On the Existence of Solutions to Quasivariational Inclusion Problems[1]

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Abstract. We establish sufficient existence conditions for general quasivariational inclusion problems, which contain most of variational inclusion problems and quasiequilibrium problems considered in the literature. These conditions are shown to extend recent existing results and sharpen even results for particular cases.

Key Words. Quasivariational inclusion problems, quasiequilibrium problems, implicit variational inequalities, the solution existence, fixed points.

1. Introduction

Ref. 1 introduced the equilibrium problem as a generalization of variational inequalities and optimization problems. This problem setting proved to be rather general, including also many other optimization-related problems such as fixed-point problems, coincidence-point problems, complementarity problems, the Nash equilibrium problems, minimax problems, traffic network problems. On the other hand, this setting proved to be suitable for applying analytic tools in consideration. For the last decade there have been a number of generalizations of the equilibrium-problem formulation. A turning point was the introduction of the quasiequilibrium problem, where the constraint set depended also on the state variable. The staring point for this kind of constraint sets was Ref. 2, where the authors investigated random impulse control problems. Further generalizations were the variational inclusion problem and quasivariational inclusion problem, see e.g. Refs. 3 - 10. It should be noted that the term "inclusion" appeared in several recent papers also in another meaning. In Refs. 11, 12 "variational inclusion" means a multivalued variational inequality. Variational inclusion problems studied in Refs. 13 - 15 are problems of finding the zeroes of maximal monotone mappings.

For the above-mentioned problems we can observe that the solution existence was always the first topic and attracted the attention of most mathematicians. Existence results for various types of equilibrium problems were the contributions of Refs. 16 - 31 among others. For the quasivariational inclusion problems, existence conditions were developed in Refs. 3 - 8.

The aim of the present paper is to establish new existence results for the quasivariational inclusion problems studied in Refs. 3, 32. This problem setting proved to include most of quasivariational inclusion problems and quasiequilibrium problems in the literature as particular cases. We try to get sufficient conditions for the solution existence so that when applied to particular cases they are stronger than some recent results. About the main tools for proving existence results in quasivariational inclusion problems and their special cases we observe the KKM-Fan theorem in the first place, see e.g. Refs. 3, 24, 25, 27, 33, 34. Several fixed-point theorems such as that of Kakutani, Tarafdar, Park, Kim - Tan are also important and convenient tools, see e.g Refs. 6, 8, 16, 26, 34 - 36. Maximal-element theorems are used in e.g. Refs. 7, 29, 37. Maximax theorems may also be applied , see e.g. Ref. 35 for applying Kneser's theorem (Ref. 38). Existence results for problems of other kinds may be applied also to get some corresponding results, e.g. in Ref. 39 an existence theorem in game theory is applied to prove existence conditions for quasivariational inequalities. Each tool has advantages in some appropriate situations. In this paper we make use of a fixed-point theorem in Ref. 40.

The layout of the paper is as follows. In the rest of this section we state our problems under consideration and supply some preliminaries. Section 2 is devoted to the main existence results. In Section 3 we discuss consequences of the main results in some particular cases as examples for others to explain their advantages and possibility of applications.

Our problem setting is as follows. In the sequel, if not stated otherwise, let X, Y and Z be real topological vector spaces; let X be Hausdorff and $A, B \subseteq X$ be nonempty closed convex subsets. Let $C : A \to 2^Y$, $S_1 : A \to 2^B$, $S_2 : A \to 2^B$and $T : A \times B \to 2^Z$ be multifunctions such that $C(x)$ is a closed convex cone with nonempty interior and $C(x) \neq Y$, for each $x \in A$. Let $F : T(A \times B) \times B \times A \to 2^Y$ and $G : T(A \times B) \times A \to 2^Y$ be multifunctions. We consider the following four kinds of quasivariational inclusion problems:

(IP1) Find $\bar{x} \in S_1(\bar{x})$, such that, $\forall y \in S_2(\bar{x})$, $\forall \bar{t} \in T(\bar{x}, y)$,

$$F(\bar{t}, y, \bar{x}) \subseteq G(\bar{t}, \bar{x});$$

(IP2) Find $\bar{x} \in S_1(\bar{x})$, such that, $\forall y \in S_2(\bar{x})$, $\exists \bar{t} \in T(\bar{x}, y)$,

$$F(\bar{t}, y, \bar{x}) \subseteq G(\bar{t}, \bar{x});$$

(IP3) Find $\bar{x} \in S_1(\bar{x})$, such that, $\forall y \in S_2(\bar{x})$, $\forall \bar{t} \in T(\bar{x}, y)$,

$$F(\bar{t}, y, \bar{x}) \cap G(\bar{t}, \bar{x}) \neq \emptyset;$$

(IP4) Find $\bar{x} \in S_1(\bar{x})$, such that, $\forall y \in S_2(\bar{x})$, $\exists \bar{t} \in T(\bar{x}, y)$,

$$F(\bar{t}, y, \bar{x}) \cap G(\bar{t}, \bar{x}) \neq \emptyset.$$

To ensure the generality of the problem setting we discuss several particular cases.

(a) Let $A = B$, $C(x) \equiv C$, $G(t,x) = F(t,x,x) + C$, where $C \subseteq Y$ is a closed convex cone. Then (IP1) collapses to the quasivariational inclusion problem studied in Ref. 4:

(IP) Find $\bar{x} \in S_1(\bar{x})$ such that, $\forall y \in S_2(\bar{x})$, $\forall \bar{t} \in T(\bar{x}, y)$,

$$F(\bar{t}, y, \bar{x}) \subseteq F(\bar{t}, \bar{x}, \bar{x}) + C.$$

If T has the special form $(x,y) \mapsto T(x,x)$, (IP) is a quasivariational inclusion problem of the Minty type (MP). While if T is of the form $(x,y) \mapsto T(y,y)$, (IP) is a quasivariational inclusion problem of the Stampacchia type. For instance, if $Y = R, C = R_+$ and $G(t,x) \equiv R_+$, then (IP) becomes the quasiequilibrium problem of the Minty type dealt with in Ref. 19.

