Approximations as generalized differentiability and optimality conditions without continuity assumptions in multivalued optimization

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Abstract

We extend the notion of approximations as generalized derivatives for mappings to the case of multivalued mappings. Using this notion we establish both necessary conditions and sufficient conditions of both order 1 and 2 for various kinds of efficiency in multivalued vector optimization without convexity and even continuity. Compactness assumptions are also relaxed. Our theorems include recent existing results in the literature as special cases.

1. Introduction and preliminaries

Differentiability assumptions are often crucial for a classical problem in all areas of continuous mathematics, since derivatives are local linear approximations for the involved nonlinear mappings and then supply a much simpler approximated linear problem, replacing the original nonlinear problem. However, such differentiability assumptions are too severe and not satisfied in many practical situations. In particular, relaxing these assumptions has been one of the main idea in optimization for more than three decades now and constituted an important field of research called nonsmooth optimization. Most of contributions in this field are based on using generalized derivatives which are local approximations bearing not the whole linearity but still parts of linearity. A lot of notions of generalized derivatives have been proposed. Each of them is suitable for a class of problems. The Clarke derivative [5] is introduced for locally Lipschitz mappings; the quasidifferentiability of Demyanov and Rubinov [6] requires directionally differentiability to be defined; the approximate Jacobian proposed in [8] exists only for continuous mappings, etc. The approximations, introduced in [11] for order 1 and in [1] for order 2, is defined for general mappings which are even discontinuous. In this note we extend these definitions to the case of multifunctions.

The main goal for generalized derivatives to be proposed is establishing optimality conditions in nonsmooth optimization problems. We see from the very beginning of classical optimization that derivatives play a fundamental role in the Fermat Theorem, the first necessary optimality condition. We would say that all generalized derivatives are used in similar ways as the classical derivative was in the Fermat Theorem. In the literature we observe only [11, 1, 2, 12-14] which deal with the approximation as a generalized derivative. This notion was used in [11] to study metric regularity, in [1] for establishing second-order necessary optimality conditions in compactness case. Second-order approximations of scalar functions are used for support functions in [2] to scalarize vector problems so that second-order optimality conditions can be established, but under strict (first-order) differentiability and compactness assumptions. In [12-14] we used first and second-order approximations of general mappings to derive first and second-order necessary conditions and sufficient conditions for various kinds of efficiency in nonsmooth vector optimization problems of several types.

In this paper, after extending the notions of first and second-order approximations of a mapping to the case of a multivalued mapping, we use these notions to establish both necessary conditions and sufficient conditions of both orders 1 and 2 for various kinds of efficiency in multivalued vector optimization without continuity and convexity assumptions. The optimization problem under our consideration is as follows. Throughout this paper, unless otherwise specified, let X and Y be normed spaces, $S \subseteq X$ be a nonempty subset and $F : X \to 2^Y$ be a multifunction (i.e. a multivalued mapping). Consider the problem

$$\text{(P)} \qquad \min F(x), \text{ subject to } x \in S.$$

The layout of the paper is as follows. In the rest of this section we recall definitions and preliminaries needed for our later investigation. Section 2 is devoted to defining first and second-order approximations of a multivalued mapping. In Section 3 we establish both necessary conditions and sufficient conditions of order 1 for suitable kinds of efficiency of problem (P) so that these conditions are valid for many other efficiency kinds. We develop such conditions for these kinds of efficiency, but of order 2, in the final Section 4.

Our notation are rather standard. $\mathbb{N} = \{1, 2, ...n, ...\}$ and $\|.\|$ stands for the norm in any normed space (the context will make it clear what space is concerned). $B_X(x, r) = \{z \in X \mid \|x - z\| < r\}$; X^* is the topological dual of X with $\langle ., . \rangle$ being the canonical pairing. $L(X, Y)$ denotes the space of all bounded linear mappings from X into Y and $B(X, X, Y)$ that of all bounded bilinear mappings from $X \times X$ into Y. For a cone $C \subseteq X$, $C^* = \{x^* \in X^* \mid \langle x^*, c\rangle \geq 0, \forall c \in C\}$ is the positive polar cone of C. For $A \subseteq X$, int$A$, cl$A$ and bd$A$ denote the interior, closure and boundary of A, respectively. For $t > 0$ and $k \in \mathbb{N}$, $o(t^k)$ stands for a moving point (in a normed space) such that $o(t^k)/t^k \to 0$ as $t \to 0^+$. We will use the following tangent sets of $A \subseteq X$ at $x_0 \in A$:

(a) the contingent (or Bouligand) cone of A at x_0, see [3], is

$$T(A, x_0) = \{v \in X \mid \exists t_n \to 0^+, \exists v_n \to v, \forall n \in \mathbb{N}, x_0 + t_n v_n \in A\};$$

(b) the second-order contingent set of A at (x_0, v), see [3], is

$$T^2(A, x_0, v) = \{w \in X \mid \exists t_n \to 0^+, \exists w_n \to w, \forall n \in \mathbb{N}, x_0 + t_n v + \tfrac{1}{2}t_n^2 w_n \in A\};$$

(c) the asymptotic second-order tangent cone of A at (x_0, v), see [4, 16], is

$$T^{''}(A, x_0, v) = \{w \in X \mid \exists (t_n, r_n) \to (0^+, 0^+) : \tfrac{t_n}{r_n} \to 0, \exists w_n \to w,$$
$$\forall n \in \mathbb{N}, x_0 + t_n v + \tfrac{1}{2}t_n r_n w_n \in A\}.$$

