := \{(u, v) \in \boldsymbol{R}^2 | L(u, v) \leq c\} \supset [u_+, u_-] \times \{0\}.$$

Then, $L(u, v)$ is continuously differentiable and satisfies

$$\dot{L}(u, v) = h(u)v + v(-\alpha v - h(u)) = -\alpha v^2 \leq 0,$$

so that $\dot{L}$ is negative semi-definite. Therefore, it follows from the proof of Lyapunov stability theorem that once a solution of (3.6) crosses the surface $\{(u, v) | L(u, v) = c\}$, it will stay inside the set Ω_c forever. The set Ω_c is thus invariant with respect to (3.6). And we can therefore take

$$\Omega = \Omega_c.$$

Next, we find $E = \{(u, v) \in \boldsymbol{R}^2 \mid \quad \dot{L}(u, v) = 0\}$. Suppose

$$\dot{L}(u, v) = 0.$$

Then

$$\alpha v^2 = 0$$

or

$$v = 0.$$

Thus,

$$E = \{(u, v) \mid \quad v = 0\}. \tag{3.18}$$

Let (u, v) be a solution that stays identically in E. Then,

$$v(y) \equiv 0,$$

which implies

$$v'(y) \equiv 0.$$

Hence,

$$h(u(y)) \equiv 0.$$

By the assumptions, u must be constant and coincides with some equilibrium. In other words, there exists $i \in I$ such that

$$u(y) \equiv u_i. \tag{3.19}$$

We can deduce from the conclusion (3.19) that no solution can stay identically in E, except trivial solutions. This implies that the largest invariant set M in E is the set of equilibria

$$M = \{(u_i, 0) \; i \in I\}. \tag{3.20}$$

Thus, every every trajectory of (3.6) starting in Ω must approach M in both directions $y \to \pm\infty$. □

Corollary 3.3. *Suppose E contains a finite number of isolated equilibria of (3.8) and that Ω is a nonempty invariant set with respect to (3.8). Then, the limits*

$$\lim_{y \to \pm\infty} (u(y), v(y)) \tag{3.21}$$

exist, and equal to some of the equilibria of (3.8).

As a consequence, whenever the limits in (3.21) coincide with the boundary conditions (3.3), we obtain a travelling wave of (1.1).

4. Some remarks on the uniqueness of Riemann problem

From the construction of diffusive-dispersive travelling waves in the last section, we can see that travelling waves connecting two states u_-, u_+ can exist even when there are several inflection points of the flux function in the interval between u_- and u_+. This also implies the existence of nonclassical shocks connecting the two states u_-, u_+.

The question is: how to select a unique nonclassical solution of the Riemann problem when the flux function admits several inflection points? Let us illustrate the above situation by considering the Riemann problem for the following conservation law

$$\partial_t u + \partial_x (u^2 - 1)^2 = 0. \tag{4.1}$$

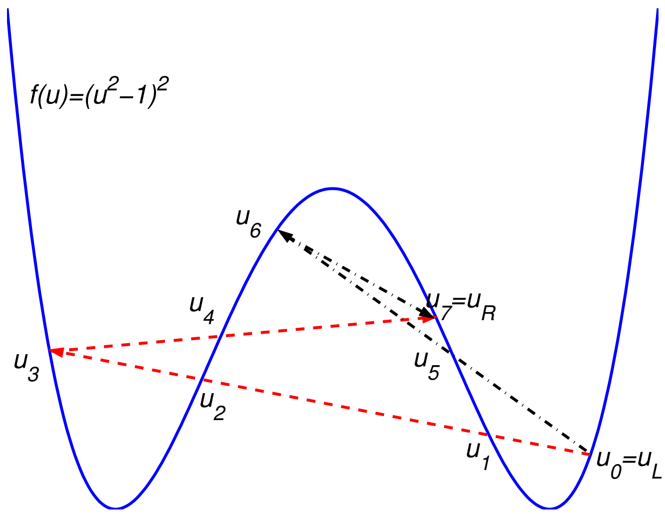

Figure 2. Two distinct nonclassical Riemann solutions

As the solution of the Riemann problem can be selected using the equal-area rule, see [15, 21], we can construct two different nonclassical Riemann solutions between two states u_L, u_R as follows. The Figure 2 describes two district nonclassical Riemann solutions for (4.1).

The first nonclassical solution from $u_L = u_0$ to u_3 using first a nonclassical shock. The line between u_0 and u_3 cuts the graph of the flux function at four points u_0, u_1, u_2, u_3. Then, the solution continues by another nonclassical shock from u_3 to $u_7 = u_R$.

The second nonclassical solution using first a nonclassical shock from u_L to u_6, followed by a classical shock from u_6 to u_R.

Even if we exclude nonclassical shocks where the line connecting two states crosses the graph of the flux function at four points as above, the Riemann problem still lacks the uniqueness. This was observed in [21].

Therefore, the question on the uniqueness of the solutions of the Riemann problem for conservation laws when the flux has several inflection points is still open. It seems that to answer this question, one needs more conditions or some kind of admissibility criteria on the solutions of the Riemann problem.

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