(b) With the following special forms of the involved multifunction: $A = B$, $S_1(x) = S_2(x) := S(x)$ and $C(x) \equiv C$, $G(t,x) = F(t,x,x) + C$ and T is given by $(x,y) \mapsto T(x,x) := T(x)$, (IP1) becomes the upper variational inclusion problem investigated in Ref. 5:

(UIP) Find $\bar{x} \in S(\bar{x})$ such that, $\forall y \in S(\bar{x})$, $\forall \bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \subseteq F(\bar{t}, \bar{x}, \bar{x}) + C.$$

(c) Consider the following general quasiequilibrium problem studied by many authors:

(QEP) Find $\bar{x} \in S(\bar{x})$ such that, $\forall y \in S(\bar{x})$, $\forall \bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \subseteq C.$$

It is clear that (IP) and (UIP) do not include (QEP), without severe assumptions on F. However, our (IP1) does contain it (by choosing special forms of involved multifunctions as in (b), except G which now is $G(t, x) \equiv C$).

(d) If $A = B, S_1(x) = S_2(x) := S(x), T$ is of the form $(x, y) \mapsto T(x, x) := T(x)$ and $G(t, x) = C(x)$, then (IP1) and (IP4) coincide with the following generalized quasiequilibrium problems, respectively, considered in Ref. 28:

(EP1) Find $\bar{x} \in A$, such that, $\bar{x} \in S(\bar{x})$ and, $\forall y \in S(\bar{x})$, $\forall \bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \subseteq C(\bar{x});$$

(EP2) Find $\bar{x} \in A$, such that, $\bar{x} \in S(\bar{x})$ and, $\forall y \in S(\bar{x})$, $\forall \bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \cap C(\bar{x}) \neq \emptyset.$$

If we replace the above form of G by $G(t, x) = Y \backslash -\text{int}C(x)$, then (IP1) and (IP4) collapse to the other equilibrium problems in Ref. 28:

(EP3) Find $\bar{x} \in A$, such that, $\bar{x} \in S(\bar{x})$ and, $\forall y \in S(\bar{x})$, $\forall \bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \subseteq Y \backslash -\text{int}C(\bar{x});$$

(EP4) Find $\bar{x} \in A$, such that, $\bar{x} \in S(\bar{x})$ and, $\forall y \in S(\bar{x})$, $\forall \bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \cap Y \backslash -\text{int}C(\bar{x}) \neq \emptyset.$$

For the more special case, where $A = B \equiv S_1(x) = S_2(x) := S(x)$, we have the corresponding four problems (EP1)-(EP4) investigated in Ref. 18.

(e) If $A = B = S_1(x) = S_2(x)$, T has the form $(x, y) \mapsto T(x, x) := T(x)$, $Z = L(X, Y)$ (the space of the linear continuous mappings of X into Y equipped with either the topology of pointwise convergence or that of bounded convergence), F is single-valued and $G(t, x) = Y \backslash -\text{int}C(x)$, then (IP2) and (IP4) become the implicit vector variational inequality investigated in Refs. 35, 36:

(IVI) Find $\bar{x} \in A$ such that, $\forall y \in A$, $\exists \bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \notin -\text{int } C(\bar{x}).$$

(f) The following quasivariational inequality commonly interested in the literature, see e.g. Ref. 30:

(QVI) Find $\bar{x} \in S(\bar{x})$ such that, $\forall y \in S(\bar{x})$, $\exists \bar{t} \in T(\bar{x})$,

$$(\bar{t}, y - \bar{x}) \notin -\text{int}C(\bar{x}).$$

where (t, x) denotes the image of $t \in L(X, Y)$ at x, is clear a particular case of our problems (IP2) and (IP4).

We recall semicontinuity properties of a multifunction $I : X \to 2^Y$, where X and Y are topological spaces. I is said to be upper semicontinuous (usc) at $x_0 \in X$ if, for each open subset U containing $I(x_0)$, there is a neighborhood N of x_0 such that $I(N) \subseteq U$. $I : X \to 2^Y$ is called lower semicontinuous (lsc) at

$x_0 \in X$ if, for each open subset U with $I(x_0) \cap U \neq \emptyset$, there is a neighborhood N of x_0 such that, $\forall x \in N, I(x) \cap U \neq \emptyset$. $I : X \to 2^Y$ is termed closed at x_0 if, $\forall x_\alpha \to x_0, \forall y_\alpha \in I(x_\alpha)$: $y_\alpha \to y_0$, $y_0 \in I(x_0)$. If I closed at $\forall x \in A \subseteq X$ we say that I is closed in A. In particular, if $A = \mathrm{dom} I := \{x \in X : I(x) \neq \emptyset\}$ we say simply that I is closed. A similar saying is adopted for the other properties of I.

Our main tool for proving the existence conditions in this paper is the following fixed-point theorem.

Theorem 1.1 (Ref. 40). Let X be a Hausdorff topological vector space, $A \subseteq X$ be nonempty convex and $D \subseteq A$ be a nonempty compact subset. Let $S : A \to 2^A$ and $L : A \to 2^A$ be multifunctions. Assume that

(a) $\forall x \in A, L(x)$ is convex and $S(x) \subseteq L(x)$;

(b) $\forall x \in D, S(x) \neq \emptyset$;

(c) $\forall y \in A :\ S^{-1}(y)$ is open in A;

(d) for each finite subset N of A and, there is a compact, convex subset L_N such that, $N \subseteq L_N \subseteq A$ and, $\forall x \in L_N \backslash D$, $S(x) \cap L_N \neq \emptyset$.

Then L has fixed points.

Remark 1.1. The coercivity condition (d) in Theorem 1.1 can be replaced by the following coercivity assumption:

(d') There is a nonempty compact convex subset $K \subseteq A$ such that, $\forall x \in$

$A \backslash D, \exists y \in K, x \in S^{-1}(y)$.

Indeed, assume (d') and let $N \subseteq A$ be finite. Take $L_N = \operatorname{co}(K \cup N)$, then $\forall x \in L_N \backslash D \subseteq A \backslash D, \exists y \in K \subseteq L_N$ with $x \in S^{-1}(y)$. Hence $y \in S(x) \cap K \subseteq S(x) \cap L_N$, i.e. (d) is satisfied.