Lemma 1.1 [10]. Assume that X is a finite dimensional space $\mathbb{R}^m$ and $x_0 \in A \subseteq X$. If $x_n \in A \setminus \{x_0\}$ tends to x_0, then there exists $v \in T(A, x_0) \setminus \{0\}$ and a subsequence, denoted again by x_n, such that, for $t_n = \|x_n - x_0\|$,

(i) $\frac{1}{t_n}(x_n - x_0) \to v$;

(ii) either $z \in T^2(A, x_0, v) \cap v^\perp$ exists such that $(x_n - x_0 - t_n v)/\frac{1}{2}t_n^2 \to z$ or $z \in T^{''}(A, x_0, v) \cap v^\perp \setminus \{0\}$ and $r_n \to 0^+$ with $\frac{t_n}{r_n} \to 0^+$ exist such that and $(x_n - x_0 - t_n v)/\frac{1}{2}t_n r_n \to z$, where $v^\perp = \{y \in \mathbb{R}^m \mid \langle y, v\rangle = 0\}$.

Recall now notions of efficiency in vector optimization . Consider a subset V of the objective space Y ordered by the ordering cone C with int $C \neq \emptyset$ a point $y_0 \in V$ is called an efficient point (weakly efficient point) of V if

$$(V - y_0) \cap -C \subseteq (-C) \cap C$$
$$((V - y_0) \cap - \text{ int } C \neq \emptyset, \text{ respectively}).$$

The set of efficient and weakly efficient points are denoted by $\text{Min}_C V$ and $\text{WMin}_C V$, respectively.

Apply now these notions to problem (P). A point (x_0, y_0) with $x_0 \in S$ and $y_0 \in F(x_0)$ is said to be a local weakly efficient solution of (P) if there is a neighborhood U of x_0 such that, $\forall x \in S \cap U$,

$$(F(x) - y_0) \cap - \text{int}C = \emptyset, \qquad (1)$$

while (x_0, y_0) is called a local efficient solution if (1) is replaced by

$$(F(x) - y_0) \cap -C \subseteq (-C) \cap C.$$

We extend the firm efficiency notion, see [9, 15], to the case of multivalued optimization as follows.

Definition 1.1. Let $x_0 \in S$, $y_0 \in F(x_0)$ and $m \in \mathbb{N}$. Then (x_0, y_0) is said to be a local firm efficient solution of order m if there are a neighborhood U of x_0 and $\gamma > 0$ such that, $\forall x \in S \cap U \setminus \{x_0\}$,

$$(F(x) - y_0) \cap (B_Y(0, \gamma\|x - x_0\|^m) - C) = \emptyset$$

and $y_0 \in \mathrm{Min}_C F(x_0)$.

In the sequel let LWE(P), LE(P) and LFE(m, P) denote the sets of the local weakly efficient solutions, of the local efficient solutions and of the local firm efficient solutions of order m, respectively. Then it is clear that, for $p, m \in \mathbb{N}$ with $p \geq m$,

$$\mathrm{LFE}(m, \mathrm{P}) \subseteq \mathrm{LFE}(p, \mathrm{P}) \subseteq \mathrm{LE}(\mathrm{P}) \subseteq \mathrm{LWE}(\mathrm{P}).$$

Hence, necessary conditions for the right-most term are valid also for the others and sufficient conditions for the left-most term hold true for the others as well.

In the sequel a multifunction H from X to Y is denoted by $H : X \rightsquigarrow Y$ and the domain of H is defined as

$$\mathrm{dom}H = \{x \in X \mid H(x) \neq \emptyset\}.$$

H is said to be upper semicontinuous (usc) at $x_0 \in \mathrm{dom}H$ if for all open set $V \supseteq H(x_0)$, there is a neighborhood U of x_0 such that $V \supseteq H(U)$. H is termed lower semicontinuous (lsc) at $x_0 \in \mathrm{dom}H$ if for all open set $V \cap H(x_0) \neq \emptyset$, there is a neighborhood U of x_0 such that for all $x \in U$, $V \cap H(x) \neq \emptyset$.

2. First and second-order approximations of multifunctions

Consider a multifunction $H : X \rightsquigarrow Y$, $x_0 \in \mathrm{dom}H$ and $y_0 \in F(x_0)$.

Definition 2.1

(i) A subset $A_H(x_0, y_0)$ of $L(X, Y)$ is said to be a first-order approximation of H at (x_0, y_0) if there exists neighborhood U of x_0 such that, $\forall x \in U\cap$ dom$H$,

$$(H(x)-y_0)\cap(A_H(x_0, y_0)(x-x_0)+o(\|x-x_0\|) \neq \emptyset. \tag{2}$$

(ii) A subset $A^S_H(x_0, y_0)$ of $L(X, Y)$ is called a first-order strong approximation of H at (x_0, y_0) if (2.1) is replaced by

$$(H(x) - y_0) \subseteq A^S_H(x_0, y_0)(x - x_0) + o(\|x - x_0\|).$$

(iii) A pair $(A_H(x_0, y_0), B_H(x_0, y_0))$, where $A_H(x_0, y_0) \subseteq L(X, Y)$ and $B_H(x_0, y_0) \subseteq B(X, X, Y)$, is called a second-order approximation of H at (x_0, y_0) if

