The existence of ε-solutions to general quasiequilibrium problems

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Abstract

We consider three types of approximate solutions of multivalued quasiequilibrium vector problems. Sufficient conditions for the ε-solution existence are established for variants of such problems. Several applications are provided as examples to show that our results can imply consequences about approximate solutions of many optimization-related problems.

Keywords: Quasiequilibrium problems, ε-solutions, W-quasiconvexity relative to a set, quasivariational inequalities, quasioptimization problems.

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1. Introduction and preliminaries

The equilibrium problem was proposed by Blum and Oettli [4] as a generalization of variational inequalities and optimization problems and includes also other problems such as the complementarity problem, the Nash equilibrium, the fixed point and coincidence point problems, the traffic equilibrium problem, etc. On the other hand Bensoussan, Goursat, and Lions [3], considering random impulse control problems, observed the necessity to investigate the constraint set depending on the state variable. This paper led to the birth of the quasivariational inequality, and later, of the quasiequilibrium problem. The solution existence was often of interest first, see e.g. recent papers [2,5-18,21-23] and references therein. However, the conditions for the existence of exact solutions are often restricted. Moreover, some problems in practice do not have exact solutions, but possess ε-solutions (approximate solutions with ε-tolerance), see e.g. Examples 1.1 and 1.2. Such solutions make sense in practical situations, since the data of the problem under consideration is obtained approximately by measurements or statistical ways and hence the mathematically exact solutions are in fact also approximate ones. Therefore, the demand on the existence of exact solutions may be too costly.

To the best of our knowledge, there are not papers dealing with the existence of approximate solutions of equilibrium or quasiequilibrium problems in the literature (the only paper [1] considers the semicontinuity of approximate solution sets). This motivates our aim in this note: to establish sufficient conditions for the existence of ε-solutions to quasiequilibrium problems in general spaces. It appears that if $\varepsilon = 0$, i.e. for exact solutions, our results are also new, and shown by examples to be more applicable than existing ones in some cases.

We now outline the remainder of the paper. The rest of this section is devoted to the problem setting and some preliminaries. The main results are presented in Section 2. In the final Section 3, some applications are provided.

Throughout the paper, if not otherwise stated, let X and Z be Hausdorff topological vector spaces and Y be a linear metric space with invariant metric $d(.,.)$. Let $A \subseteq X$ and $B \subseteq Z$ be nonempty compact convex sets. Let $C \subseteq Y$ be closed with the interior $\mathrm{int}C \neq \emptyset$ and $C \neq Y$. Let the multifunctions $K : A \to 2^X$, $T : A \to 2^B$ and $F : T(A) \times X \times A \to 2^Y$ have nonempty values. We consider the following four quasiequilibrium problems:

(QEP1) Find $\bar{x} \in A \cap \mathrm{cl}K(\bar{x})$ such that $\forall y \in K(\bar{x})$, $\exists \bar{t} \in T(\bar{x})$,
$$F(\bar{t}, y, \bar{x}) \cap (Y \backslash -\mathrm{int}C) \neq \emptyset;$$

(QEP2) Find $\bar{x} \in A \cap \mathrm{cl}K(\bar{x})$ such that $\exists \bar{t} \in T(\bar{x})$, $\forall y \in K(\bar{x})$,
$$F(\bar{t}, y, \bar{x}) \cap (Y \backslash -\mathrm{int}C) \neq \emptyset;$$

(QEP3) Find $\bar{x} \in A \cap \mathrm{cl}K(\bar{x})$ such that $\forall y \in K(\bar{x})$, $\exists \bar{t} \in T(\bar{x})$,
$$F(\bar{t}, y, \bar{x}) \subseteq Y \backslash -\mathrm{int}C;$$

(QEP4) Find $\bar{x} \in A \cap \mathrm{cl}K(\bar{x})$ such that $\exists \bar{t} \in T(\bar{x})$, $\forall y \in K(\bar{x})$,
$$F(\bar{t}, y, \bar{x}) \subseteq Y \backslash -\mathrm{int}C.$$

Observe that any solution of (QEP2) or of (QEP3) solves (QEP1) and any solution of (QEP4) solves all three other problems. The converses are not valid. Also, the solutions of (QEP2) and (QEP3) may in general be different.

Let us use the notations

$$\mathrm{comp}(-\mathrm{int}C)_1^\varepsilon = \{y \in Y \mid d(y, Y\backslash -\mathrm{int}C) \le \varepsilon\},$$
$$\mathrm{comp}(-\mathrm{int}C)_2^\varepsilon = (Y\backslash -\mathrm{int}C) + \bar{B}_Y^\varepsilon,$$
$$\mathrm{comp}(-\mathrm{int}C)_3^\varepsilon = \{y \in Y \mid d(y, Y\backslash -\mathrm{int}C) < \varepsilon\},$$

where $d(y, V) :=\inf_{v\in V} d(y, v)$ is the distance between the point y and the set V, $\bar{B}_Y^\varepsilon := \{y \in Y \mid d(0, y) \le \varepsilon\}$ and $B_Y^\varepsilon := \{y \in Y \mid d(0, y) < \varepsilon\}$. The notation "comp(.)" is related to the word "complement".