2. The main results

Theorem 2.1. For (IP1) assume the existence of a multifunction $H : T(A \times B) \times B \times A \to 2^Y$ satisfying the following conditions

(i) if, $\forall t \in T(x, y), H(t, y, x) \subseteq G(t, x)$, then, $\forall t \in T(x, y), F(t, y, x) \subseteq G(t, x)$;

(ii) $\forall x \in A$, the set $\{y \in A \mid \exists t \in T(x, y), H(t, y, x) \not\subseteq G(t, x)\}$ is convex and, $\forall t \in T(x, x), H(t, x, x) \subseteq G(t, x)$;

(iii) $\forall y \in A$, the set $\{x \in A \mid \forall t \in T(x, y), H(t, y, x) \subseteq G(t, x)\}$ is closed;

(iv) $S_1(.)$ is closed and, $\forall x, y \in A$, $\operatorname{co}(S_2(x)) \subseteq S_1(x)$, $S_2(x) \cap A \neq \emptyset$ and $S_2^{-1}(y)$ is open in A;

(v) there is a nonempty, compact subset $D \subseteq A$ such that, for each finite subset N of A, a compact convex subset L_N with $N \subseteq L_N \subseteq A$ exists satisfying, $\forall x \in L_N \backslash D$, $S_2(x) \cap L_N \neq \emptyset$ and, for $x \in S_1(x) \cap (L_N \backslash D)$, there is $y \in S_2(x) \cap L_N$ with $H(t, y, x) \not\subseteq G(t, x)$ for some $t \in T(x, y)$.

Then problem (IP1) has solutions.

Proof. For $x \in A$ and $i = 1, 2$, set

$$E = \{x \in A \mid x \in \ S_1(x)\},$$

$$P_1(x) = \{y \in A \mid \exists t \in T(x,y), F(t,y,x) \not\subseteq G(t,x)\},$$

$$P_2(x) = \{y \in A \mid \exists t \in T(x,y), H(t,y,x) \not\subseteq G(t,x)\},$$

$$\Phi_i(x) = \begin{cases} S_2(x) \cap P_i(x), & \text{if } x \in E, \\ A \cap S_2(x), & \text{if } x \in A \backslash E, \end{cases}$$

$$Q(x) = \begin{cases} (\text{co}S_2(x)) \cap P_2(x), & \text{if } x \in E, \\ A \cap \text{co}S_2(x), & \text{if } x \in A \backslash E. \end{cases}$$

We will apply Theorem 1.1 with $L = Q$ and $S = \Phi_2$, showing that Q has no fixed point and assumptions (a), (c) and (d) of this theorem are satisfied and hence assumption (b) must be violated. For (a) we see from (i) of Theorem 2.1 that, $\forall x \in A, P_1(x) \subseteq P_2(x)$, whence $\Phi_1(x) \subseteq \Phi_2(x) \subseteq Q(x)$ by the definition of Q. Moreover, $Q(x)$ is convex by (ii).

For (c) we have, $\forall y \in A$,

$$\begin{aligned} \Phi_2^{-1}(y) &= [E \cap S_2^{-1}(y) \cap P_2^{-1}(y)] \cup [(A \backslash E) \cap S_2^{-1}(y)] \\ &= [(A \backslash E) \cup P_2^{-1}(y)] \cap S_2^{-1}(y). \end{aligned}$$

Consequently,

$$A \backslash \Phi_2^{-1}(y) = [E \cap (A \backslash P_2^{-1}(y))] \cup (A \backslash S_2^{-1}(y)). \qquad (1)$$

It suffices to verify that this set is closed. By the closedness of $S_1(.)$ assumed in

(iv) it is not hard to see that E is closed. $A \backslash S_2^{-1}(y)$ is closed also by (iv). The rest term in (1) is

$$A \backslash P_2^{-1}(y) = \{x \in A \mid \forall t \in T(x, y), H(t, y, x) \subseteq G(t, x)\},$$

which is closed by (iii). Thus, $A \backslash \Phi_2^{-1}(y)$ is closed.

To see (d) we have D and L_N for each N by assumption (v). Let $x \in L_N \backslash D$ be arbitrary. If $x \in A \backslash E$ then $\Phi_2(x) \cap L_N = A \cap S_2(x) \cap L_N = S_2(x) \cap L_N \neq \emptyset$ by (v). If $x \in E$, then $x \in S_1(x) \cap (L_N \backslash D)$ and, by (v), there is $y \in S_2(x) \cap L_N$ such that $y \in P_2(x)$. Hence $y \in \Phi_2(x)$ and $\Phi_2(x) \cap L_N \neq \emptyset$.

Finally, suppose that Q has a fixed point $x_0 \in A$. If $x_0 \in E$, then $x_0 \in P_2(x_0)$, i.e. $\exists t \in T(x_0, x_0), H(t, x_0, x_0) \nsubseteq G(t, x_0)$, contradicting (ii). If $x_0 \in A \backslash E$, then $x_0 \in \mathrm{co}(S_2(x_0)) \subseteq S_1(x_0)$, i.e. $x_0 \in E$, a contradiction.

The above argument implies that (b) of Theorem 1.1 must be violated, i.e. there is $x_0 \in D \subseteq A$ such that $\Phi_2(x_0) = \emptyset$ and hence $\Phi_1(x_0) = \emptyset$. If $x_0 \in A \backslash E$ then $A \cap S_2(x_0) = \Phi_1(x_0) = \emptyset$, contradicting (iv). So $x_0 \in E$ and $\emptyset = \Phi_1(x_0) = S_2(x_0) \cap P_1(x_0)$. Consequently, $\forall y \in S_2(x_0), y \notin P_1(x_0)$, i.e., $\forall t \in T(x_0, y), F(t, y, x_0) \subseteq G(t, x_0)$, which means that x_0 is a solution of (IP1). $\square$

Remark 2.1.

(i) (v) is a coercivity condition. If A is compact, (v) is satisfied with $D = A$. So we can omit (v). Moreover, due to Remark 1.1, assumption (v) can be replaced by

(v') there are nonempty compact convex subset $K \subseteq A$ and nonempty compact subset $D \subseteq A$ such that, $\forall x \in A \backslash D$, $S_2(x) \cap K \neq \emptyset$ and, if $x \in S_1(x) \cap (A \backslash D)$, $\exists y \in S_2(x) \cap K$, $\exists t \in T(x, y)$, $H(t, y, x) \not\subseteq G(t, x)$.