(a) $A_H(x_0, y_0)$ is a first-order approximation of H at (x_0, y_0);

(b) there is a neighborhood U of x_0 such that, $\forall x \in U\cap$ dom$H$,

$$(H(x) - y_0) \cap (A_H(x_0, y_0)(x - x_0) + B_H(x_0, y_0)(x - x_0, x - x_0) + o(\|x - x_0\|^2)) \neq \emptyset. \tag{3}$$

(iv) If (3) is replaced by

$$(H(x) - y_0) \subseteq A^S_H(x_0, y_0)(x - x_0) + B^S_H(x_0, y_0)(x - x_0, x - x_0) + o(\|x - x_0\|^2),$$

then $(A^S_H(x_0, y_0), B^S_H(x_0, y_0))$ is term a second-order strong approximation of H at (x_0, y_0).

In this paper we will impose on these approximations the following relaxed compactness.

Definition 2.2

(i) Let M_n and M be in $L(X, Y)$. The sequence M_n is said to pointwisely converge to M and written as $M_n \xrightarrow{\mathrm{p}} M$ or $M = \mathrm{p\text{-}lim}\, M_n$ if $\lim M_n(x) = M(x)$ for all $x \in X$. A similar definition is adopted for $N_n, N \in B(X, X, Y)$.

(ii) A subset $A \subseteq L(X, Y)$ ($B \subseteq B(X, X, Y)$, respectively) is called (sequentially) asymptotically pointwisely compact, or (sequentially) asymptotically p-compact if

(a) each norm bounded sequence $(M_n) \subseteq A$ ($\subseteq B$, respectively) has a subsequence (M_{n_k}) and $M \in L(X, Y)$ ($M \in B(X, X, Y)$, respectively) such that $M = \mathrm{p\text{-}lim}\, M_{n_k}$,

(b) for each sequence $(M_n) \subseteq A$ ($\subseteq B$, respectively) with $\lim \|M_n\| = \infty$, the sequence $(M_n/\|M_n\|)$ has a subsequence which pointwisely converges to some $M \in L(X, Y) \setminus \{0\}$

($M \in B(X, X, Y) \setminus \{0\}$, respectively).

(iii) If in (ii), pointwise convergence, i.e. p-lim, is replaced by convergence, i.e. lim, a subset $A \subseteq L(X, Y)$ (or $B \subseteq B(X, X, Y)$) is called (sequentially) asymptotically compact.

Since only sequential convergence is met in this paper, we will omit the word "sequentially" for short.

For $A \subseteq L(X, Y)$ and $B \subseteq B(X, X, Y)$ we adopt the notations:

$$\text{p-cl}\, A = \{M \in L(X, Y) : \exists (M_n) \subseteq A, M = \text{p-lim}\, M_n\}, \tag{4}$$

$$\text{p-cl}\, B = \{N \in B(X, X, Y) : \exists (N_n) \subseteq B, N = \text{p-lim}\, N_n\}, \tag{5}$$

$$A_\infty = \{M \in L(X, Y) : \exists (M_n) \subseteq A, \exists t_n \to 0^+, M = \lim t_n M_n\}, \tag{6}$$

$$\text{p-}A_\infty = \{M \in L(X, Y) : \exists (M_n) \subseteq A, \exists t_n \to 0^+, M = \text{p-lim}\, t_n M_n\}, \tag{7}$$

$$\text{p-}B_\infty = \{N \in B(X, X, Y) : \exists (N_n) \subseteq B, \exists t_n \to 0^+, N = \text{p-lim}\, t_n N_n\}. \tag{8}$$

The sets (4), (5) are pointwise closures; (6) is just the definition of the recession cone of A. So (7), (8) are pointwise recession cones.

Remark 2.1

(i) If X is finite dimensional, a convergence occurs if and only if the corresponding pointwise convergence does, but in general the "if" does not hold, see [12, Example 3.1].

(ii) If X and Y are finite dimensional, every subset is asymptotically p-compact and asymptotically compact but in general the asymptotical compactness is stronger, as shown by [12, Example 3.2].

(iii) Assume that $\{M_n\} \subseteq L(X, Y)$ is norm bounded. If $x_n \to x$ in X and $M_n \xrightarrow{\text{p}} M$ in $L(X, Y)$, then $M_n x_n \to Mx$ in Y. Similarly, if $x_n \to x$, $y_n \to y$ in X, $N_n \xrightarrow{\text{p}} N$ in $B(X, X, Y)$ and $\{N_n\}$ is norm bounded then $N_n(x_n, y_n) \to N(x, y)$ in Y.

Proof. The conclusion is derived from the following evaluations.

$$\|M_n x_n - Mx\| \le \|M_n x_n - M_n x\| + \|M_n x - Mx\| \le \|M_n\|\|x_n - x\| + \|M_n x - Mx\|;$$

$$\|N_n(x_n, y_n) - N(x, y)\| \le \|N_n(x_n, y_n) - N_n(x_n, y)\| + \|N_n(x_n, y) - N_n(x, y)\| +$$

$$\|N_n(x, y) - N(x, y)\| \le \|N_n\|\|x_n\|\|y_n - y\| + \|N_n\|\|x_n - x\|\|y\| + \|N_n(x, y) - N(x, y)\|. \quad \square$$

The following example gives a multivalued map F, which is neither usc nor lsc at x_0, but has even second-order strong approximations.