Remark 1.1. (i) We have, for $\varepsilon > 0$,

$$\{y \in Y \mid d(y, Y\backslash -\mathrm{int}C) < \varepsilon\} = (Y\backslash -\mathrm{int}C) + B_Y^\varepsilon.$$

Indeed, we prove more generally that $\{y \in Y \mid d(y, Q) < \varepsilon\} := Q^\varepsilon$ is equal to $Q + B_Y^\varepsilon$, for any $\emptyset \ne Q \subseteq Y$. To see "$\supset$" let $y = q + z$ for some $q \in Q$ and $z \in B_Y^\varepsilon$. Then, $d(y, q) = d(y - q, 0) = d(z, 0) < \varepsilon$, i.e. $y \in Q^\varepsilon$. For the inverse inclusion "$\subset$", let $y \in Q^\varepsilon$. Then $d(y, Q) := d_y < \varepsilon$. Hence, there is $q \in Q$ with $d_y < d(y, q) < \varepsilon$. Consequently, $y - q \in B_Y^\varepsilon$ and $y \in Q + B_Y^\varepsilon$.

(ii) For $\emptyset \ne Q \subseteq Y$, denote $\bar{Q}^\varepsilon = \{y \in Y \mid d(y, Q) \le \varepsilon\}$. Then, following Remark 1.1 of [1], $Q + \bar{B}_Y^\varepsilon \subseteq \bar{Q}^\varepsilon$ and one has an equality if Y is finite dimensional and Q is closed. However, while Y is infinite dimensional $Q + \bar{B}_Y^\varepsilon$ may be properly contained in $\bar{Q}^\varepsilon$ since $Q + \bar{B}_Y^\varepsilon$ may be not closed even for a closed set Q, see Examples 1.1 and 1.2 of [1].

According to Remark 1.1 we have the following definition of three kinds of

ε-solutions.

Definition 1.1. Each of our four problems (QEP1)-(QEP4) has three kinds of ε-solutions corresponding to the above three sets $\mathrm{comp}(-\mathrm{int}C)_i^\varepsilon$. For instance $\bar{x} \in A \cap \mathrm{cl}K(\bar{x})$ is said to be an ε-solution of type $i, i = 1, 2, 3$, of problem (QEP1) if $\forall y \in K(\bar{x})$, $\exists \bar{t} \in T(\bar{x})$ such that

$$F(\bar{t}, y, \bar{x}) \cap \mathrm{comp}(-\mathrm{int}C)_i^\varepsilon \neq \emptyset.$$

Note that the ε-solutions of type 1 and 2 were proposed in [1]. Following Remark 1.1 (ii) each ε-solution of type 2 is an ε-solution of type 1, but the converse is not true if Y is infinite dimensional.

The following example shows that problem (QEP1) may be unsolvable (in the exact sense) but ε-solutions exist.

Example 1.1. Let $X = Y = Z = R$, $A = [0, 1]$, $K(x) \equiv [0, 1]$, $C = R_+$, $T(x) = [0, x]$ and $F(t, y, x) = [-0.1, -0.1 + 0.05x]$. Then it is clear that the exact solution of (QEP1) does not exist. However, for $\varepsilon \geq 0.1$, each $\bar{x} \in [0, 1]$ is an ε-solution of type 1.

ε-solution sets depend, in general, on ε as shown in the following example.

Example 1.2. Let X, Y, Z, A, K and C be as in Example 1.1. Let $T(x) = \{x\}$ and $F(t, y, x) = [-0.1 + x, 1]$. Then it is easy to check that the ε-solution set of (QEP1) is $[0.1 - \varepsilon, 1]$ for $0 \leq \varepsilon < 0.1$ and [0,1] for $\varepsilon \geq 0.1$.

Our main tool in this paper is the following fixed point theorem, which is a slightly weaker version (suitable for our use) of the corresponding theorem in [20].

Theorem 1.1. *Let X be a Hausdorff topological vector space, $A \subseteq X$ be nonempty compact convex and $\varphi : A \to 2^X$ be a multifunction with nonempty convex values. Assume that, for each $x \in A, \varphi^{-1}(x)$ is open in A. Then there is a fixed point $\hat{x} \in A$ of φ, i.e. $\hat{x} \in \varphi(\hat{x})$.*

We recall now the semicontinuity of multifunctions needed in the sequel. Let X and Y be topological spaces and $H : X \to 2^Y$ be a multifunction. H is called lower semicontinuous (lsc) at $x_0 \in X$ if, for any open subset U such that $U \cap H(x_0) \neq \emptyset$, there exists a neighborhood N of x_0 such that, $\forall x \in N, U \cap H(x) \neq \emptyset$. H is termed upper semicontinuous (usc) at $x_0 \in X$ if, for any open subset U such that $U \supseteq H(x_0)$, there exists a neighborhood N of x_0 such that $U \supseteq H(N)$. H is called lsc (or usc) if H is lsc (usc, respectively) at every point $x \in \mathrm{dom}H := \{x \in X : H(x) \neq \emptyset\}$. H is said to be closed if the graph $\mathrm{gr}H := \{(x,y) \in X \times Y \mid y \in H(x)\}$ is closed.

