Optimality Conditions Using Approximations for Nonsmooth Vector Optimization Problems under General Inequality Constraints*

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First and second-order necessary conditions and sufficient conditions for optimality in nonsmooth vector optimization problems with general inequality constraints are established. We use approximations as generalized derivatives and avoid even continuity assumptions. Convexity conditions are not imposed explicitly. Examples are provided to show that our theorems are easily applied in situations where many known results cannot be.

1. Introduction

Applying generalized derivatives to establish optimality conditions in nonsmooth optimization has been one of the most interesting issues with world-wide enormous efforts and contributions for several last decades. There have been various notions of generalized derivatives of mappings with different requirements on the regularity of the mappings. Some kinds of derivatives need the mappings under consideration to be locally Lipschitz. Other ones are developed on continuous mappings, etc. Each of these generalized derivatives is appropriate for a range of problems. In this note we use the notion of approximations as a generalized derivative, which was introduced in [12] and extended to the second-order in [1]. Second-order optimality conditions under strict (first-order) differentiability and compactness assumptions were obtained in [2]. In [13, 14], using first and second-order approximations we established both necessary conditions and sufficient conditions of order 1 and 2 for set-constrained vector problems. The reason for us to make use of approximations as generalized derivatives is that even discontinuous mappings may have second-order approximations (see [13, Remark 2.1]) and hence the assumptions for getting optimality conditions are rather relaxed.

The aim of this note is to obtain such optimality conditions, but for nonsmooth vector problems under general inequality constraints. So the problem under our consideration is as follows. Let throughout the paper, if not otherwise stated, X, Y and Z be normed spaces, C and K be closed convex cones in Y and Z, respectively. Let $f : X \to Y$ and $g : X \to Z$ be mappings. Consider the vector optimization problem

$$\text{(P)} \qquad \min f(x), \text{ s.t. } g(x) \in -K.$$

We will develop Lagrange multiplier rules of orders 1 and 2, with the Lagrange multipliers depending on the directions, as necessary conditions and sufficient conditions for problem (P). Comparisons, especially by examples, will show advantages of our results. Note that our optimality conditions are developed without continuity assumptions. Convexity assumptions are not necessarily imposed explicitly.

Our notations are basically standard. $\mathbb{N} = \{1, 2, ...n, ...\}$. For a normed space X, X^* stands for the topological dual of X; $\langle ., . \rangle$ is the canonical pairing; $\|.\|$ is used for the norm in any normed space (from the context no confusion occurs); $B_X(x, r) = \{z \in X \mid \|x - z\| < r\}$; $L(X, Y)$ denotes the space of the bounded linear mappings from X into Y and $B(X, X, Y)$ is the space of the bounded bilinear mappings from $X \times X$ into Y. For a cone $C \subseteq X$, $C^* = \{c^* \in X^* \mid \langle c^*, c \rangle \geq 0, \forall c \in C\}$ is the polar cone of C. For $A \subseteq X$, int$A$, cl$A$ and

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[0*] This work was supported in part by the National Basic Research Program in Natural Sciences of Vietnam.

co$A$ stand for the interior, closure and convex hull of A, respetively; cone$A$ and span$A$ denote the cone generated by A and the linear hull of A, i.e.

$$\mathrm{cone}A = \{\lambda a \mid \lambda \geq 0, a \in A\},$$
$$\mathrm{span}A = \{\alpha a + \beta b \mid \alpha, \beta \in \mathbb{R}, a, b \in A\}.$$

For $u \in X$ and a closed convex cone $C \subset X$, set

$$C(u) = \mathrm{cone}\ (C + u).$$

$o(t^k)$ for $t > 0$ and $k \in \mathbb{N}$ denotes a moving point such that $o(t^k)/t^k \to 0$ as $t \to 0^+$. $C^{0,1}$ is used for the space of the locally Lipschitz mappings (between two given spaces, which are clear from the context) and $C^{1,1}$ for the space of the Fréchet differentiable mappings whose Fréchet derivative is locally Lipschitz.

The layout of the rest of the paper is as follows. Basic definitions and preliminaries are given in Section 2. First-order optimality conditions are established in Section 3, while second-order ones are the content of the final Section 4.

2. Preliminaries

Recall that X, Y and Z are normed spaces throughout the paper, if not otherwise specified. Recall further that a multivalued mapping $H : X \to 2^Y$ is said to be upper semicontinuous (usc, for short) at $x_0 \in X$ if for all open set $V \supseteq H(x_0)$, there is a neighborhood U of x_0 such that $V \supseteq H(U)$. A mapping $h : X \to Y$ is called locally Lipschitz at $x_0 \in X$ if there are a neighborhood U of x_0 and $L > 0$ such that, $\forall x_1, x_2 \in U$,

$$\|h(x_1) - h(x_2)\| \leq L\|x_1 - x_2\|.$$

$h : X \to Y$ is termed calm (see [18]) at $x_0 \in X$ if there are a neighborhood U of x_0 and $L > 0$ such that, $\forall x \in U$,

$$\|h(x) - h(x_0)\| \leq L\|x - x_0\|.$$

(The term ”calm” is sometimes replaced by ”weak Lipschitz” in the literature.)

Definition 2.1 [12, 1]. *Let $x_0 \in X$ and $h : X \to Y$.*

(i) A set $A_h(x_0) \subseteq L(X, Y)$ is called a first-order approximation of h at x_0 if there exists a neighborhood U of x_0 such that, for all $x \in U$,

$$h(x) - h(x_0) \in A_h(x_0)(x - x_0) + o(\|x - x_0\|).$$

(ii) A pair $(A_h(x_0), B_h(x_0))$, with $A_h(x_0) \subseteq L(X, Y)$ and $B_h(x_0) \subseteq B(X, X, Y)$, is said to be a second-order approximation of h at x_0 if $A_h(x_0)$ is a first-order approximation of h at x_0 and

$$h(x) - h(x_0) \in A_h(x_0)(x - x_0) + B_h(x_0)(x - x_0, x - x_0) + o(\|x - x_0\|^2).$$

Remark 2.2. If h has second Fréchet derivative $h^{''}(x_0)$ then $(h^{'}(x_0), \frac{1}{2}h^{''}(x_0))$ is a second-order approximation of h.

Proposition 2.3 [12, 1].

(i) If $h : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz at x_0 then the Clarke Jacobian (see [3]) $\partial_C h(x_0)$ is a first-order approximation of h at x_0.

(ii) If $h : \mathbb{R}^n \to \mathbb{R}^m$ is in $C^{1,1}$ at x_0 then $(h^{'}(x_0), \frac{1}{2}\partial_C^2 g(x_0))$ is a second-order approximation of h at x_0, where $\partial_C^2 h(x_0)$ is the Clarke Hessian of h at x_0 (see [8]).

Proposition 2.4 [13].

(i) If $h : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and has an approximate Jacobian mapping $\partial h(.)$ (see [9]) which is usc at x_0, then co$\partial h(x_0)$ is a first-order approximation of h at x_0.

(ii) If $h : \mathbb{R}^n \to \mathbb{R}^m$ is continuously Fréchet differentiable in a neighborhood of x_0 and has an approximate Hessian mapping $\partial^2 h(.)$ (see [10]) which is usc at x_0, then

$(h^{'}(x_0), \frac{1}{2}\mathrm{co}\partial^2 h(x_0))$ *is a second-order approximation of* h *at* x_0.

Note that as shown in [9, 10], the approximate Jacobian and Hessian include many other generalized derivatives of orders 1 and 2, respectively, as special cases. So by Proposition 2.4 the first and second-order approximations also do. Furthermore, Examples 2.1-2.5 in [13] show that the converse of Proposition 2.4 is not true and under its assumptions we still have other approximations beside the mentioned one.