Example 2.1. Let $F : \mathbb{R} \rightsquigarrow \mathbb{R}$ be defined by

$$F(x) = \begin{cases} \{y \in \mathbb{R} \mid y \ge \sqrt{x}\} & \text{if } x > 0, \\ \{y \in \mathbb{R} \mid y \le \frac{1}{x}\} & \text{if } x < 0, \\ \{0\} & \text{if } x = 0. \end{cases}$$

Let $(x_0, y_0) = (0, 0)$. Then F is neither usc nor lsc at x_0 but F has the following approximations, for $\alpha, \beta > 0$ fixed,

$$A_F(x_0, y_0) = (\alpha, +\infty),\ A_F^S(x_0, y_0) = (\beta, +\infty),$$
$$B_F(x_0, y_0) = B_F^S(x_0, y_0) = \{0\}.$$

In the next example F is not usc at x_0 but $A_F(x_0, y_0)$ is even a singleton.

Example 2.2. Let $F : \mathbb{R}^2 \rightsquigarrow \mathbb{R}$ be defined by

$$F(x_1, x_2) = \begin{cases} \{y \in \mathbb{R} \mid \frac{2}{3}|x_1|^{\frac{3}{2}} + x_2^2 \le y \le \frac{1}{|x_1| + |x_2|}\} & \text{if } (x_1, x_2) \ne (0, 0), \\ \{0\} & \text{if } (x_1, x_2) = (0, 0). \end{cases}$$

Then F is not usc at $x^0 = (0, 0)$. But for $y_0 = 0$ we have

$$A_F(x^0, y_0) = \{0\},\ A_F^S(x^0, y_0) = (0, +\infty),$$

$$B_F(x^0, y_0) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \mid \alpha > 1 \right\},\ B_F^S(x^0, y_0) = \{0\}.$$

This example gives a possible particular property of a multifunction which is not usc at x_0. Because a single-valued mapping F has a first-order approximation at x_0 being a singleton if and only if F is Fréchet differentiable at x_0.

3. First-order optimality conditions

Theorem 3.1 (Necessary condition). Consider problem (P). Assume that $A_F(x_0, y_0)$ is an asymptotically p-compact first-order approximation of F at (x_0, y_0). If $(x_0, y_0) \in \text{LWE(P)}$ then, $\forall v \in T(S, x_0), \exists M \in \text{p-cl}A_F(x_0, y_0) \bigcup (\text{p-}A_F(x_0, y_0)_\infty \setminus \{0\})$ such that

$$Mv \notin -\text{int } C.$$

Proof. Let $v \in T(S, x_0)$ be arbitrary and fixed. By the definition of a contingent cone, there is $(t_n, v_n) \to (0^+, v)$ such that, $\forall n \in \mathbb{N}$, $x_0 + t_n v_n \in S$. By the weak efficiency of (x_0, y_0) one has, for large n,

$$(F(x_0 + t_n v_n) - y_0) \cap - \text{ int } C = \emptyset.$$

On the other hand, as $A_F(x_0, y_0)$ is a first-order approximation,

$$(F(x_0 + t_n v_n) - y_0) \cap (A_F(x_0, y_0)(t_n v_n) + o(t_n)) \neq \emptyset.$$

Therefore, $M_n \in A_F(x_0, y_0)$ exists such that

$$M_n(t_n v_n) + o(t_n) \notin - \text{ int } C. \tag{9}$$

If $\{M_n\}$ is norm bounded, one can assume that $M_n \xrightarrow{\text{p}} M \in \text{p-cl}A_F(x_0, y_0)$. Passing to the limit one gets $Mv \notin - \text{ int } C$. If $\{M_n\}$ is unbounded, one can assume that $\|M_n\| \to \infty$ and $\frac{M_n}{\|M_n\|} \xrightarrow{\text{p}} M \in \text{p-}A_F(x_0, y_0)_\infty \setminus \{0\}$. Dividing (9) by $\|M_n\| t_n$ one obtains in the limit $Mv \notin -$ int C. □

If F is single-valued, Theorem 3.1 becomes Theorem 3.3 of [13]. The following example shows that, for F being multivalued, Theorem 3.1 is easy to be applied.

Example 3.1. Let $X = Y = \mathbb{R}$, $S = [0, +\infty)$, $C = \mathbb{R}_+$, $x_0 = y_0 = 0$ and

$$F(x) = \begin{cases} \{y \in \mathbb{R} \mid y \leq \frac{1}{\sqrt[3]{x}}\} & \text{if } x > 0, \\ \{y \in \mathbb{R} \mid y \geq \sqrt[3]{-x}\} & \text{if } x < 0, \\ \{0\} & \text{if } (x_1, x_2) = (0, 0). \end{cases}$$

Then $T(S, x_0) = S$ and for a fixed $\alpha < 0$ we have $A_F(x_0, y_0) = (-\infty, \alpha)$, $\text{cl}A_F(x_0, y_0) = (-\infty, \alpha]$, $A_F(x_0, y_0)_\infty = (-\infty, 0]$. Taking $v = 1 \in T(S, x_0)$ one sees that, $\forall M \in \text{cl}A_F(x_0, y_0) \cup (A_F(x_0, y_0)_\infty \setminus \{0\} = (-\infty, 0)$,

$$Mv = M \in - \text{ int } C.$$

Due to Theorem 3.1, (x_0, y_0) is not a local (weakly efficient) solution of problem (P).