The convexity assumptions imposed in our theorems are the following relaxed property. Let X be a vector space and $D \subseteq X$ be nonempty and convex. Let $P, Q, V$and $W \subseteq V$ be nonempty sets. Let $T : P \to 2^Q$ and $F : Q \times D \to 2^V$ be multifunctions. For $x \in P$, F is said to be W-quasiconvex relative to $T(x)$ of type 1 if, $\forall \xi, \eta \in D, \forall \lambda \in [0,1]$,

$$
\begin{aligned}
&[F(t,\xi) \not\subseteq W \text{ and } F(t,\eta) \not\subseteq W, \forall t \in T(x)] \\
&\Rightarrow [F(t,(1-\lambda)\xi + \lambda\eta) \not\subseteq W, \forall t \in T(x)]. \qquad (1)
\end{aligned}
$$

F is called W-quasiconvex relative to $T(x)$ of type 2 if (1) is replaced by

$[F(t,\xi)\cap W=\emptyset$ and $F(t,\eta)\cap W=\emptyset, \forall t\in T(x)]$

$\Rightarrow [F(t,(1-\lambda)\xi+\lambda\eta)\cap W=\emptyset, \forall t\in T(x)]$.

To see the nature of these definitions, consider the simplest case, where $X = D = V = P = Q = R, T(x)\equiv\{x_0\}, W = R_+$ and $F:\{x_0\}\times X\to R$ is single-valued, depending only on $x\in X$. Then the above two types of relaxed convexity coincide and become: $\forall\xi,\eta\in R, \forall\lambda\in[0,1]$,

$[F(\xi)<0$ and $F(\eta)<0] \Rightarrow [F((1-\lambda)\xi+\lambda\eta)<0]$.

This property is a relaxed 0-level quasiconvexity, since F is called quasiconvex if $\forall\xi,\eta\in R, \forall\lambda\in[0,1], F((1-\lambda)\xi+\lambda\eta)\le\max\{F(\xi),F(\eta)\}$.

For the special case of the above general quasiconvexity, where $T(x)\equiv\{x_0\}$, i.e. F depends on only one variable x, we simply say that F is W-quasiconvex of type 1 or type 2.

2. Main results

In principle we have to investigate the existence of ε-solutions of three types for four problems (QEP1) - (QEP4). However, it is easy to imagine some similarities. So we only prove sufficient conditions for the existence of several among these twelve types of ε-solutions.

Theorem 2.1. *For problem* (QEP1) *assume that*

(i_1^1) *for each $x\in A, F(.,.,x)$ is comp$(-intC)_1^\varepsilon$-quasiconvex relative to $T(x)$ of type 2 and $F(t,x,x)\cap comp(-intC)_1^\varepsilon\neq\emptyset$ for some $t\in T(x)$;*

(ii$_1^1$) *for each* $y \in A, F(.,y,.)$ *and* $T(.)$ *are usc and map compact sets to compact sets;*

(iii$_1^1$) *cl*$K(.)$ *is usc; for each* $x \in A, A \cap K(x) \neq \emptyset$ *and* $K(x)$ *is convex; for each* $y \in A, K^{-1}(y)$ *is open in* A.

Then, problem (QEP1) *has* ε*-solutions of type 1.*

Proof. For $x \in A$ set

$$P(x) = \{z \in A \mid \forall t \in T(x), F(t,z,x) \cap \mathrm{comp}(-\mathrm{int}C)_1^{\varepsilon} = \emptyset\},$$

$$E = \{z \in A \mid z \in \mathrm{cl}K(z)\}.$$

By virtue of (i$_1^1$), $P(x)$ is convex for all $x \in A$. By (iii$_1^1$) cl$K(.)$ is closed and hence E is a closed set.

We now prove that $P^{-1}(y)$ is open in $A, \forall y \in A$. Assume that $\{x_\alpha\} \subseteq A \backslash P^{-1}(y)$ and $x_\alpha \to \bar{x}$. Then, $t_\alpha \in T(x_\alpha)$ exists such that we are ensured the existence of

$$z_\alpha \in F(t_\alpha, y, x_\alpha) \cap \mathrm{comp}(-\mathrm{int}C)_1^{\varepsilon}.$$

Setting $L = \{x_\alpha\} \cup \{\bar{x}\}$ we see that $T(L)$ is compact and hence we can assume that $t_\alpha \to \bar{t}$, for some $\bar{t} \in T(L)$. By (ii$_1^1$), $T(.)$ is closed and then $\bar{t} \in T(\bar{x})$. Setting now $N = \{(t_\alpha, y, x_\alpha)\} \cup \{(\bar{t}, y, \bar{x})\}$ we see that $F(N)$ is compact and then we can assume that $z_\alpha \to \bar{z}$, for some $\bar{z} \in F(N)$. By the closedness of the map $F(.,y,.) \cap \mathrm{comp}(-\mathrm{int}C)_1^{\varepsilon}$, $\bar{z} \in F(\bar{t}, y, \bar{x}) \cap \mathrm{comp}(-\mathrm{int}C)_1^{\varepsilon}$. Consequently, $A \backslash P^{-1}(y)$ is closed, i.e. $P^{-1}(y)$ is open in A.