(ii) If, $\forall y \in A, T(., y)$ and $H(., y, .)$ are lsc and G is closed, then the following set is closed:

$$M_1 = \{x \in A \mid \forall t \in T(x, y), H(t, y, x) \subseteq G(t, x)\}.$$

Indeed, let $x_\alpha \in M_1$ and $x_\alpha \to x_0$. By the assumed lower semicontinuity, $\forall t_0 \in T(x_0, y), \forall w_0 \in H(t_0, y, x_0), \exists t_\alpha \in T(x_\alpha, y) : t_\alpha \to t_0, \exists w_\alpha \in H(t_\alpha, y, x_\alpha) \subseteq G(t_\alpha, x_\alpha) : w_\alpha \to w_0$. Since G is closed, $w_0 \in G(t_0, x_0)$. Hence, $\forall\, t_0 \in T(x_0, y)$, $H(t_0, y, x_0) \subseteq G(t_0, x_0)$, i.e. $x_0 \in M_1$.

(iii) If we replace, in assumptions (ii), (iii) and (v) of Theorem 2.1, multifunction H by F, then we can omit assumption (i) to get a consequence, called Theorem 2.1_F for our convenience in the later use. This Theorem 2.1_F is different from Theorem 3.1 of Ref. 3 for the same problem (IP1) and may be more applicable in some cases as shown by the following example.

Example 2.1. Let $X = Y = Z = R$, $A = B = (-\infty, 1]$, $S_1(x) = S_2(x) \equiv (-\infty, 1]$, $T(x, y) = \{x\}, G(t, x) \equiv R_+$ and

$$F(t, y, x) = \begin{cases} y^2, & \text{if } y < 0, \\ xy, & \text{if } 0 \leq y \leq 1. \end{cases}$$

Since A is not compact, Theorem 3.1 of Ref. 3 cannot be applied. For assumptions of Theorem 2.1_F, only the coercivity condition is not clear and needs to be checked. Take $D = [0, 1]$. For any finite subset $N \subseteq A$, choose $L_N = \{x \in A \mid 1 \geq x \geq \min N\}$. Then for each $x \in L_N \backslash D, S_2(x) \cap L_N = L_N \neq \emptyset$ and, for $y = 1 \in S_2(x) \cap L_N, F(t, y, x) = xy \nsubseteq G(t, x) = R_+$, as $x < 0$. Now that all assumptions of Theorem 2.1_F are satisfied, (IP1) has solutions. (Direct computations give the solution set being [0,1].)

Moreover, with assumption (v') replacing (as mentioned in Remark 2.1 (i)) the coercivity condition of Theorem 2.1_F, by Remark 2.1 (ii) we see that Theorem 2.1_F sharpens Theorem 4.2 of Ref. 28, since the semicontinuity assumptions of Theorem 4.2 are stronger than the corresponding assumptions of Theorem 2.1_F. The following example yields a special case of quasiequilibrium problems where Theorem 2.1_F can be applied but Theorem 4.2 of Ref. 28 cannot.

Example 2.2. Let $X = Y = Z = R$, $A = B = [0, 1]$, $S_1(x) = S_2(x) \equiv [0, 1]$, $G(t, x) \equiv R_+$,

$$T(x, y) = \begin{cases} [0, 0.5], & \text{if } 0 \leq x < 0.5, \\ [0.5, 1], & \text{if } 0.5 \leq x \leq 1, \end{cases}$$

$$F(t, y, x) = \begin{cases} [0, 0.5], & \text{if } 0 \leq y < 0.5, \\ [1, 1.5], & \text{if } 0.5 \leq y \leq 1. \end{cases}$$

It clear that, for any $y \in [0, 1], T(., y)$ and $F(., y, .)$ is not lsc in [0,1]. Consequently, Theorem 4.2 of Ref. 28 cannot be employed. On the other hand, it is equally evident that all assumptions of Theorem 2.1_F are fulfilled. By direct calculations we see that the solution set of (IP1) is [0,1].

By a similar technique of proof we obtain the following existence conditions for our problems (IP2) - (IP4).

Theorem 2.2. For (IP2) assume that $H : T(A \times B) \times B \times A \to 2^Y$ exists satisfying (iv) of Theorem 2.1 and the following conditions

(i) if, $\exists t \in T(x, y), H(t, y, x) \subseteq G(t, x)$, then, $\exists t \in T(x, y), F(t, y, x) \subseteq G(t, x)$;

(ii) $\forall x \in A$, the set $\{y \in A \mid \forall t \in T(x, y), H(t, y, x) \not\subseteq G(t, x)\}$ is convex and, $\exists t \in T(x, x), H(t, x, x) \subseteq G(t, x)$;

(iii) $\forall y \in A$, the set $\{x \in A \mid \exists t \in T(x, y), H(t, y, x) \subseteq G(t, x)\}$ is closed;

(v) it is (v) of Theorem 2.1 with "for some t" replaced by "for all t".

Then problem (IP2) is solvable.

Proof. Use similar arguments as that for Theorem 2.1 with the following new multifunctions P_1 and P_2:

$$P_1(x) = \{y \in A \mid \forall t \in T(x, y), F(t, y, x) \not\subseteq G(t, x)\},$$

$$P_2(x) = \{y \in A \mid \forall t \in T(x, y), H(t, y, x) \not\subseteq G(t, x)\}. \qquad \square$$

Remark 2.2.

(i) Assumption (v) of Theorem 2.2 can be replaced by

(v') it is (v') in Remark 2.1 (i) with "$\exists t \in T(x, y)$" replaced by "$\forall t \in T(x, y)$".

(ii) If, $\forall y \in A, T(., y)$ is usc and has compact images; $H(., y, .)$ is lsc and G is closed, then the following set is closed:

$$M_2 = \{x \in A \mid \exists t \in T(x, y), H(t, y, x) \subseteq G(t, x)\}.$$

(iii) We have a consequence of Theorem 2.2 called Theorem 2.2_F when replacing H by F in assumptions (ii), (iii), (v) and omitting assumption (i) of Theorem 2.2. For the data given in Example 2.1 we consider problem (IP2). Then similarly as reasoning in Example 2.1, Theorem 3.2 of Ref. 3 cannot be employed but Theorem 2.2_F can.