Theorem 3.2 (Sufficient condition). Consider problem (P) with X being finite dimensional. Assume that $A_F^S(x_0, y_0)$ is an asymptotically p-compact first-order strong approximation of F at (x_0, y_0). Assume that $x_0 \in S$ and $y_0 \in \text{Min}_C F(x_0)$. Assume further that $\forall v \in T(S, x_0) \setminus \{0\}, \forall M \in \text{p-cl}A_F^S(x_0, y_0) \bigcup (\text{p-}A_F^S(x_0, y_0)_\infty \setminus \{0\})$,

$$Mv \notin -\text{cl } C.$$

Then $(x_0, y_0) \in \text{LFE}(1, \text{P})$.

Proof. Reasoning ad absurdum, suppose the existence of $x_n \in S \cap B_X(x_0, \frac{1}{n}) \setminus \{x_0\}$ such that, $\forall n \in \mathbb{N}$,

$$(F(x_n) - y_0) \cap (B_Y(0, \tfrac{1}{n}\|x_n - x_0\|) - C) \neq \emptyset.$$

As X is finite dimensional, we can assume that $\frac{x_n - x_0}{\|x_n - x_0\|}$ tends to a point v in $T(S, x_0) \setminus \{0\}$. On the other hand, for large n,

$$F(x_n) - y_0 \subseteq A_F^S(x_0, y_0)(x_n - x_0) + o(\|x_n - x_0\|).$$

Hence, there is $M_n \in A_F^S(x_0, y_0)$ such that

$$M_n(x_n - x_0) + o(\|x_n - x_0\|) \in B_Y(0, \tfrac{1}{n}\|x_n - x_0\|) - C.$$

Arguing similarly as in the final part of the proof of Theorem 3.1, we obtain $M \in$ p-cl$A^S_F(x_0,y_0) \bigcup$(p-$A^S_F(x_0,y_0)_\infty \setminus \{0\}$) such that $Mv \in -$ cl C, a contradiction. □

Theorem 3.2 includes Theorem 3.4 of [13] as a special case where F is single-valued. The following example explains how to employ Theorem 3.2.

Example 3.2. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $S = [0, +\infty)$, $C = \mathbb{R}^2_+$, $x_0 = 0$, $(y_0, z_0) = (0, 0) \in Y$ and $F(x) = \{(y, z) \in \mathbb{R}^2 \mid y \geq \sqrt[3]{x}, z = x\}$. Then $(y_0, z_0) \in \mathrm{Min}_C F(x_0)$ and for any fixed $\alpha > 0$ we have

$$A^S_F(x_0,(y_0,z_0)) = \{(y,z) \in \mathbb{R}^2 \mid y > \alpha, z = 1\},$$
$$\mathrm{cl}A^S_F(x_0,(y_0,z_0)) = \{(y,z) \in \mathbb{R}^2 \mid y \geq \alpha, z = 1\},$$
$$A^S_F(x_0,(y_0,z_0))_\infty = \{(y,z) \in \mathbb{R}^2 \mid y \geq 0, z = 0\}.$$

It is clear that, $\forall v \in T(S, x_0) \setminus \{0\} = (0, +\infty)$, one has, $\forall M \in \mathrm{cl}A^S_F(x_0,(y_0,z_0))$, $Mv = (yv, v) \notin -C$, as $y \geq v > \alpha$. Furthermore, $\forall M \in A^S_F(x_0,(y_0,z_0))_\infty \setminus \{0\}$, $Mv = (yv, 0) \notin -C$. By Theorem 3.2, $(x_0,(y_0,z_0)) \in$ LFE(1, P).

4. Second-order optimality conditions

Theorem 4.1 (Necessary condition). For problem (P) assume that $(A_F(x_0,y_0), B_F(x_0,y_0))$ is an asymptotically p-compact second-order approximation of F at (x_0,y_0) with $A_F(x_0,y_0)$ being norm bounded. Assume further that $(x_0,y_0) \in$ LWE(P). Then

(i) $\forall v \in T(S,x_0)$, $\exists M \in$ p-cl$A_F(x_0,y_0)$, $Mv \notin -$int C;

(ii) $\forall v \in T(S,x_0)$ with $A_F(x_0,y_0)v \subseteq -$bd C one has

(a) $\forall w \in T^2(S,x_0,v)$, either $\exists\, \overline{M} \in$ p-cl$A_F(x_0,y_0)$, $\exists \overline{N} \in$ p-cl$B_F(x_0,y_0)$ such that

$$\overline{M}w + 2\overline{N}(v,v) \notin -\text{ int } C,$$

or $\exists \overline{N} \in$ p-$B_F(x_0,y_0)_\infty \setminus \{0\}$ such that

$$\overline{N}(v,v) \notin -\mathrm{int}C;$$

(b) $\forall w \in T^{''}(S,x_0,v)$, either $\exists\, M^{'} \in$ p-cl$A_F(x_0,y_0)$, $\exists N^{'} \in$ p-$B_F(x_0,y_0)_\infty$ such that

$$M^{'}w + N^{'}(v,v) \notin -\text{ int}C,$$

or $\exists N^{'} \in$ p-$B_F(x_0,y_0)_\infty \setminus \{0\}$ such that

$$N^{'}(v,v) \notin -\mathrm{int}C.$$

Proof. (i) The assertion follows from Theorem 3.1.