Later, if P_n and P are in $L(X,Y)$ and P_n converges to P pointwisely, then we write $P_n \xrightarrow{\mathrm{p}} P$ or $P = \text{p-lim}\, P_n$. A similar notation is adopted for $M_n, M \in B(X,X,Y)$.
We recall [14] that a subset $A \subseteq L(X,Y)$ $(B \subseteq B(X,X,Y))$ is called (sequentially) asymptotically pointwisely compact (p-compact, for short) if

- each norm bounded sequence $(M_n) \subseteq A$ $(\subseteq B$, respectively) has a pointwisely convergent subsequence with a limit in A (B, respectively).
- if $(M_n) \subseteq A$ $(\subseteq B$, respectively) with $\lim \|M_n\| = \infty$, then $(M_n/\|M_n\|)$ has a subsequence which pointwisely converges with a nonzero limit.

If the "pointwise convergence" in the above definition is replaced by "convergence" then we say that A (or B) is (sequentially) asymptotically compact. Note that if $Y = \mathbb{R}$, then the poinwise convergence coincides with the star-weak convergence. The poinwise convergence is corresponding to a nonmetrizable topology. Hence the mentioned sequential compactness is different from p-compactness. However, the latter notion is not used in this paper and we omit the term "sequentially" for short. Note that the asymptotical p-compactness here is equivalent to the relative p-compactness and asymptotical p-compactness together defined in [13].

For $A \subseteq L(X,Y)$ and $B \subseteq B(X,X,Y)$ we adopt the notations:

$$\text{p-cl}\, A = \{P \in L(X,Y) \mid \exists (P_n) \subseteq A, P = \text{p-lim}\, P_n\}, \tag{1}$$

$$\text{p-cl}\, B = \{M \in B(X,X,Y) \mid \exists (M_n) \subseteq B, M = \text{p-lim}\, M_n\}, \tag{2}$$

$$A_\infty = \{P \in L(X,Y) \mid \exists (P_n) \subseteq A, \exists t_n \to 0^+, P = \lim t_n P_n\}, \tag{3}$$

$$\text{p-}A_\infty = \{P \in L(X,Y) \mid \exists (P_n) \subseteq A, \exists t_n \to 0^+, P = \text{p-lim}\, t_n P_n\}, \tag{4}$$

$$\text{p-}B_\infty = \{M \in B(X,X,Y) \mid \exists (M_n) \subseteq B, \exists t_n \to 0^+, M = \text{p-lim}\, t_n M_n\}. \tag{5}$$

The sets (1), (2) are pointwise closures; (3) is just the known definition of the recession cone of a set A (not necessarily convex). So (4), (5) are pointwise recession cones.

Remark 2.5. (i) Assume that $\{P_n\} \subseteq L(X,Y)$ is norm bounded. If $x_n \to x$ and $P_n \xrightarrow{\mathrm{p}} P$, then $P_n x_n \to Px$. Similarly, if $x_n \to x$, $z_n \to z$, $\{M_n\} \subseteq B(X,X,Y)$ is norm bounded and $M_n \xrightarrow{\mathrm{p}} M$, then $N_n(x_n, z_n) \to N(x,z)$. Indeed, the conclusions follow directly from the following inequalities

$$\|P_n x_n - Px\| \le \|P_n x_n - P_n x\| + \|P_n x - Px\| \le \|P_n\|\|x_n - x\| + \|P_n x - Px\|;$$

$$\|M_n(x_n,y_n) - M(x,y)\| \le \|M_n(x_n,y_n) - M_n(x_n,y)\| + \|M_n(x_n,y) - M_n(x,y)\| +$$

$$\|M_n(x,y) - M(x,y)\| \le \|M_n\|\|x_n\|\|y_n - y\| + \|M_n\|\|x_n - x\|\|y\| + \|M_n(x,y) - M(x,y)\|.$$

(ii) If X is finite dimensional, a convergence occurs if and only if the corresponding pointwise convergence does, but in general the "if" does not hold, see [13, Example 3.1].

Definition 2.6. *Let* $x_0, v \in X$ *and* $S \subseteq X$.

(a) The contingent (or Bouligand) cone of S *at* x_0 *is*

$$T(S,x_0) = \{v \in X \mid \exists t_n \to 0^+, \exists v_n \to v, \forall n \in \mathbb{N}, x_0 + t_n v_n \in S\}.$$

(b) The second-order contingent set of S *at* (x_0, v)*, see e.g. [11], is*

$$T^2(S,x_0,v) = \{w \in X \mid \exists t_n \to 0^+, \exists w_n \to w, \forall n \in \mathbb{N}, x_0 + t_n v + \tfrac{1}{2}t_n^2 w_n \in S\}.$$

(c) The asymptotic second-order tangent cone of S *at* (x_0, v) *[11, 17] (the name is proposed*

by Penot in [17]) is

$$T^{''}(S,x_0,v)=\{w\in X\mid \exists(t_n,r_n)\to(0^+,0^+):\tfrac{t_n}{r_n}\to 0,\exists w_n\to w,$$
$$\forall n\in\mathbb{N}, x_0+t_nv+\tfrac{1}{2}t_nr_nw_n\in S\}.$$

The following assertion can be proved similarly as for Lemma 2.3 of [15].

Lemma 2.7. *If $K\subseteq Z$ is a closed convex cone with int $K\neq\emptyset$, $z_0\in -K$, $z\in -$ int $K(z_0)$ and $\frac{1}{t_n}(z_n-z_0)\to z$ as $t_n\to 0^+$, then $z_n\in -$ int K for all n large enough.*

Lemma 2.8 [11]. *Assume that $X=\mathbb{R}^m$ and $x_0\in S\subseteq X$. If $x_n\in S\setminus\{x_0\}$ tends to x_0, then there exists $u\in T(S,x_0)\setminus\{0\}$ and a subsequence, denoted again by x_n, such that*

(i) $\frac{1}{t_n}(x_n-x_0)\to u$, *where $t_n=\|x_n-x_0\|$;*

(ii) *either $z\in T^2(S,x_0,u)\cap u^{\perp}$ exists such that $(x_n-x_0-t_nu)/\frac{1}{2}t_n^2\to z$ or $z\in T^{''}(S,x_0,u)\cap u^{\perp}\setminus\{0\}$ and $r_n\to 0^+$ exist such that $\frac{t_n}{r_n}\to 0^+$ and $(x_n-x_0-t_nu)/\frac{1}{2}t_nr_n\to z$, where $u^{\perp}$ is the orthogonal compliment of $u\in\mathbb{R}^m$.*

Let us now recall notions of solutions to problem (P). A point $x_0\in g^{-1}(-K)$ is said to be a local weakly efficient solution (local efficient solution) of (P) if there exists a neighborhood U of x_0 such that, $\forall x\in U\cap g^{-1}(-K)$,

$$f(x)-f(x_0)\notin -\text{int } C$$
$$(f(x)-f(x_0)\notin(-C)\setminus C, \text{ respectively}).$$

The set of all local weakly efficient solutions of (P) is denoted by LWE(f,g) and that of local efficient ones by LE(f,g). These sets are basic solution sets considered in vector optimization.