Theorem 2.3. For problem (IP3) assume the existence of $H : T(A \times B) \times B \times A \to 2^Y$ satisfying (iv) of Theorem 2.1 and the following conditions

(i) if, $\forall t \in T(x, y), H(t, y, x) \cap G(t, x) \neq \emptyset$, then, $\forall t \in T(x, y), F(t, y, x) \cap G(t, x) \neq \emptyset$;

(ii) $\forall x \in A$, the set $\{y \in A \mid \exists t \in T(x, y), H(t, y, x) \cap G(t, x) = \emptyset\}$ is convex and, $\forall t \in T(x, x), H(t, x, x) \cap G(t, x) \neq \emptyset$;

(iii) $\forall y \in A$, the set $\{x \in A \mid \forall t \in T(x, y), H(t, y, x) \cap G(t, x) \neq \emptyset\}$ is closed;

(v) it is (v) of Theorem 2.1 with "$H(t, y, x) \not\subseteq G(t, x)$" replaced by "$H(t, y, x) \cap G(t, x) = \emptyset$".

Then problem (IP3) has a solution.

Proof. The arguments are similar to that for Theorem 2.1 with the following new P_1 and P_2:

$$P_1(x) = \{y \in A \mid \exists t \in T(x,y), F(t,y,x) \cap G(t,x) = \emptyset\},$$

$$P_2(x) = \{y \in A \mid \exists t \in T(x,y), H(t,y,x) \cap G(t,x) = \emptyset\}. \qquad \square$$

Remark 2.3.

(i) Assumption (v) of Theorem 2.3 can be replaced by

(v') It is (v') in Remark 2.1 (i) with "$H(t,y,x) \not\subseteq G(t,x)$" replaced by "$H(t,y,x) \cap G(t,x) = \emptyset$".

(ii) If, $\forall y \in A, T(.,y)$ is lsc; $H(.,y,.)$ is usc and has compact images and G is closed, then the following set:

$$M_3 = \{x \in A \mid \forall t \in T(x,y), H(t,y,x) \cap G(t,x) \neq \emptyset\}.$$

is closed.

(iii) We have a consequence of Theorem 2.3 called Theorem 2.3_F while replacing H by F in assumptions (ii), (iii), (v) and removing assumption (i) of Theorem 2.3. Similarly as in Remark 2.2, Example 2.1 gives also a case where Theorem 2.3_F can be applied but Theorem 3.3 of Ref. 3 cannot.

Theorem 2.4. For (IP4) assume that $H : T(A \times B) \times B \times A \to 2^Y$ exists satisfying (iv) of Theorem 2.1. Assume further assumptions (i), (ii) and (v) of

Theorem 2.3 with the following changes:

(i) both "$\forall t$" are replaced by "$\exists t$";

(ii) "$\exists t$" and "$\forall t$" change the places for each other;

(iii) "$\forall t$" is replaced by "$\exists t$";

(v) "for some t" is replaced by "for all t".

Then problem (IP4) has solutions.

Proof. Use arguments similar to that for Theorem 2.1 with the following new multifunctions P_1 and P_2:

$$P_1(x) = \{y \in A \mid \forall t \in T(x, y), F(t, y, x) \cap G(t, x) = \emptyset\},$$

$$P_2(x) = \{y \in A \mid \forall t \in T(x, y), H(t, y, x) \cap G(t, x) = \emptyset\}. \qquad \square$$

Remark 2.4.

(i) Assumption (v') replacing (v) is (v') in Remark 2.3 with "$\exists t$" replaced by "$\forall t$";

(ii) If, $\forall y \in A, T(., y)$ and $H(., y, .)$ are usc and have compact images and G is closed, then the following set is closed:

$$M_4 = \{x \in A \mid \exists t \in T(x, y), H(t, y, x) \cap G(t, x) \neq \emptyset\}.$$

(iii) The corresponding consequence of Theorem 2.4 called Theorem 2.4_F is obtained from Theorem 2.4 when replacing H by F in assumptions (ii), (iii), (v) and omitting assumption (i). Example 2.1 yields also a case, where Theorem 2.4_F

can be used to prove the existence of solutions of (IP4) but Theorem 3.4 of Ref. 3 cannot.

3. Particular cases

The main results in Section 2 imply clearly existence conditions for various particular cases mentioned in Section 1. Here we derive only several consequences and compare them with recent papers to see advantages of our existence conditions.

We first discuss equilibrium problems (EP1) and (EP2) encountered in Section 1 (d) and studied in Ref. 28.

Corollary 3.1. For problem (EP1) assume the existence of $H : T(A) \times A \times A \to 2^Y$ satisfying the following conditions

(i) if, $\forall t \in T(x), H(t, y, x) \subseteq C(x)$, then, $\forall t \in T(x), F(t, y, x) \subseteq C(x)$;

(ii) $\forall x \in A$, the set $\{y \in A \mid \exists t \in T(x), H(t, y, x) \nsubseteq C(x)\}$ is convex and, $\forall t \in T(x), H(t, x, x) \subseteq C(x)$;

(iii) $\forall y \in A$, the set $\{x \in A \mid \forall t \in T(x), H(t, y, x) \subseteq C(x)\}$ is closed;

(iv) $S(.)$ is closed and, $\forall y \in A$, $S^{-1}(y)$ is open in A;

(v) there is a nonempty compact subset $D \subseteq A$ such that, for each finite subset N of A, a compact convex subset L_N of A exists containing N and satisfying, $\forall x \in L_N \backslash D$, $\exists y \in L_N$ with $H(t, y, x) \nsubseteq C(x)$ for some

$t \in T(x)$.

Then problem (EP1) is solvable.

Proof. We simply apply Theorem 2.1 with $A = B, S_1(x) = S_2(x), T(x, y) = T(x)$ and $G(t, x) = C(x)$. $\square$

Corollary 3.2. For problem (EP2) assume assumption (iv) of Corollary 3.1 and that $H : T(A) \times A \times A \to 2^Y$ exists satisfying

(i) if, $\exists t \in T(x), H(t, y, x) \cap C(x) \neq \emptyset$, then, $\exists t \in T(x), F(t, y, x) \cap C(x) \neq \emptyset$;

(ii) $\forall x \in A$, the set $\{y \in A \mid \forall t \in T(x), H(t, y, x) \cap C(x) \neq \emptyset\}$ is convex and, $\exists t \in T(x), H(t, x, x) \subseteq C(x)$;

(iii) $\forall y \in A$, the set $\{x \in A \mid \exists t \in T(x), H(t, y, x) \cap C(x) \neq \emptyset\}$ is closed;

(v) it is (v) of Corollary 3.1 with "$H(t, y, x) \not\subseteq C(x)$ for some $t \in T(x)$" replaced by "$H(t, y, x) \cap C(x) = \emptyset$ for all $t \in T(x)$".