We define multifunction $Q : A \to 2^A$ by

$$Q(x) = \begin{cases} K(x) \cap P(x) & \text{if}\, x \in E, \\ A \cap K(x) & \text{if}\, x \in A \backslash E. \end{cases}$$

Then, $\forall x \in A, Q(x)$ is convex. We have, for $y \in A$,

$$\begin{aligned} Q^{-1}(y) &= \{x \in E \mid x \in K^{-1}(y) \cap P^{-1}(y)\} \cup \{x \in A \backslash E \mid x \in K^{-1}(y)\} \\ &= K^{-1}(y) \cap [P^{-1}(y) \cup (A \backslash E)]. \end{aligned}$$

Therefore,

$$A \backslash Q^{-1}(y) = [A \backslash K^{-1}(y)] \cup [(A \backslash P^{-1}(y)) \cap E].$$

Since $K^{-1}(y)$ and $P^{-1}(y)$ are open in A, this implies the openness of $Q^{-1}(y)$ in A, for all $y \in A$. From (i_1^1), $x \notin P(x)$ and then $x \notin Q(x)$, for all $x \in A$. Applying Theorem 1.1 to multifunction Q, one gets $\hat{x} \in A$ such that $Q(\hat{x}) = \emptyset$. Since $A \cap K(\hat{x}) \neq \emptyset$, $\hat{x} \in E$ and $K(\hat{x}) \cap P(\hat{x}) = \emptyset$. Thus, $\hat{x} \in A \cap \mathrm{cl} K(\hat{x})$ and , $\forall y \in K(\hat{x})$, $y \notin P(\hat{x})$, i.e., $F(\hat{t}, y, \hat{x}) \cap \mathrm{comp}(-\mathrm{int} C)_1^{\varepsilon} \neq \emptyset$ and $\hat{x}$ is an ε-solution of type 1 of (QEP1). $\square$

Remark 2.1

(a) If the set C in problem (QEP1) depends on $x \in A$, i.e. $C : A \to 2^Y$ is a multifunction, then it is not hard to check that Theorem 2.1 is still valid with C replaced by $C(x)$, for each x and with the additional assumption that $Y \backslash -\mathrm{int} C(.)$ is usc.

(b) If, A is not compact but the following coersive assumption is additionally imposed:

(iv_1^1) there exists a nonempty compact subset $D \subseteq A$ such that for each finite subset $M \subseteq A$, there is a compact convex subset L_M of A, containing M, such that $\forall x \in L_M \backslash D, \exists y \in L_M \cap K(x)$, $\forall t \in T(x)$,

$$F(t, y, x) \cap \text{comp}(-\text{int}C)_1^{\varepsilon} = \emptyset;$$

then Theorem 2.1 is still valid.

(c) The special case of Theorem 2.1 with $\varepsilon = 0$ and the additional assumptions mentioned in (a) and (b) is a result stronger than Theorem 4.13 of [16], since our quasiconvexity assumption is more relaxed than the corresponding assumption there.

The next example indicates that an ε-solution of type 1 may exist even though ε-solutions of type 3 do not exist.

Example 2.1. Let $X = Y = Z = R$, $A = [0, 1]$, $K(x) \equiv [0, 1]$, $C = R_+$, $T(x) = \{x\}$ and $F(t, y, x) \equiv \{-0.1\}$. Then it is evident that problem (QEP1) does not have ε-solutions of type 3 for $\varepsilon = 0.1$. However, all assumptions of Theorem 2.1 are fulfilled and hence ε-solutions of type 1 exist.

Using similar techniques we can prove the following sufficient conditions for the existence of the other two types of solutions of problem (QEP1).

Theorem 2.2. *For problem* (QEP1) *assume (iii_1^1) as in Theorem 2.1 and assume further*

(i_1^2) *for each $x \in A, F(., ., x)$ is comp$(-intC)_2^{\varepsilon}$-quasiconvex relative to $T(x)$*

of type 2 and $F(t,x,x)\cap comp(-intC)_2^\varepsilon \neq \emptyset$ *for some* $t \in T(x)$*;*

(ii_1^2) *for each* $y \in A$*, the set* $\{x \in A \mid \exists t \in T(x), F(t,y,x)\cap comp(-intC)_2^\varepsilon \neq \emptyset\}$ *is closed.*

Then, ε*-solutions of type 2 exist for* (QEP1).

Theorem 2.3. *For problem* (QEP1) *assume (*iii_1^1*) as in Theorem 2.1 and replace the other assumptions by*

(i_1^3) *for each* $x \in A, F(.,.,x)$ *is* $comp(-intC)_3^\varepsilon$*-quasiconvex relative to* $T(x)$ *of type 2 and* $F(t,x,x)\cap comp(-intC)_3^\varepsilon \neq \emptyset$ *for some* $t \in T(x)$*;*

(ii_1^3) *for each* $y \in A$*, the set* $\{x \in A \mid \exists t \in T(x), F(t,y,x)\cap comp(-intC)_3^\varepsilon \neq \emptyset\}$ *is closed.*

Then, problem (QEP1) *has* ε*-solutions of type 3.*

Now we pass to problem (QEP3) before the other two problems, since it is closer to (QEP1).