For $m\in\mathbb{N}$, $x_0\in g^{-1}(-K)$ is called a local firm efficient solution of order m, denoted by $x_0\in$ LFE(m,f,g) if there are $\gamma>0$ and a neighborhood U of x_0 such that, $\forall x\in U\cap g^{-1}(-K)\setminus\{x_0\}$,

$$(f(x)+C)\cap B_Y(f(x_0),\gamma\|x-x_0\|^m)=\emptyset,$$

or, equivalently,

$$d(f(x)-f(x_0),-C)\geq\gamma\|x-x_0\|^m.$$

Note that, in the literature, instead of "firm efficient", other terms as "strict efficient" and "isolated efficient" are also used. The term "firm" was suggested by an anonymous referee of our paper [15]. Note also that, for $p\geq m$,

$$\text{LFE}(m,f,g)\subseteq\text{LFE}(p,f,g)\subseteq\text{LE}(f,g)\subseteq\text{LWE}(f,g).$$

Hence, necessary conditions for a point to be in the right-most set hold true also for all other sets and a similar assertion is valid for sufficient conditions and the left-most set.

3. First-order optimality conditions

Theorem 3.1 (Necessary condition). *Consider problem (P) with* int $C\neq\emptyset$ *and* int $K\neq\emptyset$. *Assume that $A_f(x_0)$ and $A_g(x_0)$ are asymptotically p-compact first-order approximations of f and g, respectively, at x_0 with $A_g(x_0)$ being normed bounded.*

If $x_0\in$ LWE(f,g), i.e. x_0 is a local weakly efficient solution of (P), then $\forall u\in X,\exists P\in$ p-cl$A_f(x_0)\bigcup($p-$A_f(x_0)_\infty\setminus\{0\})$, $\exists Q\in$ p-cl$A_g(x_0)$, $\exists(y^*,z^*)\in C^*\times K^*\setminus\{(0,0)\}$ such that

$$\langle y^*,Pu\rangle+\langle z^*,Qu\rangle\geq 0,$$
$$\langle z^*,g(x_0)\rangle=0.$$

Proof. For arbitrary fixed $u\in X$ and $t_n\to 0^+$, by Definition 2.1, there are $P_n\in A_f(x_0)$

and $Q_n \in A_g(x_0)$ such that

$$f(x_0 + t_n u) - f(x_0) = t_n P_n u + o(t_n),$$
$$g(x_0 + t_n u) - g(x_0) = t_n Q_n u + o(t_n).$$

By the boundedness of $A_g(x_0)$, assume that $Q_n \xrightarrow{\text{p}} Q$, for some $Q \in$ p-cl$A_g(x_0)$. Then

$$\tfrac{1}{t_n}(g(x_0 + t_n u) - g(x_0) \to Qu.$$

If $\{P_n\}$ is normed bounded, then we can assume that $P_n \xrightarrow{\text{p}} P \in$p-cl$A_f(x_0)$ and

$$\tfrac{1}{t_n}(f(x_0 + t_n u) - f(x_0) \to Pu.$$

Suppose that $(Pu, Qu) \in -\ \text{int}C \times K(g(x_0))$. Then, for large $n \in \mathbb{N}$,

$$f(x_0 + t_n u) - f(x_0) \in -\ \text{int}\ C, \tag{6}$$
$$\tfrac{1}{t_n}(g(x_0 + t_n u) - g(x_0) \to Qu \in -\ \text{int}\ K(g(x_0)),$$

as $n \to \infty$. Taking Lemma 2.7 into account, one sees that $g(x_0 + t_n u) \in -\ \text{int}\ K$ for all large n. This together with (6) contradicts the local weak efficiency of x_0. Therefore, $(Pu, Qu) \notin -\ \text{int}\ C \times K(g(x_0))$.

If $\{P_n\}$ is unbounded one can assume $\|P_n\| \to \infty$ and $\frac{P_n}{\|P_n\|} \xrightarrow{\text{p}} P \in$ p-$A_f(x_0)_\infty \setminus \{0\}$ and

$$\tfrac{1}{t_n\|P_n\|}(f(x_0 + t_n u) - f(x_0) \to Pu.$$

By an argument similar to that for the above boundedness case, one obtains $(Pu, Qu) \notin -\ \text{int}\ C \times K(g(x_0))$. Now employing the separation theorem one gets the conclusion. □

Note that the Lagrange multipliers mentioned in Theorem 3.1 depend on the given direction $u \in X$. In the following example, Theorem 3.1 rejects x_0, a suspected point when finding local weakly efficient solutions, while many known results cannot be applied.

Example 3.2. Let $X = Y = Z = \mathbb{R}, C = K = \mathbb{R}_+, x_0 = 0, g(x) = x^2 - 2x$ and

$$f(x) = \begin{cases} -1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Let $\alpha < 0$ be arbitrary and fixed. Then we have the following first-order approximations of f and g, which satisfy the assumptions of Theorem 3.1: $A_f(x_0) = (-\infty, \alpha)$ and $A_g(x_0) = \{-2\}$. Hence cl$A_f(x_0) = (-\infty, \alpha]$, $A_f(x_0)_\infty = (-\infty, 0]$. For $u = 1 \in X$ we see that $\forall P \in \text{cl}A_f(x_0) \bigcup (A_f(x_0)_\infty \setminus \{0\})$, $\forall Q \in \text{cl}A_g(x_0)$, $\forall (y^*, z^*) \in C^* \times K^* \setminus \{(0,0)\}$ with $\langle z^*, g(x_0)\rangle = 0$ one has, since $P < 0$,

$$\langle y^*, Pu\rangle + \langle z^*, Qu\rangle = y^*P - 2z^* < 0.$$

According to Theorem 3.1, $x_0 \notin$ LWE(f, g). However, f is not locally Lipschitz at x_0. Hence, necessary optimality conditions using the Clarke generalized derivative or the Dini directional derivative, e.g. in [3, 5], do not work. The Hadamard upper directional derivative (see [15]) of f at x_0 in the direction u defined by

$$Df(x_0, u) := \limsup_{t \to 0^+, v \to u} \tfrac{1}{t}[f(x_0 + tv) - f(x_0)]$$
$$:= \{y \in Y \mid \exists (t_n, u_n) \to (0^+, u), y = \lim_{n \to \infty} \tfrac{1}{t_n}(f(x_0 + t_n u_n) - f(x_0))\},$$

is empty in this case and then Theorem 3.1 of [15] cannot be employed. Furthermore, f is not continuous at x_0 and hence results which make use of the approximate Jacobian, e.g. in [9, 16] cannot be applied. f does not have directional derivative $f'(x_0, u)$ and then results using quasidifferentiability [4] cannot be used either.

Theorem 3.3 (Sufficient condition). *Assume that X is finite dimensional, $x_0 \in g^{-1}(-K)$ and $A_f(x_0)$ and $A_g(x_0)$ are asymptotically p-compact first-order approximations of f and g, respectively, at x_0. Assume that $\forall u \in X : \|u\| = 1, u \in T(g^{-1}(-K), x_0)$, $\forall P \in$ p-cl$A_f(x_0) \bigcup ($p-$A_f(x_0)_\infty \setminus \{0\})$, $\forall Q \in$ p-cl$A_g(x_0) \bigcup ($p-$A_g(x_0)_\infty \setminus \{0\})$, $\exists (y^*, z^*) \in$*

$C^* \times K^* \setminus \{(0,0)\}$,

$$\langle y^*, Pu\rangle + \langle z^*, Qu\rangle > 0,$$
$$\langle z^*, g(x_0)\rangle = 0.$$

Then $x_0 \in \mathrm{LFE}(1, f, g)$, *i.e.* x_0 *is a local firm efficient solution of order 1 of (P).*

Proof. Reasoning by contraposition, suppose the existence of $x_n \in B_X(x_0, \frac{1}{n}) \setminus \{x_0\}$ and $c_n \in C$ such that $g(x_n) \in -K$ and

$$f(x_n) - f(x_0) + c_n \in B_Y(0, \tfrac{1}{n}\|x_n - x_0\|).$$

Then, by Definition 2.1, there is $P_n \in A_f(x_0)$ such that, for $n \in \mathbb{N}$ large enough,

$$P_n(x_n - x_0) + o(\|x_n - x_0\|) + c_n \in B_Y(0, \tfrac{1}{n}\|x_n - x_0\|). \tag{7}$$