Then problem (EP2) has solutions.

Remark 3.1. Corollaries 3.1 and 3.2 sharpen Theorems 4.11, 4.12, respectively, of Ref. 18. Corollary 3.1 improves Theorem 4.2 of Ref. 28. The convexity and semicontinuity assumptions in these theorems are stricter than the corresponding assumptions in our corollaries. That is why these theorems are not applicable in the following example while our corollaries are.

Example 3.1. Let $X = Y = Z = R$, $A = [0, 1]$, $C(x) \equiv R_+, T(x) = \{x\}$ and $F(t, y, x) = \{1 - (y - 0.5)^2\}$. Then F is not $C(x)-$quasiconvex for $x \in A$ as assumed in Ref. 28. (Recall that, see Ref. 31, a multifunction $y \mapsto Q(y, x)$ is called $C(x)-$quasiconvex if, $\forall y_1, y_2 \in A, \forall\lambda \in [0, 1]$,

$$Q(y_1, x) \subseteq Q((1 - \lambda)y_1 + \lambda y_2, x) + C(x)$$

or

$$Q(y_2, x) \subseteq Q((1 - \lambda)y_1 + \lambda y_2, x) + C(x).)$$

Indeed, choose $y_1 = 0, y_2 = 1$ and $\lambda = 0.5$. Then

$$F(t, y_1, x) = \{0.75\} \not\subseteq F(t, 0.5y_1 + 0.5y_2, x) + C(x) \equiv [1, +\infty],$$

$$F(t, y_2, x) = \{0.75\} \not\subseteq [1, +\infty],$$

i.e. F is not $C(x)-$quasiconvex. Hence Theorem 4.2 of Ref. 28 is not applicable. Now all the assumptions of Corollary 3.1 are easily seen to be satisfied with $H(t, y, x) = F(t, y, x)$. (For assumption (ii) note that $\{y \in A \mid \exists t \in T(x), H(t, y, x) \not\subseteq C(x)\}$ is empty and then convex.) So by Corollary 3.1, in this case (EP1) has solutions. Direct computations show that the solution set is [0,1].

Pass now to the equilibrium problem of the Minty type (MP) mentioned in Section 1 (a) and investigated in Ref. 19.

Corollary 3.3. For problem (MP) assume assumption (iv) of Theorem 2.1

and that $H : T(A \times A) \times A \times A \to 2^R$ exists satisfying

(i) if, $\forall s \in T(x,y), H(s,y,x) \subseteq R_+$, then, $\forall t \in T(x,y), F(t,y,x) \subseteq R_+$;

(ii) $\forall x \in A$, the set $\{y \in A \mid \exists t \in T(x,y), H(t,y,x) \not\subseteq R_+\}$ is convex and $H(t,x,x) \subseteq R_+$ for all $t \in T(x,x)$;

(iii) $\forall y \in A$, the set $\{x \in A \mid \forall t \in T(x,y), H(t,y,x) \subseteq R_+\}$ is closed;

(v) there is a nonempty compact subset $D \subseteq A$ such that, for each finite subset N of A, a compact convex subset L_N with $N \subseteq L_N \subseteq A$ and, $\forall x \in L_N \backslash D$, $S_2(x) \cap L_N \neq \emptyset$ and furthermore, for $x \in S_1(x) \cap (L_N \backslash D)$, $\exists y \in S_2(x) \cap L_N$ so that $H(t,y,x) \not\subseteq R_+$ for some $t \in T(x,y)$.

Then (MP) has a solution.

Proof. Employ Theorem 2.1 with $G(t,x) = R_+$. □

Remark 3.2. When applied to the case where $H(t,y,x) = \{\sup F(T(x,x),y,x)\}$, Corollary 3.3 is stronger than Theorem 4.1 of Ref. 19, since its assumptions are more relaxed. The example below gives a case where this Theorem 4.1 cannot be employed but our Corollary 3.3 can be easily.

Example 3.2. Let $X = R$, $A = (-\infty, 3]$, $S_1(x) = S_2(x) \equiv A$, $T(x,y) \equiv R$ and

$$F(t,y,x) = \begin{cases} [x-y+1, 6], & \text{if } y \geq 0, \\ 0, & \text{if } y < 0. \end{cases}$$

Then, we have

$$\inf F(T(x,y),y,x) = \min\{0, x-y+1\},$$

$$\sup F(T(x,x),y,x) = \begin{cases} 6, & \text{if } y \geq 0, \\ 0, & \text{if } y < 0. \end{cases}$$

Then, $\inf F(T(x,y),y,x) < 0$ does not imply $\sup F(T(x,x),y,x) < 0$. Moreover, $0 \notin F(t,1,1)$. Hence, the assumptions of Theorem 4.1 of Ref. 19 are not satisfied. As opposed to this, the assumptions of Corollary 3.3 are easy to be checked with $D = [0,3]$ and

$$H(t,y,x) = \begin{cases} [x-y, 6], & \text{if } y \geq 0, \\ 0, & \text{if } y < 0. \end{cases}$$

Passing to a particular case where F is single-valued, we apply a result in Section 2 to the implicit variational inequality (IVI) stated in Section 1 (e) and studied in Refs. 35, 36.