Theorem 2.4. *For problem* (QEP3) *assume that*

(i_3^1) *for each* $x \in A, F(.,.,x)$ *is* $comp(-intC)_1^\varepsilon$*-quasiconvex relative to* $T(x)$ *of type 1 and* $F(t,x,x) \subseteq comp(-intC)_1^\varepsilon$ *for some* $t \in T(x)$*;*

(ii_3^1) *for each* $y \in A, F(.,y,.)$ *is lsc;* $T(.)$ *is usc and* $T(H)$ *is compact if* $H \subseteq A$ *is compact;*

(iii_3^1) $clK(.)$ *is usc; for each* $x \in A, A \cap K(x) \neq \emptyset$ *and* $K(x)$ *is convex; for each* $y \in A, K^{-1}(y)$ *is open in* A*.*

Then, there exist ε-solutions of type 1 of problem (QEP3).

Proof. For $x \in A$ set

$$P(x) = \{ z \in A \mid \forall t \in T(x), F(t,z,x) \not\subseteq \text{comp}(-\text{int}C)_1^{\varepsilon} = \emptyset\},$$

$$E = \{z \in A \mid z \in \text{cl}K(z)\}.$$

By (i_3^1) and (iii_3^1), $P(x)$ is convex for each $x \in A$ and E is closed. We claim that $P^{-1}(y)$ is open in $A, \forall y \in A$. Indeed, if a net $\{x_\alpha\} \subseteq M_y := A \backslash P^{-1}(y)$ with $x_\alpha \to \bar{x}$. Then, by the definition of P, there is $t_\alpha \in T(x_\alpha)$ such that

$$F(t_\alpha, y, x_\alpha) \subseteq \text{comp}(-\text{int}C)_1^{\varepsilon}. \tag{2}$$

Let $L = \{x_\alpha\} \cup \{\bar{x}\}$. By the compactness of $T(L)$ we can assume that $t_\alpha \to \bar{t}$ for some $\bar{t} \in T(L)$. By (ii_3^1), $\bar{t} \in T(\bar{x})$. As $F(.,y,.)$ is lsc, for any $\bar{z} \in F(\bar{t}, y, \bar{x})$, there is $z_\alpha \in F(t_\alpha, y, x_\alpha)$ such that $z_\alpha \to \bar{z}$. By (2) and the closedness of $\text{comp}(-\text{int}C)_1^{\varepsilon}$, $\bar{z} \in \text{comp}(-\text{int}C)_1^{\varepsilon}$. Hence

$$F(\bar{t}, y, \bar{x}) \subseteq \text{comp}(-\text{int}C)_1^{\varepsilon},$$

i.e. $\bar{x} \in M_y$. So M_y is closed and $P^{-1}(y)$ is open in A.

Defining now multifunction $Q : A \to 2^A$ as in the proof of Theorem 2.1 we also see that $Q(x)$ is convex for each $x \in A$, that $Q^{-1}(y)$ is open in A for each $y \in A$ and that $x \notin Q(x)$, for each $x \in A$. Theorem 1.1 implies that there exists $\hat{x} \in A$ with $Q(\hat{x}) = \emptyset$. Since $A \cap K(\hat{x}) \neq \emptyset$ by (iii_3^1), one sees that $\hat{x} \in E$ and $K(\hat{x}) \cap P(\hat{x}) = \emptyset$. Hence, $\hat{x} \in A \cap \text{cl}K(\hat{x})$ and $y \notin P(\hat{x})$, for each $y \in K(\hat{x})$. The later means that, for some $\hat{t} \in T(\hat{x})$,

$$F(\hat{t}, y, \hat{x}) \subseteq \text{comp}(-\text{int}C)_1^{\varepsilon},$$

i.e. $\hat{x}$ is an ε-solution of type 1 of (QEP3). $\square$

For the other two types of solutions of this problem we have the following sufficient conditions for the existence. The proof are analogous to that of Theorem 2.4.

Theorem 2.5. *For problem* (QEP3) *assume* (iii_3^1) *as in Theorem 2.4 and replace* (i_3^1), (ii_3^1) *by the following, respectively,*

(i_3^2) *for each* $x \in A, F(.,.,x)$ *is* $comp(-intC)_2^{\varepsilon}$*-quasiconvex relative to* $T(x)$ *of type 1 and* $F(t,x,x) \subseteq comp(-intC)_2^{\varepsilon}$ *for some* $t \in T(x)$*;*

(ii_3^2) *for each* $y \in A$*, the set* $\{x \in A \mid \exists t \in T(x), F(t,y,x) \subseteq comp(-intC)_2^{\varepsilon}\}$ *is closed.*

Then, ε*-solutions of type 2 exist for* (QEP3).

Theorem 2.6. ε*-solutions of type 3 for problem* (QEP3) *exist under assumption* (iii_3^1) *as in Theorem 2.4 and under the following assumptions, replacing* (i_3^1), (ii_3^1),

(i_3^3) *for each* $x \in A, F(.,.,x)$ *is* $comp(-intC)_3^{\varepsilon}$*-quasiconvex relative to* $T(x)$ *of type 1 and* $F(t,x,x) \subseteq comp(-intC)_3^{\varepsilon}$ *for some* $t \in T(x)$*;*

(ii_3^3) *for each* $y \in A$*, the set* $\{x \in A \mid \exists t \in T(x), F(t,y,x) \subseteq comp(-intC)_3^{\varepsilon}\}$ *is closed.*

Passing to problems (QEP2) and (QEP4), which are close to each other, we investigate in details another solution type, namely ε-solutions of type 3 for problem (QEP4).