(ii) (a) Let $v \in T(S,x_0)$ with $A_F(x_0,y_0)v \subseteq -$bd C and $w \in T^2(S,x_0,v)$. Then, there are $x_n \in S$ and $t_n \to 0^+$ such that

$$w_n := (x_n - x_0 - t_n v)/\tfrac{1}{2}t_n^2 \to w.$$

By the definition of the second-order approximation, there are $M_n \in A_F(x_0,y_0)$ and $N_n \in B_F(x_0,y_0)$ such that, for large n,

$$M_n(x_n - x_0) + N_n(x_n - x_0, x_n - x_0) + o(\|x_n - x_0\|^2) \in F(x_n) - y_0.$$

The weak efficiency of (x_0,y_0) implies then

$$M_n w_n + 2N_n(v + \tfrac{1}{2}t_n w_n, v + \tfrac{1}{2}t_n w_n) + o(t_n^2)/\tfrac{1}{2}t_n^2 \notin -\text{int } C. \tag{10}$$

We can assume that $M_n \xrightarrow{\text{p}} \overline{M}$ for some $\overline{M} \in$ p-cl$A_F(x_0,y_0)$. If $\{N_n\}$ is norm bounded then $N_n \xrightarrow{\text{p}} \overline{N}$ for some $\overline{N} \in$ p-cl$B_F(x_0,y_0)$. From (10) we get in the limit

$$\overline{M}w + 2\overline{N}(v,v) \notin -\text{ int } C,$$

If $\{N_n\}$ is unbounded, we can assume $\|N_n\| \to \infty$ and $\frac{N_n}{\|N_n\|} \xrightarrow{\text{p}} \overline{N}$ for some $\overline{N} \in$ p-$B_f(x_0)_\infty \setminus \{0\}$. Dividing (10) by $\|N_n\|$ and passing to the limit gives $\overline{N}(v,v) \notin -$int C.

(b) For any $w \in T^{''}(S,x_0,v)$, there are $x_n \in S$ and $(t_n, r_n) \to (0^+, 0^+)$ with $\frac{t_n}{r_n} \to 0^+$ such that

$$w_n := (x_n - x_0 - t_n v)/\tfrac{1}{2}t_n r_n \to w.$$

Similarly as in (a) we have $M_n^{'}$ and $N_n^{'}$ satisfying the following relation, corresponding to (10),

$$M_n^{'} w_n + (\tfrac{2t_n}{r_n})N_n^{'}(v + \tfrac{1}{2}t_n w_n, v + \tfrac{1}{2}t_n w_n) + o(t_n^2)/\tfrac{1}{2}t_n r_n \notin -\text{int } C. \tag{11}$$

We can assume that $M_n^{'} \xrightarrow{\text{p}} M^{'} \in \text{p-cl}A_F(x_0, y_0)$. There are three possibilities.

(α) $(\frac{2t_n}{r_n})N_n^{'} \to 0$. From (11) we get in the limit

$$M^{'} w \notin -\text{int } C.$$

(β) If $(\frac{2t_n}{r_n})\|N_n^{'}\| \to a > 0$, then $\|N_n\| \to \infty$ and we can assume that $\frac{N_n^{'}}{\|N_n^{'}\|} \xrightarrow{\text{p}} N^{'} \in$ p-$B_F(x_0, y_0)_\infty \setminus \{0\}$. Passing (11) to the limit yields

$$M^{'} w + aN^{'}(v, v) \notin -\text{ int } C.$$

(γ) If $(\frac{2t_n}{r_n})\|N_n^{'}\| \to \infty$, then dividing (11) by $(\frac{2t_n}{r_n})\|N_n^{'}\|$ and passing to the limit gives

$$N^{'}(v, v) \notin -\text{int } C.$$ □

If F is single-valued, Theorem 4.1 collapses to Theorem 4.10 of [13]. The example below gives an application of Theorem 4.1 to a multivalued case.

Example 4.1. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $S = \{(x, z) \in \mathbb{R}^2 \mid z = |x|^{\frac{3}{2}}\}$, $C = \mathbb{R}_+$, $(x_0, z_0) = (0, 0) \in X$, $y_0 = 0 \in Y$ and

$$F(x, z) = \begin{cases} \{y \in \mathbb{R} \mid -\frac{2}{3}|x|^{\frac{3}{2}} + z^2 - z \le y \le \frac{1}{x^2+z^2}\} & \text{if } (x, z) \neq 0, \\ \{0\} & \text{if } (x, z) = (0, 0). \end{cases}$$

Then, for a fixed $\alpha < 0$,

$$T(S, (x_0, z_0)) = \{(x, z) \in \mathbb{R}^2 \mid z = 0\},$$
$$A_F((x_0, z_0), y_0) = \{(0, -1)\},$$
$$B_F((x_0, z_0), y_0) = \left\{ \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \mid t < \alpha \right\},$$
$$\text{cl}B_F((x_0, z_0), y_0) = \left\{ \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \mid t \le \alpha \right\},$$
$$B_F((x_0, z_0), y_0)_\infty = \left\{ \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \mid t \le 0 \right\}.$$

Taking $v = (1, 0) \in T(S, (x_0, z_0))$ one has

$$A_F((x_0, z_0), y_0)v = \{0\} \subseteq -\text{bd}C,$$
$$T^2(S, (x_0, z_0), v) = \emptyset,$$
$$T^{''}(S, (x_0, z_0), v) = \mathbb{R} \times \mathbb{R}_+.$$

Hence, for $w = (0, 1) \in T^{''}(S, (x_0, z_0), v)$ one obtains

$$(0, -1)w + N(v, v) = -1 + t < 0$$

for all $N \in \text{cl}B_F((x_0, z_0), y_0)$ and

$$N(v, v) = t < 0$$

for all $N \in B_F((x_0, z_0), y_0)_\infty \setminus \{0\}$.