Since X is finite dimensional, one can assume that $\frac{x_n - x_0}{\|x_n - x_0\|} \to u$ for some $u \in T(g^{-1}(-K), x_0)$ with norm one. Then (7) implies the existence of $P \in \text{p-cl}A_f(x_0) \bigcup (\text{p-}A_f(x_0)_\infty \setminus \{0\})$ such that $Pu \in -C$. (For details one can split the consideration into two cases depending on $\{P_n\}$ is bounded or not, similarly as in the proof of Theorem 3.1.)
On the other hand,

$$g(x_n) - g(x_0) \in -K - g(x_0) \subseteq -K(g(x_0)).$$

Hence, there is $Q_n \in A_g(x_0)$ such that

$$Q_n(x_n - x_0) + o(\|x_n - x_0\|) \in -K(g(x_0)). \tag{8}$$

Similarly as for $\{P_n\}$, from (8) it follows the existence of $Q \in \text{p-cl}A_g(x_0) \bigcup (\text{p-}A_g(x_0)_\infty \setminus \{0\})$ such that $Qu \in -K(g(x_0))$.
Therefore, for each $(y^*, z^*) \in C^* \times K^* \setminus \{(0,0)\}$ with $\langle z^*, g(x_0)\rangle = 0$, one has

$$\langle y^*, Pu\rangle + \langle z^*, Qu\rangle \leq 0,$$

which is absurd. □

The following example explains advantages of Theorem 3.3.

Example 3.4. Let $X = Z = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, K = \mathbb{R}_+, x_0 = 0, f(x) = (x, (\mathrm{sgn}x)\sqrt{|x|})$, $g(x) = \sqrt[3]{x^4} - 2x$. Then $T(g^{-1}(-K), x_0) = [0, \infty)$ and, for any fixed $\alpha > 0$, f and g admit first-order approximations $A_f(x_0) = \{(1, y) \in \mathbb{R}^2 \mid y > \alpha\}$ and $A_g(x_0) = \{-2\}$, respectively. One has $\mathrm{cl}A_f(x_0) = \{(1, y) \in \mathbb{R}^2 \mid y \geq \alpha\}$ and $A_f(x_0)_\infty = \{(0, y) \in \mathbb{R}^2 \mid y \geq 0\}$. Then one sees that $\forall u \in T(g^{-1}(-K), x_0)$ with $\|u\| = 1, \forall P \in \mathrm{cl}A_f(x_0) \bigcup (A_f(x_0)_\infty \setminus \{0\})$, $\forall Q \in \mathrm{cl}A_g(x_0) \bigcup (A_g(x_0)_\infty \setminus \{0\})$, for $(y^*, z^*) = ((0, 1), 0) \in C^* \times K^* \setminus \{(0, 0)\}$ one obtains

$$\langle y^*, Pu\rangle + \langle z^*, Qu\rangle = y > 0,$$
$$\langle z^*, g(x_0)\rangle = 0.$$

By virtue of Theorem 3.3, x_0 is a local firm efficient solution of order 1 of (P). Clearly f is not locally Lipschitz at x_0 hence results using this property cannot be applied. f is not calm at x_0, so Theorem 3.2 of [15] cannot either.

4. Second-order optimality conditions

In the sequel we admit the following notations for problem (P). For $z^* \in K^*$, set

$$G(z^*) = \{x \in X \mid g(x) \in -K, \langle z^*, g(x)\rangle = 0\}.$$

If f and g have Fréchet derivatives $f'(x_0)$ and $g'(x_0)$ then set

$$C_0^* \times K_0^* = \{(y^*, z^*) \in C^* \times K^* \setminus \{(0,0)\} \mid y^* \circ f'(x_0) + z^* \circ g'(x_0) = 0, \langle z^*, g(x_0)\rangle = 0\}.$$

If f and g have first-order approximations $A_f(x_0)$ and $A_g(x_0)$, respectively, then for $x_0 \in$

$X, (y^*, z^*) \in C^* \times K^*$, we set

$$P(x_0, y^*, z^*) = \{v \in X \mid \langle y^*, Pv\rangle + \langle z^*, Qv\rangle = 0, \forall P \in A_f(x_0), \forall Q \in A_g(x_0)\}.$$

4.1. The first-order differentiable case

In this subsection we consider the case where f and g are Fréchet differentiable at x_0.

Theorem 4.1 (Necessary condition for the first-order differentiable case). *Assume that* $(y^*, z^*) \in C_0^* \times K_0^*$. *Assume further that* $(f^{'}(x_0), B_f(x_0))$ *and* $(g^{'}(x_0), B_g(x_0))$ *are asymptotically p-compact second-order approximations of* f *and* g*, respectively, at* x_0 *with norm-bounded* $B_g(x_0)$.

If $x_0 \in \mathrm{LWE}(f, g)$ *then, for any* $v \in T(G(z^*), x_0)$, *either* $\exists M \in \text{p-cl}B_f(x_0)$, $\exists N \in \text{p-cl}B_g(x_0)$ *such that*

$$\langle y^*, M(v, v)\rangle + \langle z^*, N(v, v)\rangle \geq 0,$$

or $\exists M \in \text{p-}B_f(x_0)_\infty \setminus \{0\}$ *such that*

$$\langle y^*, M(v, v)\rangle \geq 0.$$

Proof. Fix any $v \in T(G(z^*), x_0)$. Then $\exists t_n \to 0^+$, $\exists v_n \to v$ such that $x_0 + t_n v_n \in G(z^*), \forall n \in \mathbb{N}$. On the other hand, $(y^* \circ f^{'}(x_0) + z^* \circ g^{'}(x_0), y^* \circ B_f(x_0) + z^* \circ B_g(x_0))$ is a second-order approximation of the Lagrangian $L(., y^*, z^*) := \langle y^*, f(.)\rangle + \langle z^*, g(.)\rangle$ for $y^* \in Y^*$ and $z^* \in Z^*$. Hence, by Definition 2.1, there are $M_n \in B_f(x_0)$ and $N_n \in B_g(x_0)$ such that for large $n \in \mathbb{N}$,

$$\begin{aligned} L(x_0 + t_n v_n, y^*, z^*) - L(x_0, y^*, z^*) = t_n\langle y^* \circ f^{'}(x_0) + z^* \circ g^{'}(x_0), v_n\rangle + t_n^2\langle y^*, M_n(v_n, v_n)\rangle \\ + t_n^2\langle z^*, N_n(v_n, v_n)\rangle + o(t_n^2). \end{aligned}$$

On the other hand, as $x_0 + t_n v_n \in G(z^*)$, for large n,

$$\begin{aligned} L(x_0 + t_n v_n, y^*, z^*) - L(x_0, y^*, z^*) = \langle y^*, f(x_0 + t_n v_n) - f(x_0)\rangle \\ + \langle z^*, g(x_0 + t_n v_n) - g(x_0)\rangle \geq 0, \\ y^* \circ f^{'}(x_0) + z^* \circ g^{'}(x_0) = 0. \end{aligned}$$

Consequently, for large n,

$$\langle y^*, M_n(v_n, v_n)\rangle + \langle z^*, N_n(v_n, v_n)\rangle + \frac{o(t_n^2)}{t_n^2} \geq 0. \tag{9}$$

By the boundedness of $B_g(x_0)$, we can assume that $N_n \xrightarrow{\text{p}} N \in \text{p-cl}B_g(x_0)$.