Theorem 2.7. *For problem* (QEP4) *assume that*

(i_4^3) *for each* $(t,x) \in B \times A, F(t,.,x)$ *is comp*$(-intC)_3^\varepsilon$*-quasiconvex of type 1 and* $F(t,x,x) \subseteq$ *comp*$(-intC)_3^\varepsilon$*;*

(ii_4^3) F *is usc; the* F*-image of a compact set is compact;* T *has nonempty convex values and* $T^{-1}(z)$ *is open in* A *for all* $z \in B$*; for each* $y \in A$*, the set* $\{(t,x) \in B \times A \mid F(t,y,x) \subseteq$*comp*$(-intC)_3^\varepsilon\}$ *is closed;*

(iii_4^3) *cl*$K(.)$ *is usc;* $\forall x \in A, A \cap K(x) \neq \emptyset$ *and* $K(x)$ *is nonempty convex and* $K(M)$ *is compact if* $M \subseteq A$ *is compact;* $\forall y \in A, K^{-1}(y)$ *is open in* A*.*

Then, there exist ε*-solutions of type 3 of problem* (QEP4).

Proof. For $t, x \in B \times A$ set

$$P(t,x) = \{z \in A \mid F(t,z,x) \not\subseteq \mathrm{comp}(-\mathrm{int}C)_3^\varepsilon\},$$

$$E = \{(t,x) \in B \times A \mid K(x) \cap P(t,x) \neq \emptyset\},$$

$$S(t,x) = \begin{cases} K(x) \cap P(t,x) & \text{if } (t,x) \in E, \\ A \cap K(x) & \text{if } (t,x) \in (B \times A) \backslash E, \end{cases}$$

$$Q(t,x) = (T(x), S(t,x)).$$

By (i_4^3), $P(t,x)$ is convex and so is $Q(t,x)$ for all $(t,x) \in B \times A$. We claim that E is closed. Indeed,

$$E = \{(t,x) \in B \times A \mid \exists y \in K(x), F(t,y,x) \not\subseteq \mathrm{comp}(-\mathrm{int}C)_3^\varepsilon\}.$$

Assume that $(t_\alpha, x_\alpha) \in E$ and $(t_\alpha, x_\alpha) \to (t^*, x^*)$. Then, there is $y_\alpha \in K(x_\alpha)$ such that we have a point

$$z_\alpha \in F(t_\alpha, y_\alpha, x_\alpha) \cap (Y \backslash \mathrm{comp}(-\mathrm{int}C)_3^\varepsilon).$$

Set $L = \{x_\alpha\} \cup \{x^*\}$. Then $T(L)$ is compact and we can assume that $y_\alpha \to y^*$. By the closedness of $K(.)$, $y^* \in K(x^*)$. Set $N = \{(t_\alpha, y_\alpha, x_\alpha)\} \cup \{(t^*, y^*, x^*)\}$. Then $F(N) \cap (Y \backslash \mathrm{comp}(-\mathrm{int}C)_3^\varepsilon)$ is compact, and hence we can assume that $z_\alpha \to z^*$. Since $F(.,.,.) \cap (Y \backslash \mathrm{comp}(-\mathrm{int}C)_3^\varepsilon)$ is a closed multifunction,

$$z^* \in F(t^*, y^*, x^*) \cap (Y \backslash \mathrm{comp}(-\mathrm{int}C)_3^\varepsilon),$$

which indicates that $(t^*, x^*) \in E$ and hence E is closed.

Now we have, for $y \in A$ and $z \in B$,

$$\begin{aligned}
S^{-1}(y) &= \{(t,x) \in E \mid x \in K^{-1}(y), (t,x) \in P^{-1}(y)\} \\
&\quad \cup \{(t,x) \in (B \times A) \backslash E \mid x \in K^{-1}(y)\} \\
&= [E \cap P^{-1}(y) \cap (B \times K^{-1}(y))] \cup [((B \times A) \backslash E) \cap (B \times K^{-1}(y))] \\
&= [((B \times A) \backslash E) \cup P^{-1}(y)] \cap [B \times K^{-1}(y)],
\end{aligned}$$

$$\begin{aligned}
Q^{-1}(z,y) &= \{(t,x) \in B \times A \mid x \in T^{-1}(z), (t,x) \in S^{-1}(y)\} \\
&= S^{-1}(y) \cap (B \times T^{-1}(z)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(B \times A) \backslash Q^{-1}(z,y) &= [(B \times A) \backslash S^{-1}(y)] \cup [(B \times A) \backslash (B \times T^{-1}(z))] \\
&= [(B \times A) \backslash S^{-1}(y)] \cup [B \times (A \backslash T^{-1}(z))] \\
&= [E \cap ((B \times A) \backslash P^{-1}(y))] \cup [B \times (A \backslash K^{-1}(y))] \\
&\quad \cup [B \times (A \backslash T^{-1}(z))]. \qquad (3)
\end{aligned}$$