Corollary 3.4. For problem (IVI) assume that the dual topological spaces X^* and Y^* of X and Y, respectively, separate points and that $H : L(X,Y) \times A \times A \to 2^Y$ exists such that

(i) if, $\exists t \in T(x), H(t,y,x) \subseteq Y \setminus -\mathrm{int}C(x)$, then, $\exists t \in T(x), F(t,y,x) \in Y \setminus -\mathrm{int}C(x)$;

(ii) $\forall x \in A$, the set $\{y \in A \mid \forall t \in T(x), H(t,y,x) \not\subseteq Y \setminus -\mathrm{int}C(x)\}$ is convex and, $\exists t \in T(x), H(t,x,x) \subseteq Y \setminus -\mathrm{int}C(x)$;

(iii) $\forall y \in A$, the set $\{x \in A \mid \exists t \in T(x), H(t,y,x) \subseteq Y\backslash -\mathrm{int}C(x)\}$ is closed;

(v) there is a nonempty compact subset $D \subseteq A$ such that, for each finite subset $N \subseteq A$, there is compact convex subset L_N with $N \subseteq L_N \subseteq A$ and, $\forall x \in L_N\backslash D$, $\exists y \in L_N, \forall t \in T(x), H(t,y,x) \not\subseteq Y\backslash -\mathrm{int}C(x)$.

Then problem (IVI) is solvable.

Proof. One simply employs Theorem 2.2 with $A = B, S_1(x) = S_2(x) = A$, $T(x,y) = T(x)$ and $G(t,x) = Y\backslash -\mathrm{int}C(x)$. $\square$

Remark 3.3. Theorem 3.2 of Ref. 35 and Theorem 3.1 of Ref. 36 are weaker than Corollary 3.4, since the convexity and semicontinuity assumptions there are stricter than our corresponding assumptions as explained now. Recall first some notions used in Refs. 35, 36. Let A, T, C and F be as in the formulation of (IVI). T is said to be generalized upper hemicontinuous (guhc) with respect to (wrt) F if, $\forall x, y \in A, \forall \alpha \in [0,1]$, the multifunction $\alpha \mapsto F(T((1-\alpha)x + \alpha y), x, y)$ is usc at 0^+. For $t \in L(X,Y)$ and $x \in A, F(t,.,x)$ is called $C(x)-$convex if, $\forall y, z \in A, \forall \lambda \in [0,1]$,

$$F(t,(1-\lambda)y + \lambda z, x) \in (1-\lambda)F(t,y,x) + \lambda F(t,z,x) - C(x).$$

T is termed generalized $C-$pseudomonotone wrt F if, $\forall x, y \in A$,

$$[\exists t \in T(x), F(t,y,x) \not\in -\mathrm{int}C(x)] \Longrightarrow [\forall t \in T(y), -F(t,x,y) \not\in -\mathrm{int}C(x)].$$

Proposition 3.1. Let A, T, C and F be as in the formulation of (IVI). Let

$H(t,y,x) = \{-F(s,y_\alpha,x) \mid s \in T(y_\alpha), \alpha \in [0,1]\}$, where $y_\alpha = (1-\alpha)x + \alpha y$. (Then H does not depend on t.)

(a) Assume that

(i) T is guhc with respect to F;

(ii) for each $t \in L(X,Y)$ and $x \in A, F(t,.,x)$ is $C(x)-$convex;

(iii) $\forall x, y \in A, \forall t \in T(x), F(t,y,y) \in C(x)$;

(iv) $\forall t \in L(X,Y), \forall x, y \in A, \forall \lambda \in [0,1]$,

$$F(t,y,(1-\lambda)x + \lambda y) = (1-\lambda)F(t,y,x).$$

Then assumption (i) of Corollary 3.4 is satisfied.

(b) In addition to the assumptions in (a), assume that T is generalized $C-$pseudomonotone wrt F. Then, assumption (ii) of Corollary 3.4 is fulfilled.

(c) If $Y \setminus -\text{int}C(.)$ is closed and, $\forall t \in L(X,Y), \forall y \in A, F(t,y,.)$ is continuous then assumption (iii) of Corollary 3.4 is satisfied.

Proof. Suppose to the contrary that, $\exists t \in T(x)$,

$$H(t,y,x) \subseteq Y \setminus -\text{int}C(x) \tag{2}$$

but, $\forall t \in T(x), F(t,y,x) \in -\text{int}C(x)$. By (2) and the definition of H one has, $\forall \lambda \in [0,1], \forall s \in T(y_\alpha)$,

$$-F(s,y_\alpha,x) \notin -\text{int}C(x). \tag{3}$$

Define $I : [0,1] \to 2^Y$ by

$$I(\alpha) = \{F(t,y,x) \mid t \in T(y_\alpha)\}.$$

Due to (2), $I(0) \subseteq -\text{int}C(x)$. Assumption (i) implies the existence of $\alpha_0 \in (0, 1]$ such that, $\forall \alpha \in [0, \alpha_0), I(\alpha) \subseteq -\text{int}C(x)$. Hence, $\forall \alpha \in (0, \alpha_0), \forall s \in T(y_\alpha)$,

$$F(s, y, x) \in -\text{int}C(x). \tag{4}$$

For any fixed $\alpha \in (0, \alpha_0)$, from (ii) one has, $\forall s \in T(y_\alpha)$,

$$F(s, y_\alpha, y_\alpha) \in (1 - \alpha)F(s, x, y_\alpha) + \alpha F(s, y, y_\alpha) - C(x). \tag{5}$$

Assumptions (iii), (iv) together with (4), (5) imply, $\forall s \in T(y_\alpha)$,

$$\begin{aligned} -(1-\alpha)F(s, x, y_\alpha) &\in \alpha F(s, y, y_\alpha) - F(s, y_\alpha, y_\alpha) - C(x) \\ &\subseteq \alpha(1-\alpha)F(s, y, x) - C(x) - C(x) \\ &\subseteq -\text{int}C(x), \end{aligned}$$

which contradicts (3).