Taking into account Theorem 4.1, one sees that $((x_0, z_0), y_0)$ is not a local weakly efficient solution of problem (P) in this case.

Theorem 4.2 (Sufficient condition). Consider problem (P) with X being finite dimensional. Assume that $x_0 \in S$ and $y_0 \in \text{Min}_C F(x_0)$. Assume further that $(A_F^S(x_0, y_0), B_F^S(x_0, y_0))$ is an asymptotically p-compact second-order strong approximation of F at (x_0, y_0) with $A_F^S(x_0, y_0)$ being norm bounded. Then $(x_0, y_0) \in \text{LFE}(2, \text{P})$ if

(i) $\forall v \in T(S, x_0) \setminus \{0\}$, $A_F^S(x_0, y_0)v \subseteq \text{cl } C$;

(ii) $\forall v \in T(S,x_0)\setminus\{0\}$ with $\overline{M}v \in -\text{cl } C$ for some $\overline{M} \in$ p-cl$A_F^S(x_0,y_0)$, $\forall N \in$ p-$B_F^S(x_0,y_0)_\infty \setminus \{0\}$, $N(v,v) \notin -\text{cl } C$ and

(a) $\forall w \in T^2(S,x_0,v)\cap v^\perp$, $\forall M \in$ p-cl$A_F^S(x_0,y_0)$, $\forall N \in$ p-cl$B_F^S(x_0,y_0)$,

$$Mw + 2N(v,v) \notin -\text{cl } C,$$

(b) $\forall w \in T^{''}(S,x_0,v)\cap v^\perp \setminus \{0\}$, $\forall M \in$ p-cl$A_F^S(x_0,y_0)$, $\forall N \in$ p-$B_F^S(x_0,y_0)_\infty$,

$$Mw + N(v,v) \notin -\text{cl } C.$$

Proof. Suppose to the contrary that $x_n \in S \cap B_X(x_0, \frac{1}{n}) \setminus \{x_0\}$ exists such that

$$(F(x_n) - y_0) \cap (B_Y(0, \tfrac{1}{n}t_n^2) - C) \neq \emptyset, \tag{12}$$

where $t_n = \|x_n - x_0\|$. We can assume that $\frac{1}{t_n}(x_n - x_0) \to v \in T(S,x_0)\setminus\{0\}$. By (12) and by the definition of the first-order strong approximation, for large n, there exists $\overline{M}_n \in A_F^S(x_0,y_0)$ such that

$$\overline{M}_n(x_n - x_0) + o(t_n) \in B_Y(0, \tfrac{1}{n}t_n^2) - C. \tag{13}$$

The norm boundedness of $A_F^S(x_0,y_0)$ allows to assume that $\overline{M}_n \xrightarrow{\text{p}} \overline{M} \in$ p-cl$A_F^S(x_0,y_0)$. Dividing (13) by t_n we get, in the limit, $\overline{M}v \in -\text{cl } C$. According to Lemma 1.1, there are only the following two possibilities.

(α) One has $w_n := (x_n - x_0 - t_n v)/\frac{1}{2}t_n^2 \to w \in T^2(S,x_0,v)\cap v^\perp$. By the definition of the second-order strong approximation, (12) implies the existence of $M_n \in A_F^S(x_0,y_0)$ and $N_n \in B_F^S(x_0,y_0)$ such that, for large n,

$$M_n(x_n - x_0) + N_n(x_n - x_0, x_n - x_0) + o(\|x_n - x_0\|^2) \in B_Y(0, \tfrac{1}{n}t_n^2) - C.$$

This can be rewritten as

$$M_n w_n + 2N_n(v + \tfrac{1}{2}t_n w_n, v + \tfrac{1}{2}t_n w_n) + o(t_n^2)/\tfrac{1}{2}t_n^2 = d_n/\tfrac{1}{2}t_n^2 - c_n^{'}, \tag{14}$$

where $d_n \in B_Y(0, \frac{1}{n}t_n^2)$ and $c_n^{'} = (c_n + t_n M_n v)/\frac{1}{2}t_n^2 \in \text{cl } C$, since $c_n \in C$ and $A_F^S(x_0,y_0)v \subseteq \text{cl } C$. We can assume that $M_n \xrightarrow{\text{p}} M \in$ p-cl$A_F^S(x_0,y_0)$. If $\{N_n\}$ is norm bounded, we can assume that $N_n \xrightarrow{\text{p}} N \in$ p-cl$B_F^S(x_0,y_0)$. In the limit (14) gives the contradiction

$$Mw + 2N(v,v) \in -\text{cl } C.$$

If $\{N_n\}$ is unbounded, we can assume that $\|N_n\| \to \infty$ and $\frac{N_n}{\|N_n\|} \xrightarrow{\text{p}} N \in$ p-$B_F(x_0,y_0)_\infty \setminus \{0\}$. We divide (14) by $\|N_n\|$ and pass it to the limit to get $N(v,v) \in -\text{cl } C$, also a contradiction.