If $\{M_n\}$ is norm bounded, then we assume that $M_n \xrightarrow{\text{p}} M \in \text{p-cl}B_f(x_0)$. Letting $n \to \infty$ in (9) gives

$$\langle y^*, M(v, v)\rangle + \langle z^*, N(v, v)\rangle \geq 0.$$

If $\{M_n\}$ is unbounded, we can assume that $\|M_n\| \to \infty$ and $\frac{M_n}{\|M_n\|} \xrightarrow{\text{p}} M \in \text{p-}B_f(x_0)_\infty \setminus \{0\}$. Dividing (9) by $\|M_n\|$ and passing to the limit we obtain $\langle y^*, M(v, v)\rangle \geq 0$. □

In the following example, applying Theorem 4.1 we can reject the suspected x_0, but many recent results cannot be employed.

Example 4.2. Let $X = \mathbb{R}^2, Y = Z = \mathbb{R}, C = \mathbb{R}_+, K = \{0\}, x_0 = (0, 0)$ and

$$f(x, y) = -\tfrac{2}{3}|x|^{\frac{3}{2}} + \tfrac{1}{2}y^2,$$
$$g(x, y) = x^2 - y.$$

Then $f^{'}(x_0) = (0, 0), g^{'}(x_0) = (0, -1)$,

$$B_f(x_0) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mid \alpha < -1 \right\},$$

$$\text{cl}B_f(x_0) = \left\{ \begin{pmatrix} \beta & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mid \beta \leq -1 \right\},$$

$$B_f(x_0)_\infty = \left\{ \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix} \mid \gamma \leq 0 \right\},$$

$$B_g(x_0) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\},$$

$$C_0^* \times K_0^* = \{(y^*, 0) \mid y^* \in \mathbb{R}_+ \setminus \{0\}\},$$

$$G(z^*) = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y = 0\} \text{ for } z^* = 0.$$

Choosing $(y^*, z^*) = (1, 0) \in C_0^* \times K_0^*$ and $v = (1, 0) \in T(G(z^*), x_0) = \mathbb{R} \times \mathbb{R}_+$, we have

$$\langle y^*, M(v, v)\rangle + \langle z^*, N(v, v)\rangle = \alpha \leq -1 < 0$$

for all $M \in \mathrm{cl}B_f(x_0)$ and all $N \in \mathrm{cl}B_g(x_0)$, and

$$\langle y^*, M(v, v)\rangle = \gamma < 0$$

for all $M \in B_f(x_0)_\infty \setminus \{0\}$. Therefore, following Theorem 4.1, x_0 is not a local weakly efficient solution of problem (P). However, since int $K \neq \emptyset$, Theorem 4.1 of [15] does not work. f is not in $C^{1,1}$ at x_0 and hence the results based on this class of functions, e.g. in [6, 7], cannot be employed.

Remark 4.3. In [12], second-order approximations are used to derive second-order necessary conditions for scalar optimization problems with general inequality constraints. When applied to scalar problems our Theorem 4.1 is different from the corresponding results in [12]. However, there is some mistakes in the proof of Theorem 3.2.2, one of the main results in [12]. The following example shows that the conclusion of Theorem 3.2.2 is false.

Example 4.4. Let $X = Z = \mathbb{R}^2$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $K = \mathbb{R}^2_+$, $x_0 = (0, 0)$, $f(x_1, x_2) = x_1^2 + x_2^2 - x_2$ and $g(x_1, x_2) = (x_2, x_1^2 - 2x_2^2 + \frac{1}{2}x_2)$. Then, it is clear that x_0 is a (global efficient) solution of problem (P). The mentioned Theorem 3.2.2 states for (P) as follows. *Assume that f and g are Fréchet differentiable, that $(f'(x_0), B_f(x_0))$ and $(g'(x_0), B_g(x_0))$ are compact and that x_0 is a (efficient) solution of (P). Then $\forall z^* \in \mathbb{R}^2_+$ with $f'(x_0) + z^* \circ g'(x_0) = 0$, $\forall v \in T(G(z^*), x_0)$, $\exists M \in B_f(x_0)$, $\exists N \in B_g(x_0)$ such that*

$$M(v, v) + \langle z^*, N(v, v)\rangle \geq 0,$$

where $G(z^*) = g^{-1}(-K)$ (see [12]).

For this example, by direct computations we have

$$f'(x_0) = (0, -1),\ g'(x_0) = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & \frac{1}{2} \end{pmatrix} \right\},$$

$$G(z^*) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 - 2x_2^2 + \tfrac{1}{2}x_2 \leq 0, x_2 \leq 0\},$$

$$T(G(z^*), x_0) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq 0\},$$

$$B_f(x_0) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},\ B_g(x_0) = \{N\},$$

where $N : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is the $2 \times 2 \times 2$ matrix

$$N = \left\{ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \right\},$$

i.e. $N(v, v) = (0, u_1v_1 - 2u_2v_2)$, $\forall (u_1, u_2), (v_1, v_2) \in \mathbb{R}^2$.

We see that all assumptions of Theorem 3.2.2 are satisfied, but, for $z^* = (0, 2)$ and $v = (0, -1) \in T(G(z^*), x_0)$ we have

$$M(v, v) + \langle z^*, N(v, v)\rangle = -3 < 0.$$

Theorem 4.5 (Sufficient condition for the first-order differentiable case). *Assume that X is finite dimensional, $x_0 \in g^{-1}(-K)$ and $(f'(x_0), B_f(x_0))$ and $(g'(x_0), B_g(x_0))$ are asymptotically p-compact second-order approximations of f and g, respectively, at x_0 with $B_g(x_0)$ being norm bounded. Assume further the existence of $(y^*, z^*) \in C_0^* \times K_0^*$ such* *that,* $\forall v \in T(g^{-1}(-K), x_0)$ *with* $\|v\| = 1$ *and* $\langle y^*, f'(x_0)v\rangle = \langle z^*, f'(x_0)v\rangle = 0$,

(i) $\forall M \in \text{p-cl}B_f(x_0)$, $\forall N \in \text{p-cl}B_g(x_0)$,

$$\langle y^*, M(v,v)\rangle + \langle z^*, N(v,v)\rangle > 0,$$

(ii) $\forall M \in \text{p-}B_f(x_0)_\infty \setminus \{0\}$,

$$\langle y^*, M(v,v)\rangle > 0.$$

Then x_0 *is a local firm efficient solution of order 2, i.e.* $x_0 \in \text{LFE}(2, f, g)$.