We see also that

$$(B \times A)\backslash P^{-1}(y) = \{(t,x) \in B \times A \mid F(t,y,x) \subseteq \text{comp}(-\text{int}C)^{\varepsilon}_{3}\}$$

is closed, by (ii$^3_4$). This together with the openness in A of $K^{-1}(y)$ and $T^{-1}(z)$ and with (3) show that $Q^{-1}(z,y)$ is open in $B \times A, \forall (z,y) \in B \times A$. By virtue of Theorem 1.1, there exists a fixed point $(\bar{t},\bar{x})$, i.e. $(\bar{t},\bar{x}) \in Q(\bar{t},\bar{x})$. Suppose that $(\bar{t},\bar{x}) \in E$. Then $\bar{x} \in P(\bar{t},\bar{x})$ contradicting (i$^3_4$). Thus $(\bar{t},\bar{x}) \notin E$. Consequently, by the definitions of E, S and Q,

$$\bar{x} \in K(\bar{x}), \bar{t} \in K(\bar{x}), K(\bar{x}) \cap P(\bar{t},\bar{x}) = \emptyset.$$

Therefore, for all $y \in K(\bar{x})$, we have $y \notin P(\bar{t},\bar{x})$, i.e.

$$F(\bar{t},y,\bar{x}) \subseteq \text{comp}(-\text{int}C)^{\varepsilon}_{3},$$

which means that $\bar{x}$ is an ε-solution of type 3 of (QEP4). □

We first state without proofs similar results for the other two types of ε-solutions of problem (QEP4).

Theorem 2.8. *For problem* (QEP4) *assume that*

(i$^1_4$) *for each* $(t,x) \in B \times A, F(t,.,x)$ *is comp*$(-intC)^{\varepsilon}_{1}$*-quasiconvex of type 1 and* $F(t,x,x) \subseteq$ *comp*$(-intC)^{\varepsilon}_{1}$*;*

(ii$^1_4$) $\forall y \in A$, $F(.,y,.)$ *is lsc and the set* $\{(t,x) \in B \times A \mid \exists y \in K(x), F(t,y,x) \not\subseteq$ *comp*$(-intC)^{\varepsilon}_{1}\}$ *is closed;* T *has nonempty convex values and* $T^{-1}(z)$ *is open in* A *for all* $z \in B$*;*

(iii$^1_4$) $\forall x \in A, A \cap K(x) \neq \emptyset$ *and* $K(x)$ *is nonempty convex;* $\forall y \in A, K^{-1}(y)$ *is open in* A*.*

Then, there exist ε*-solutions of type 1 of problem* (QEP4).

Remark 2.2. For the special case $\varepsilon = 0$, i.e. we are concerned with the (exact) solution of problem (QEP4). [7,18] contain results different from Theorem 2.8. The following example gives a case where Theorem 2.8 is applicable but the mentioned results are not.

Example 2.2. Let $X = Y = Z = R$, $A = B = [0, 1]$, $C = R_+$, $T(x) = [0, x]$ and $F(t, y, x) \equiv [0.5, 1]$ and

$$K(x) = \begin{cases} 0 & \text{if x = 0 or x = 1,} \\ [0, x) & \text{if } 0 < x < 1. \end{cases}$$

Then all assumptions of Theorem 2.8 are satisfied for $\varepsilon = 0$ and hence solutions of problem (QEP4) exist. However, since $K(.)$ is not continuous, Theorem 3.1 of [18] and Theorem 1 of [7] cannot be employed.

Theorem 2.9. *For problem* (QEP4) *assume (*iii_4^1*) of Theorem 2.8 and replace (*i_4^1*) and (*ii_4^1*) by the following, respectively,*

(i$_4^2$) *for each* $(t, x) \in B \times A, F(., ., x)$ *is* $comp(-intC)_2^\varepsilon$*-quasiconvex of type 1 and* $F(t, x, x) \subseteq comp(-intC)_2^\varepsilon$*;*

(ii$_4^2$) T *has nonempty convex values and* $T^{-1}(z)$ *is open in* A *for all* $z \in B$*;* $\{(t, x) \in B \times A \mid \exists y \in K(x), F(t, y, x) \not\subseteq comp(-intC)_2^\varepsilon\}$ *and* $\{(t, x) \in B \times A \mid F(t, y, x) \subseteq comp(-intC)_2^\varepsilon\}$ *are closed,* $\forall y \in A$*.*

Then, there exist ε*-solutions of type 2 of problem* (QEP4).

Passing finally to problem (QEP2), we formulate the following three existence results, which can be proved by similar techniques as that for Theorem 2.7.

Theorem 2.10. *Consider problem* (QEP2). *Assume (*iii_4^1*) of Theorem 2.8 and assume further that*

(i_2^1) *for each* $(t,x) \in B \times A, F(t,.,x)$ *is* $comp(-intC)_1^{\varepsilon}$*-quasiconvex of type 2 and* $F(t,x,x) \cap$ $comp(-intC)_1^{\varepsilon} \neq \emptyset$*;*

(ii_2^1) T *has nonempty convex values and* $T^{-1}(z)$ *is open in* A *for all* $z \in B$*;* $\forall y \in A$*,* $\{(t,x) \in B \times A \mid F(t,y,x) \cap comp(-intC)_1^{\varepsilon} \neq \emptyset\}$ *and* $\{(t,x) \in B \times A \mid \exists y \in K(x), F(t,y,x) \cap comp(-intC)_1^{\varepsilon} = \emptyset\}$ *are closed.*