(b) First we prove that, $\forall x \in A$, the set

$$\begin{aligned} M(x) &:= \{y \in A \mid \forall t \in T(x), H(t, y, x) \not\subseteq Y \setminus -\text{int}C(x)\} \\ &= \{y \in A \mid \exists \alpha \in [0, 1], \exists s \in T(y_\alpha), -F(t, y_\alpha, x) \in -\text{int}C(x)\} \end{aligned}$$

is convex. For arbitrarily fixed $y, z \in M(x)$ and $\lambda \in [0, 1]$, we have to show that $y^* = (1 - \lambda)z + \lambda y \in M(x)$. By the definition of $M(x), \exists \alpha_1, \alpha_2 \in [0, 1], \exists s_1 \in T(y_{\alpha_1}), \exists s_2 \in T(z_{\alpha_2})$,

$$-F(s_1, y_{\alpha_1}, x) \in -\text{int}C(x),$$

$$-F(s_2, z_{\alpha_2}, x) \in -\text{int}C(x).$$

Due to the assumed $C-$pseudomonotonicity of T, one has, $\forall t \in T(x)$,

$$F(t, x, y_{\alpha_1}) \in -\text{int}C(x),$$

$$F(t,x,z_{\alpha_2}) \in -\mathrm{int}C(x).$$

This and assumption (ii) together imply that, $\forall \gamma \in [0,1]$,

$$\begin{aligned} F(t,x,(1-\gamma)z_{\alpha_2}+\gamma y_{\alpha_1}) &\in (1-\gamma)F(t,x,z_{\alpha_2})+\gamma F(t,x,y_{\alpha_1})-C(x) \\ &\subseteq -\mathrm{int}C(x). \end{aligned} \tag{6}$$

Without loss of generality assume that $\alpha_1 \geq \alpha_2$. Setting

$$\gamma_0 = \tfrac{\lambda\alpha_2}{\alpha_1+\lambda(\alpha_2-\alpha_1)},$$

$$\alpha_0 = \tfrac{\alpha_1(1-\alpha_2)+\lambda(\alpha_2-\alpha_1)}{\alpha_1+\lambda(\alpha_2-\alpha_1)},$$

we see that $\gamma_0, \alpha_0 \in [0,1]$. Set $y_0 = (1-\alpha_0)y^* + \alpha_0 x$ and substitute γ_0 into (6) we obtain, $\forall t \in T(x)$,

$$F(t,x,y_0) \in -\mathrm{int}C(x).$$

By a similar argument as that of part (a), using assumptions (i)-(iv) we see the existence of $\hat{\alpha} \in [0,1]$ such that, $\forall s \in T(y_{0\hat{\alpha}})$,

$$-F(s,y_{0\hat{\alpha}},x) \in -\mathrm{int}C(x). \tag{7}$$

Choosing $\bar{\alpha} = \hat{\alpha}(1-\alpha_0) \in [0,1]$ one gets from (7) that, $\forall s \in T(y^*_{\bar{\alpha}})$,

$$-F(s,y^*_{\bar{\alpha}},x) \in -\mathrm{int}C(x).$$

(By our convention, $y^*_{\bar{\alpha}} = (1-\bar{\alpha})x + \bar{\alpha}y^*$.) This means that $y^* \in M(x)$.

Next we have to check that,

$$H(t,x,x) \subseteq Y \setminus -\mathrm{int}C(x).$$

This is derived from the fact that, $\forall s \in T(x)$,

$$-F(t,x,x) \notin -\mathrm{int}C(x),$$

which in turn follows from assumption (iii) and the $C-$pseudomonotonicity of T.

(c) Consider arbitrarily fixed $y \in A$, and $x_\lambda \to x_0$, where x_λ is in the set

$$N(y) := \{x \in A \mid \forall t \in T(y), -F(t, y, x) \notin -\text{int}C(x)\}.$$

Then, $\forall \lambda, \forall t \in T(y)$,

$$-F(t, y, x_\lambda) \in Y \setminus -\text{int}C(x_\lambda).$$

Since $F(t, y, .)$ is continuous and $Y \setminus -\text{int}C(.)$ is closed, we have, $\forall t \in T(y)$,

$$-F(t, y, x_0) \in Y \setminus -\text{int}C(x_0),$$

i.e. $x_0 \in N(y)$ and hence $N(y)$ is closed. Now we consider the set in assumption (iii)

$$\begin{aligned}
\widetilde{M}(y) &= \{x \in A \mid \exists t \in T(x), H(t, y, x) \subseteq Y \setminus -\text{int}C(x)\} \\
&= \{x \in A \mid \forall \alpha \in [0, 1], \forall s \in T(y_\alpha), -F(s, y_\alpha, x) \notin -\text{int}C(x)\} \\
&= \textstyle\bigcap_{\alpha \in [0,1]} \{x \in A \mid \forall s \in T(y_\alpha), -F(s, y_\alpha, x) \notin -\text{int}C(x)\} \\
&:= \textstyle\bigcap_{\alpha \in [0,1]} N(y_\alpha).
\end{aligned}$$

$\widetilde{M}(y)$ is closed since, $\forall \alpha \in [0, 1]$, so is $N(y_\alpha)$. $\square$

Remark 3.4. The assumptions of Proposition 3.1 are (or are slightly weaker than) those of Theorem 3.1 of Ref. 36. Indeed, note first that, since each continuous linear mapping from X into Y with the original topologies is still continuous when X and Y are equipped with the weak topologies, the space $L(X, Y)$ is the same for these two cases. Observe next that if T is guhc with respect to F when Z is endowed with the original topology, then so is T when Z is equipped with the weak topology. Then by Proposition 3.1, Corollary 3.4, with all the topologies in X, Y and Z are the weak topologies, contains directly Theorem 3.1 of Ref. 36, since its assumptions are more relaxed.

The following example shows that this containing is proper, since it gives a case where the assumptions of Corollary 3.4 are satisfied but that of Theorem 3.1 of Ref. 36 are not.

Example 3.3. Let $X = Y = Z = R$, $A = [0, 1], C(x) \equiv R_+, F(t, y, x) = t$ and

$$T(x) = \begin{cases} [0, 0.5], & \text{if } x = 1, \\ [0.5, 1], & \text{if } 0 \leq x < 1. \end{cases}$$

Then the multifunction $\alpha \mapsto \{F(t, y, x) \mid t \in T(y_\alpha)\}$ is not usc at 0^+, since

$$\{F(t, y, x) \mid t \in T(y_\alpha)\} = T(y_\alpha)$$

$$= T(1 - \alpha) = \begin{cases} [0, 0.5], & \text{if } \alpha = 1, \\ [0.5, 1], & \text{if } 0 \leq \alpha < 1. \end{cases}$$

This means that an assumption of Theorem 3.1 of Ref. 36 is not satisfied. However, choosing $H(t, y, x) = F(t, y, x)$ it is easy to see that all assumptions of Corollary 3.4 are satisfied. Direct computations yield the solution set being [0,1].

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