Proof. Reasoning ad absurdum, suppose the existence of $x_n \in B_X(x_0, \frac{1}{n})$ and $c_n \in C$ such that $g(x_n) \in -K$ and

$$f(x_n) - f(x_0) + c_n \in B_Y(0, \tfrac{1}{n}\|x_n - x_0\|^2). \tag{10}$$

As $\dim X$ is finite we can assume the existence of $v \in T(g^{-1}(-K), x_0)$ with $\|v\| = 1$ such that $\frac{x_n - x_0}{\|x_n - x_0\|} \to v$. Dividing (10) by $\|x_n - x_0\|$ and passing to the limit we get $f'(x_0)v \in -C$ and hence

$$\langle y^*, f'(x_0)v\rangle \le 0.$$

On the other hand

$$g(x_n) - g(x_0) \in -K - g(x_0) \subseteq -K(g(x_0)). \tag{11}$$

Dividing this by $\|x_n - x_0\|$ we get in the limit $g'(x_0)v \in -K(g(x_0))$. Hence

$$\langle z^*, g'(x_0)v\rangle \le 0. \tag{12}$$

Since $(y^*, z^*) \in C_0^* \times K_0^*$, (11) and (12) together imply that

$$\langle y^*, f'(x_0)v\rangle = \langle z^*, f'(x_0)v\rangle = 0.$$

By the definition of the second-order approximation, for large n, there are $M_n \in B_f(x_0)$ and $N_n \in B_g(x_0)$ such that (setting $t_n = \|x_n - x_0\|$) we have for the Lagrangian

$$\begin{aligned} L(x_n, y^*, z^*) - L(x_0, y^*, z^*) &= \langle y^* \circ f'(x_0) + z^* \circ g'(x_0), x_n - x_0\rangle \\ &+ \langle y^*, M_n(x_n - x_0, x_n - x_0)\rangle + \langle z^*, N_n(x_n - x_0, x_n - x_0)\rangle + o(t_n^2). \end{aligned} \tag{13}$$

As $\langle z^*, g(x_0)\rangle = 0$ it follows from (10) that

$$L(x_n, y^*, z^*) - L(x_0, y^*, z^*) = \langle y^*, f(x_n) - f(x_0)\rangle + \langle z^*, g(x_n)\rangle - \langle z^*, g(x_0)\rangle \le \langle y^*, d_n\rangle$$

for some $d_n \in B_Y(0, \frac{1}{n}t_n^2)$. Hence from (13) we obtain

$$\langle y^*, M_n(x_n - x_0, x_n - x_0)\rangle + \langle z^*, N_n(x_n - x_0, x_n - x_0)\rangle + o(t_n^2) \le \langle y^*, d_n\rangle.$$

This, in a similar way as in the proof of Theorem 4.1, implies that either $\exists M \in \text{p-cl}B_f(x_0)$, $\exists N \in \text{p-cl}B_g(x_0)$ such that

$$\langle y^*, M(v,v)\rangle + \langle z^*, N(v,v)\rangle \le 0,$$

or $\exists M \in \text{p-}B_f(x_0)_\infty \setminus \{0\}$ such that

$$\langle y^*, M(v,v)\rangle \le 0,$$

both of which are impossible. □

The following example gives a case where Theorem 4.5 can be applied but the corresponding Theorem 4.2 of [15] and many other results cannot.

Example 4.6. Let $X = Z = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, K = \mathbb{R}_+, x_0 = 0$ and

$$f(x) = (|x|^{\frac{4}{3}}, x^2),$$

$$g(x) = -x + x^2.$$

Then $f'(x_0) = (0,0), g'(x_0) = -1, B_f(x_0) = \{(\alpha, 1) \mid \alpha > 1\}$, $\text{cl}B_f(x_0) = \{(\beta, 1) \mid \beta \ge 1\}$, $B_f(x_0)_\infty = \{(\gamma, 0) \mid \gamma \ge 0\}$, $B_g(x_0) = \{1\}$, $g^{-1}(-K) = [0,1]$, $C_0^* \times K_0^* = \{(y^*, 0) \mid y^* \in \mathbb{R}^2_+ \setminus \{0\}\}$, $T(g^{-1}(-K), x_0) = [0, \infty)$.

Choosing $(y^*, z^*) = ((1,0),0) \in C_0^* \times K_0^*$, $\forall v \in T(g^{-1}(-K), x_0)$ with $\|v\| = 1$, i.e. $v = 1$, we see that, $\forall M \in \mathrm{cl}B_f(x_0)$, $\forall N \in \mathrm{cl}B_g(x_0)$,

$$\langle y^*, M(v,v)\rangle + \langle z^*, N(v,v)\rangle = \beta \geq 1 > 0,$$

and, $\forall M \in B_f(x_0)_\infty \setminus \{0\}$,

$$\langle y^*, M(v,v)\rangle = \gamma > 0.$$

According to Theorem 4.5, $x_0 \in \mathrm{LFE}(2, f, g)$. However, since f' is not calm at x_0, Theorem 4.2 of [15] is out of use. $f \notin C^{1,1}$ at x_0 and hence the results in e.g. [6, 7] cannot be employed.

4.2. The nondifferentiable case

Now we pass to the general nondifferentiable case.

Theorem 4.7 (Necessary condition for the nondifferentiable case). *Assume that* $(A_f(x_0), B_f(x_0))$ *and* $(A_g(x_0), B_g(x_0))$ *are asymptotically p-compact second-order approximations of* f *and* g*, respectively, at* x_0*, with* $A_f(x_0), A_g(x_0)$ *and* $B_g(x_0)$ *being norm bounded. Assume that* $(y^*, z^*) \in C^* \times K^*$ *such that* $\langle z^*, g(x_0)\rangle = 0$. *If* $x_0 \in \mathrm{LWE}(f, g)$ *then*

(i) $\forall v \in T(G(z^*), x_0), \exists P \in \text{p-cl}A_f(x_0), \exists Q \in \text{p-cl}A_g(x_0)$ *such that*

$$\langle y^*, Pv\rangle + \langle z^*, Qv\rangle \geq 0;$$

(ii) $\forall v \in P(x_0, y^*, z^*)$*, we have*

(a) $\forall w \in T^2(G(z^*), x_0, v)$*, either* $\exists P \in \text{p-cl}A_f(x_0)$, $\exists Q \in \text{p-cl}A_g(x_0)$, $\exists M \in \text{p-cl}B_f(x_0)$, $\exists N \in \text{p-cl}B_g(x_0)$ *such that*

$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle + 2\langle y^*, M(v,v)\rangle + 2\langle z^*, N(v,v)\rangle \geq 0$$

or $\exists M \in \text{p-}B_f(x_0)_\infty \setminus \{0\}$ *such that*

$$\langle y^*, M(v,v)\rangle \geq 0;$$

(b) $\forall w \in T''(G(z^*), x_0, v)$*, either* $\exists P \in \text{p-cl}A_f(x_0)$, $\exists Q \in \text{p-cl}A_g(x_0)$, $\exists M \in \text{p-}B_f(x_0)_\infty$ *such that*

$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle + \langle y^*, M(v,v)\rangle \geq 0$$

or $\exists M \in \text{p-}B_f(x_0)_\infty \setminus \{0\}$ *such that*

$$\langle y^*, M(v,v)\rangle \geq 0.$$

Proof. (i) It is a special case of (ii) with $v = 0$.

(ii) (a) For arbitrary $v \in P(x_0, y^*, z^*)$ and $w \in T^2(G(z^*), x_0, v)$, by the definition of the second-order tangent set, $\exists t_n \to 0^+$, $w_n \to w$, $\forall n \in \mathbb{N}$,

$$x_n := x_0 + t_n v + \tfrac{1}{2}t_n^2 w_n \in G(z^*).$$

Hence, for large n,

$$L(x_n, y^*, z^*) - L(x_0, y^*, z^*) = \langle y^*, f(x_n) - f(x_0)\rangle + \langle z^*, g(x_n)\rangle - \langle z^*, g(x_0)\rangle \geq 0.$$

Consequently, by the definition of the second-order approximation and as $v \in P(x_0, y^*, z^*)$, $\exists P_n \in A_f(x_0)$, $\exists Q_n \in A_g(x_0)$, $\exists M_n \in B_f(x_0)$, $\exists N_n \in B_g(x_0)$,

$$\langle y^*, P_n w_n\rangle + \langle z^*, Q_n w_n\rangle + 2\langle y^*, M_n(v + \tfrac{1}{2}t_n w_n, v + \tfrac{1}{2}t_n w_n)\rangle$$
$$+2\langle z^*, N_n(v + \tfrac{1}{2}t_n w_n, v + \tfrac{1}{2}t_n w_n)\rangle + \frac{o(t_n^2)}{\frac{1}{2}t_n^2} \geq 0. \qquad (14)$$