(β) There is $r_n \to 0^+$ such that $\frac{t_n}{r_n} \to 0^+$ and

$$w_n := (x_n - x_0 - t_n v)/\tfrac{1}{2}t_n r_n \to w \in T^{''}(S,x_0,v)\cap v^\perp \setminus \{0\}.$$

Similarly as for the case (α), there are $M_n \in A_F^S(x_0,y_0)$ and $N_n \in B_F^S(x_0,y_0)$ such that, for large n,

$$M_n w_n + (\tfrac{2t_n}{r_n})N_n(v + \tfrac{1}{2}r_n w_n, v + \tfrac{1}{2}r_n w_n) + o(t_n^2)/\tfrac{1}{2}t_n r_n = d_n/\tfrac{1}{2}t_n r_n - c_n^{'}, \tag{15}$$

where $d_n \in B_Y(0, \frac{1}{n}t_n^2)$ and $c_n^{'} = (c_n + t_n M_n v)/\frac{1}{2}t_n r_n \in \text{cl } C$. We can assume that $M_n \xrightarrow{\text{p}} M \in$ p-cl$A_F^S(x_0,y_0)$. There are three subcases as follows.

• $(\frac{2t_n}{r_n})N_n \to 0$. Passing (15) to the limit one gets $Mw \in -\text{cl } C$, contradicting assumption (ii) (b) (with $N = 0 \in$ p-$B_F^S(x_0,y_0)_\infty$).

• $(\frac{2t_n}{r_n})\|N_n\| \to a > 0$. Then $\|N_n\| \to \infty$ and we can assume that $\frac{N_n}{\|N_n\|} \xrightarrow{\text{p}} N \in$ p-$B_F(x_0,y_0)_\infty \setminus \{0\}$. Dividing (15) by $(\frac{2t_n}{r_n})\|N_n\|$ and passing to the limit we obtain the contradiction

$$Mw + aN(v,v) \in -\text{cl } C.$$

• $(\frac{2t_n}{r_n})\|N_n\| \to \infty$. Then $\|N_n\| \to \infty$ and assume that $\frac{N_n}{\|N_n\|} \xrightarrow{\text{p}} N \in$ p-$B_F(x_0,y_0)_\infty \setminus \{0\}$. Dividing (15) by $(\frac{2t_n}{r_n})\|N_n\|$ we get in the limit $N(v,v) \notin -\text{cl } C$ which is absurd. □

Theorem 4.2 completely contains Theorem 4.12 of [13] as a special case of single-valued mappings. We interpret the use of Theorem 4.2 by the following example.

Example 4.2. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $S = [0, +\infty)$, $C = \mathbb{R}^2_+$, $x_0 = 0$, $(y_0, z_0) = (0, 0) \in Y$ and

$$F(x) = \begin{cases} \{(y,z) \in \mathbb{R}^2 \mid y = x^2, z \geq \frac{3}{4}|x|^{\frac{4}{3}}\} & \text{if } x \neq 0, \\ \{(0,0)\} & \text{if } x = 0. \end{cases}$$

Then $(y_0, z_0) \in \mathrm{Min}_C F(x_0)$, $T(S, x_0) = S$ and, for a fixed $\alpha > 0$,

$$A^S_F(x_0, (y_0, z_0)) = \{(0,0)\},$$
$$B^S_F(x_0, (y_0, z_0)) = \{(1, z) \mid z > \alpha\},$$
$$\mathrm{cl}B^S_F(x_0, (y_0, z_0)) = \{(1, z) \mid z \geq \alpha\},$$
$$B^S_F(x_0, (y_0, z_0))_\infty = \{(0, z) \mid z \geq 0\}.$$

It is easy to check that $\forall v \in T(S, x_0) \setminus \{0\}$ one has

$$A^S_F(x_0, (y_0, z_0))v = \{(0,0)\} \subseteq (-\mathrm{cl}\ C) \cap \mathrm{cl}\ C,$$
$$N(v, v) = (0, zv^2) \notin -\mathrm{cl}\ C,$$

$\forall N \in B^S_F(x_0, (y_0, z_0)) \setminus \{0\}$, and

$$(0, 0)w + 2N(v, v) = (2v^2, 2zv^2) \notin -C,$$

$\forall M \in \mathrm{cl}A^S_F(x_0, (y_0, z_0)) = \{(0,0)\}$, $\forall N \in \mathrm{cl}B^S_F(x_0, (y_0, z_0))$, and $T''(S, x_0, v) \cap v^\perp \setminus \{0\} = \emptyset$. Now that all assumptions of Theorem 4.2 are satisfied, $(x_0, (y_0, z_0)) \in$ LFE(2, P).

Summarizing it should be noted that each of the necessary conditions and sufficient conditions presented in this paper is an extension to the multivalued case of the corresponding result in [13] for the single-valued case. The latter was shown in [13] to be sharper than the corresponding theorem in [15] and better in use than many recent results in the literature, since the assumptions are very relaxed.

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