Then, problem (QEP2) *has* ε*-solutions of type 1.*

Theorem 2.11. *For problem* (QEP2) *assume (*iii_4^1*) of Theorem 2.8 and replace the other assumptions there by*

(i_2^2) *for each* $(t,x) \in B \times A, F(t,.,x)$ *is* $comp(-intC)_2^{\varepsilon}$*-quasiconvex of type 2 and* $F(t,x,x) \cap$ $comp(-intC)_2^{\varepsilon} \neq \emptyset$*;*

(ii_2^2) T *has nonempty convex values and* $T^{-1}(z)$ *is open in* A *for all* $z \in B$*;* $\forall y \in A$*,* $\{(t,x) \in B \times A \mid F(t,y,x) \cap comp(-intC)_2^{\varepsilon} \neq \emptyset\}$ *and* $\{(t,x) \in B \times A \mid \exists y \in K(x), F(t,y,x) \cap comp(-intC)_2^{\varepsilon} = \emptyset\}$ *are closed.*

Then, problem (QEP2) *has* ε*-solutions of type 2.*

Theorem 2.12. *For problem* (QEP2) *assume that* (i$_2^3$) *for each* $(t,x) \in B \times A, F(t,.,x)$ *is* $comp(-intC)_3^\varepsilon$*-quasiconvex of type 2 and* $F(t,x,x) \cap$ $comp(-intC)_3^\varepsilon \neq \emptyset$*;*

(ii$_2^3$) F *is lsc;* T *has nonempty convex values and* $T^{-1}(z)$ *is open in* A *for all* $z \in B$*;* $\forall y \in A$, $\{(t,x) \in B \times A \mid F(t,y,x) \cap comp(-intC)_3^\varepsilon \neq \emptyset\}$ *is closed;*

(iii$_2^3$) K *is usc;* $\forall x \in A, A \cap K(x) \neq \emptyset$ *and* $K(x)$ *is compact and convex;* $\forall y \in A, K^{-1}(y)$ *is open in* A.

Then, there exist ε*-solutions of type 3 of problem* (QEP2).

3. Applications

It is well-known that quasiequilibrium problems include as special cases many optimization-related problems (see Section 1). Therefore, our results in Section 2 have direct consequences for these special cases. Here we provide as examples only two such consequences.

3.1. Approximate quasivariational inequalities

Let X, Y, A, B and K be as in Section 1. Let C be a closed cone with nonempty interior. Let $Z = L(X,Y)$, the space of the bounded linear mappings from X into Y. Let $g : X \to X$ be a continuous mapping. The following quasivariational inequality has been considered by many authors, see e.g. [10,12]:

(QVI) Find $\bar{x} \in A \cap \mathrm{cl}K(\bar{x})$ such that, $\forall y \in K(\bar{x})$,

$$(T(\bar{x}), y - g(\bar{x})) \cap (Y \backslash -\mathrm{int}C) \neq \emptyset,$$

where (t, y) stands for the value of linear mapping $t \in L(X, Y)$ at $y \in X$.

Now we investigate the following approximate quasivariational inequality:

(QVI$^{\varepsilon}$) Find $\bar{x} \in A \cap \mathrm{cl}K(\bar{x})$ such that, $\forall y \in K(\bar{x})$,

$$(T(\bar{x}), y - g(\bar{x})) \cap \ \mathrm{comp}(-\mathrm{int}C)_1^{\varepsilon} \neq \emptyset.$$

Applying Theorem 2.1 with $F(t, y, x) = (t, y - g(x))$ one obtains the following new existence result.

Corollary 3.1. *For problem* (QVI$^{\varepsilon}$) *assume that* T *is usc, has nonempty compact values and comp*$(-intC)_1^{\varepsilon}$ *is convex. Then, problem* (QVI$^{\varepsilon}$) *has solutions.*

3.2. Approximate quasioptimization problems

Let X, Y, A, B, K and C be as in Subsection 3.1. Let $G : A \to B$ be a mapping. The following quasioptimization problem has been studied, e.g. in [9,19]:

(QOP) Find $\bar{x} \in K(\bar{x})$ such that, $\forall y \in K(\bar{x})$,

$$G(y) - G(\bar{x}) \in Y \backslash -\mathrm{int}C.$$

(Then $\bar{x}$ is called a weakly efficient solution.)

Let us consider the following approximate quasioptimization problem

(QOP$^{\varepsilon}$) Find $\bar{x} \in K(\bar{x})$ such that, $\forall y \in K(\bar{x})$,

$$G(y) - G(\bar{x}) \in \mathrm{comp}(-\mathrm{int}C)_3^{\varepsilon}.$$

Corollary 3.2. *For* (QOP$^{\varepsilon}$) *assume that*

(i) $\forall b \in B, G(.) - b$ *is comp*$(-intC)_3^{\varepsilon}$*-quasiconvex of type 1;*

(ii) *G is continuous;* $\forall y \in A, \{x \in A \mid G(y) - G(x) \in comp(-intC)^{\varepsilon}_{3}\}$ *is closed;*

(iii) *assume* (iii^3_4) *of Theorem 2.7.*

Then, problem (QOP$^{\varepsilon}$) *has solutions.*

Proof. The corollary is obtained from Theorem 2.7 with

$$T(x) \equiv B,$$
$$F(t, y, x) = G(y) - G(x). \qquad \square$$

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