By the boundedness of $A_f(x_0)$, $A_g(x_0)$ and $B_g(x_0)$, we can assume the existence of $P \in \text{p-cl}A_f(x_0)$, $Q \in \text{p-cl}A_g(x_0)$ and $N \in \text{p-cl}B_g(x_0)$ such that $P_n \xrightarrow{\mathrm{p}} P$, $Q_n \xrightarrow{\mathrm{p}} Q$ and $N_n \xrightarrow{\mathrm{p}} N$. If $\{M_n\}$ is norm bounded it may be assumed to converge to some $M \in \text{p-cl}B_f(x_0)$. Passing

(14) to the limit we obtain

$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle + 2\langle y^*, M(v,v)\rangle + 2\langle z^*, N(v,v)\rangle \geq 0.$$

If $\{M_n\}$ is unbounded, we can assume that $\frac{M_n}{\|M_n\|} \xrightarrow{\text{p}} M \in \text{p-}B_f(x_0)_\infty \setminus \{0\}$. From (14) we get after dividing by $\|M_n\|$ and passing to the limit

$$\langle y^*, M(v,v)\rangle \geq 0.$$

(b) For any $v \in P(x_0, y^*, z^*)$ and $w \in T^{''}(G(z^*), x_0, v)$, there are $(t_n, r_n) \to (0^+, 0^+)$ and $w_n \to w$ such that $\frac{t_n}{r_n} \to 0^+$ and, $\forall n \in \mathbb{N}$,

$$x_n := x_0 + t_n v + \tfrac{1}{2} t_n r_n w_n \in G(z^*).$$

Hence, for large n, $\exists P_n \in A_f(x_0)$, $\exists Q_n \in A_g(x_0)$, $\exists M_n \in B_f(x_0)$ and $\exists N_n \in B_g(x_0)$ such that (as $v \in P(x_0, y^*, z^*)$)

$$\begin{aligned} (\tfrac{2}{t_n r_n}(L(x_n, y^*, z^*) - L(x_0, y^*, z^*)) &= \langle y^*, P_n w_n\rangle + \langle z^*, Q_n w_n\rangle \\ +\langle y^*, (\tfrac{2t_n}{r_n}) M_n(v + \tfrac{1}{2} r_n w_n, v + \tfrac{1}{2} r_n w_n)\rangle &+ \langle z^*, (\tfrac{2t_n}{r_n}) N_n(v + \tfrac{1}{2} r_n w_n, v + \tfrac{1}{2} r_n w_n)\rangle \\ &+ \tfrac{2o(t_n^2)}{t_n r_n} \geq 0. \end{aligned} \tag{15}$$

As $B_g(x_0)$ is bounded, $(\frac{2t_n}{r_n})N_n \to 0$. Since $A_f(x_0)$ and $A_g(x_0)$ are bounded we can assume the existence of $P \in \text{p-cl}A_f(x_0)$ and $Q \in \text{p-cl}A_g(x_0)$ such that $P_n \xrightarrow{\text{p}} P$ and $Q_n \xrightarrow{\text{p}} Q$. There are three possibilities (using subsequences if necessary).

- $(\frac{2t_n}{r_n})M_n \to 0$. Passing (15) to the limit one gets
$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle \geq 0.$$
- $\|(\frac{2t_n}{r_n})M_n\| \to a > 0$. Then $\|M_n\| \to \infty$ and one can assume that $\frac{M_n}{\|M_n\|} \xrightarrow{\text{p}} M \in \text{p-}B_f(x_0)_\infty \setminus \{0\}$. From (15) one gets in the limit
$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle + \langle y^*, aM(v,v)\rangle \geq 0.$$
- $\|(\frac{2t_n}{r_n})M_n\| \to \infty$. Then $\|M_n\| \to \infty$ and one can assume that $\frac{M_n}{\|M_n\|} \xrightarrow{\text{p}} M \in \text{p-}B_f(x_0)_\infty \setminus \{0\}$. Passing (15) to limit gives
$$\langle y^*, M(v,v)\rangle \geq 0. \qquad \square$$

Example 4.8. Let $X = Y = \mathbb{R}^2$, $Z = \mathbb{R}$, $C = \mathbb{R}^2_+$, $K = \{0\}$, $x_0 = (0,0)$, $f(x,y) = (-y, x + |y|)$ and $g(x,y) = -x^3 + y^2$. Then we have the following approximations

$$A_f(x_0) = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & \pm 1 \end{pmatrix} \right\},\ B_f(x_0) = \{0\},$$

$$A_g(x_0) = \{0\},\ B_g(x_0) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Let $y^* = (1,0) \in C^*$, $z^* = 0 \in K^*$ and $v = (1,0) \in P(x_0, y^*, z^*)$. Then

$$G(z^*) = \{(x,y) \in \mathbb{R}^2 \mid -x^3 + y^2 = 0\},$$
$$T^2(G(z^*), x_0, v) = \emptyset,\ T^{''}(G(z^*), x_0, v) = \mathbb{R}^2.$$

Now for $w = (0,1) \in T^{''}(G(z^*), x_0, v)$, $\forall P \in \text{cl}A_f(x_0)$, $\forall Q \in \text{cl}A_g(x_0)$, $\forall M \in B_f(x_0)_\infty$, one has

$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle + \langle y^*, M(v,v)\rangle = -1 < 0.$$

Taking into account Theorem 4.7 one sees that $x_0 \notin \text{LWE}(f,g)$. It is not hard to check that all the corresponding necessary conditions in [7, 5, 6, 15] are satisfied and hence no conclusion about the possible weak efficiency of x_0 can be made. In this case, because int $K = \emptyset$ and f is not differentiable at x_0, Theorems 3.1 and 4.1 cannot be used.

Theorem 4.9 (Sufficient condition for the nondifferentiable case). *Let X be finite dimensional, $x_0 \in g^{-1}(-K)$, $(y^*, z^*) \in C^* \times K^*$ with $\langle z^*, g(x_0)\rangle = 0$. Let $(A_f(x_0), B_f(x_0))$*

and $(A_g(x_0), B_g(x_0))$ *are asymptotically p-compact second-order approximations of* f *and* g*, respectively, at* x_0 *such that* $A_f(x_0)$*,* $A_g(x_0)$ *and* $B_f(x_0)$ *are norm bounded. Then* $x_0 \in \mathrm{LFE}(2, f, g)$ *if the following conditions hold*

(i) $\forall v \in T(g^{-1}(-K), x_0), \forall P \in A_f(x_0), \forall Q \in A_g(x_0)$*, one has*

$$\langle y^*, Pv\rangle + \langle z^*, Qv\rangle = 0;$$

(ii) $\forall v \in T(g^{-1}(-K), x_0) : \|v\| = 1$ *such that* $\exists \overline{P} \in \text{p-cl}A_f(x_0) : \overline{P}v \in -C, \exists \overline{Q} \in \text{p-cl}A_g(x_0) : \overline{Q}v \in -K(g(x_0))$ *and* $\forall M \in p\text{-}B_f(x_0)_\infty \setminus \{0\}$*, one has*

$$\langle y^*, M(v, v)\rangle > 0$$

and

(a) $\forall w \in T^2(g^{-1}(-K), x_0, v) \cap v^\perp, \forall P \in \text{p-cl}A_f(x_0)$*,* $\forall Q \in \text{p-cl}A_g(x_0)$*,* $\forall M \in \text{p-cl}B_f(x_0)$*,* $\forall N \in \text{p-cl}B_g(x_0)$*, one has*

$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle + 2\langle y^*, M(v, v)\rangle + 2\langle z^*, N(v, v)\rangle > 0.$$

(b) $\forall w \in T^{''}(g^{-1}(-K), x_0, v) \cap v^\perp \setminus \{0\}, \forall P \in \text{p-cl}A_f(x_0)$*,* $\forall Q \in \text{p-cl}A_g(x_0)$*,* $\forall M \in \text{p-}B_f(x_0)_\infty$*, one has*

$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle + \langle y^*, M(v, v)\rangle > 0.$$

Proof. Suppose $x_n \in B_X(x_0, \frac{1}{n}) \setminus \{x_0\}$ and $c_n \in C$ exist such that $g(x_n) \in -K$ and

$$d_n := f(x_n) - f(x_0) + c_n \in B_Y(0, \tfrac{1}{n}t_n^2), \tag{16}$$

where $t_n = \|x_n - x_0\|$. We can assume that $\frac{1}{t_n}(x_n - x_0) \to v$ for some $v \in T(g^{-1}(-K), x_0)$ with norm one. For large n, there are $P_n^{'} \in A_f(x_0)$ and $Q_n^{'} \in A_g(x_0)$ such that

$$f(x_n) - f(x_0) = P_n^{'}(x_n - x_0) + o(t_n),$$
$$g(x_n) - g(x_0) = Q_n^{'}(x_n - x_0) + o(t_n) \in -K(g(x_0)). \tag{17}$$

We can assume the existence of $P^{'} \in \text{p-cl}A_f(x_0)$ and $Q^{'} \in \text{p-cl}A_g(x_0)$ such that $P_n^{'} \xrightarrow{\text{p}} P^{'}$ and $Q_n^{'} \xrightarrow{\text{p}} Q^{'}$. Dividing (16), (17) by t_n we get in the limit

$$P^{'}v \in -C,\ Q^{'}v \in -K(g(x_0)).$$

On the other hand

$$\begin{aligned} L(x_n, y^*, z^*) - L(x_0, y^*, z^*) &= \langle y^*, f(x_n) - f(x_0)\rangle + \langle z^*, g(x_n) - \langle z^*, g(x_0)\rangle \\ &\leq \langle y^*, d_n - c_n\rangle \leq \langle y^*, d_n\rangle. \end{aligned} \tag{18}$$

By Lemma 2.8, there are two possibilities now.

(α) $w_n := \frac{(x_n - x_0 - t_n v)}{\frac{1}{2}t_n^2} \to w \in T^2(g^{-1}(-K), x_0, v) \cap v^\perp$. By virtue of (18) and the definition of the second-order approximation for large n, there are $P_n \in A_f(x_0), Q_n \in A_g(x_0), M_n \in B_f(x_0)$ and $N_n \in B_g(x_0)$ such that

$$\begin{aligned} &\langle y^*, P_n(t_n v + \tfrac{1}{2}t_n^2 w_n)\rangle + \langle z^*, Q_n(t_n v + \tfrac{1}{2}t_n^2 w_n)\rangle + \langle y^*, M_n(t_n v + \tfrac{1}{2}t_n^2 w_n, t_n v + \tfrac{1}{2}t_n^2 w_n)\rangle \\ &\quad + \langle z^*, N_n(t_n v + \tfrac{1}{2}t_n^2 w_n, t_n v + \tfrac{1}{2}t_n^2 w_n)\rangle + o(t_n^2) \leq \langle y^*, d_n\rangle. \end{aligned}$$

Therefore, by assumption (i),

$$\begin{aligned} &\langle y^*, P_n w_n\rangle + \langle z^*, Q_n w_n\rangle + 2\langle y^*, M_n(v + \tfrac{1}{2}t_n w_n, v + \tfrac{1}{2}t_n w_n)\rangle \\ &+ 2\langle z^*, N_n(v + \tfrac{1}{2}t_n w_n, v + \tfrac{1}{2}t_n w_n)\rangle + \frac{o(t_n^2)}{\frac{1}{2}t_n^2} \leq \frac{\langle y^*, d_n\rangle}{\frac{1}{2}t_n^2}. \end{aligned} \tag{19}$$

We can assume $P_n \xrightarrow{\text{p}} P \in \text{p-cl}A_f(x_0)$, $Q_n \xrightarrow{\text{p}} Q \in \text{p-cl}A_g(x_0)$ and $N_n \xrightarrow{\text{p}} N \in \text{p-cl}B_g(x_0)$ by the assumed norm boundedness.

Now if $\{M_n\}$ is also norm bounded then $M_n \xrightarrow{\text{p}} M \in \text{p-cl}B_f(x_0)$, then from (19) we get in the limit the following contradiction with assumption (ii) (a):

$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle + 2\langle y^*, M(v,v)\rangle + 2\langle z^*, N(v,v)\rangle \leq 0.$$

If $\{M_n\}$ is unbounded, without loss of generality assume that $\|M_n\| \to \infty$ and $\frac{M_n}{\|M_n\|} \xrightarrow{\text{p}} M \in \text{p-}B_f(x_0)_\infty \setminus \{0\}$. Dividing (19) by $\|M_n\|$ we obtain in the limit $\langle y^*, M(v,v)\rangle \leq 0$, contradicting (ii).

(β) $w_n := \frac{(x_n - x_0 - t_n v)}{\frac{1}{2}t_n r_n} \to w \in T''(g^{-1}(-K), x_0, v) \cap v^\perp \setminus \{0\}$, where $r_n \to 0^+$ and $\frac{t_n}{r_n} \to 0^+$. Consequently, by (i), we get similarly as (19)

$$\begin{aligned}&\langle y^*, P_n w_n\rangle + \langle z^*, Q_n w_n\rangle + (\tfrac{2t_n}{r_n})\langle y^*, M_n(v + \tfrac{1}{2}r_n w_n, v + \tfrac{1}{2}r_n w_n)\rangle \\ &\quad + (\tfrac{2t_n}{r_n}\langle z^*, N_n(v + \tfrac{1}{2}r_n w_n, v + \tfrac{1}{2}r_n w_n)\rangle + \frac{o(t_n^2)}{\frac{1}{2}t_n r_n} \leq \frac{\langle y^*, d_n\rangle}{\frac{1}{2}t_n r_n}. \qquad (20)\end{aligned}$$

Similarly as before, $P_n \xrightarrow{\text{p}} P \in \text{p-cl}A_f(x_0)$ and $Q_n \xrightarrow{\text{p}} Q \in \text{p-cl}A_g(x_0)$. As $B_g(x_0)$ is bounded, $(\frac{2t_n}{r_n})N_n \to 0$. Now we have the following three subcases.

- $(\frac{2t_n}{r_n})M_n \to 0$. From (20) we get in the limit the contradiction with (ii) (b)

$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle \leq 0.$$

- $\|(\frac{2t_n}{r_n})M_n\| \to a > 0$. Then, since we can assume that $\|M_n\| \to \infty$ and $\frac{M_n}{\|M_n\|} \xrightarrow{\text{p}} M \in \text{p-}B_f(x_0)_\infty \setminus \{0\}$, (20) implies that

$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle + \langle y^*, aM(v,v)\rangle \leq 0.$$

Since $\text{p-}B_f(x_0)_\infty$ is a cone, this contradicts (ii) (b).

- $\|(\frac{2t_n}{r_n})M_n\| \to \infty$. Then, $\|M_n\| \to \infty$ and $\frac{M_n}{\|M_n\|} \xrightarrow{\text{p}} M \in \text{p-}B_f(x_0)_\infty \setminus \{0\}$. Dividing (20) by $(\frac{2t_n}{r_n})\|M_n\|$ and passing to the limit we get a contradiction that $\langle y^*, M(v,v)\rangle \leq 0$. ☐

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