SEMICONTINUITY OF THE SOLUTION SETS TO PARAMETRIC QUASIVARIATIONAL INCLUSION PROBLEMS WITH APPLICATIONS TO TRAFFIC NETWORK PROBLEMS *

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Abstract. We propose some notions related to semicontinuity of a multivalued mapping and provide a clear insight for various semicontinuity - related definitions. We establish sufficient conditions for the solution set of a general quasivariational inclusion problem to have these semicontinuity - related properties. Our results are proved to include and improve recent ones in the literature by corollaries and examples. For applications we discuss in details a traffic network problem as a sample for employing the main results in practical situations.

Key words. quasivariational inclusion problems, lower and upper semicontinuities, Hausdorff lower and upper semicontinuities, U-lower-level closedness, U-upper-level closedness, quasiequilibrium problems, quasivariational inequalities, traffic network problems

1. Introduction. Stability of a solution or solution set of a parametric optimization - related problem has been intensively studied, where stability can be understood as (Fréchet or generalized) differentiability, Lipschitz or Hölder continuity, continuity and semicontinuity. Of course, differentiability of a solution is more desirable than its semicontinuity. However, as usual, to obtain a regularity property of the solutions, one has to impose properties of the same kind on the data of the problem under consideration. Such properties may be

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too restrictive and are not satisfied for the data in practical situations. Fortunately, in many cases in practice, semicontinuity of the solutions may be sufficient for the problem to be meaningful. For instance, in a competitive economy, an equilibrium for the economy in the Walras - Ward model and Arrow - Deubreu - Mckenzie model exists under the upper semicontinuity of multifunction $Y(p)$ of the out put vectors and $W(p)$ of the factor prices, with respect to the vector of good prices p, see e.g. [31]. Note that $Y(p)$ and $W(p)$ are the optimal solution sets of a pair of linear programming problems, which are dual to each other. We observe in the literature an increasing number of works devoted to semicontinuity of the solution set of a variational inequality or equilibrium problem, see [3, 4, 8, 27, 28, 32, 36]. This growing tendency inspires us to go in details to get a clear insight for semicontinuity of a multivalued mapping. Namely, our first aim is to propose and investigate some semicontinuity - related notions in connection with classical semicontinuity definitions. However, our main purpose is to study these semicontinuity properties of solution sets.

On the other hand, one of the problem classes of optimization - related problems, which has attracted attentions of mathematicians all over the world for the last decade is the quasiequilibrium problem and its generalizations. The equilibrium was introduced in [9] as a generalization of the variational inequality and optimization problem and proved to contain as particular cases many other problems such as the complementarity problem, the fixed - point problem and coincidence - point problem, the traffic network equilibrium and the Nash equilibrium problems. [7], considering random impulse control problems, noticed the need of using constraint sets which depend on the state variables. This led to quasivariational inequalities. The extension of equilibrium problems to include them is the quasiequilibrium problem. The next step of generalizing the problems setting is variational and quasivariational inclusion problems, see [20, 21, 33, 34, 42, 43]. It should be noted here that the term "variational inclusion" is understood in different ways in several recent papers. In [2, 15, 17, 23, 24, 29, 39] it means simply general variational inequalities. Variational inclusion problems in [12, 15, 18, 37, 38, 40, 44, 45] are problems of finding zeroes of maximal monotone mappings. In this note the terminology is similar to [20, 21, 33, 34, 42, 43]. The main efforts have been devoted to the solution existence. To the best of our knowledge there have not been papers in the literature dealing with stability of variational or quasivariational inclusion problems. This observation motivates our main aim in the present paper, which is considering various kinds of semicontinuity of the solution sets to parametric quasivariational inclusion problems, especially in terms of semicontinuity - related notions proposed in this paper.

The outline of the paper is as follows. In the rest of this section we state the quasivariational inclusion problem under our consideration and recall notions and preliminaries needed in the sequel. In Section 2 we introduce some semicontinuity - related notions and discuss the connections between these notions and classical ones. Section 3 is devoted to lower semicontinuity properties of the solution set of our problem. Upper semicontinuity properties are the goal of the next Section 4. In the final Section 5 we discuss applications of our results in several particular problems as examples. We choose two problems. The first one is the quasiequilibrium problem ($\mathrm{QEP}_{\mathrm{r}\alpha}$), which is still rather general and includes many problems interested recently in the literature, for convenient comparisons. In the second application we deal with a traffic equilibrium problem, which is rather practical, to get new results on the semicontinuity of the solution set. The general problem under our consideration in this paper is stated as follows.

Let Y, Λ, M and N be Hausdorff topological spaces, let X and Z be a Hausdorff vector topological spaces, let $D \subseteq X, K \subseteq Y$ be nonempty subsets. Given the following multifunctions:

$$S_i : D \times \Lambda \to 2^D, i = 1, 2,$$
$$T : D \times D \times N \to 2^K,$$
$$F : K \times D \times D \times M \to 2^Z,$$
$$G : K \times D \times D \times M \to 2^Z.$$

For the sake of simplicity we adopt the following notations. Letters w and s are used for a weak and strong, respectively, kinds of considered problems. For subsets A and B under consideration we adopt the notations

$$(u, v)\,\mathrm{w}\,A \times B \quad \text{means} \quad \forall u \in A, \exists v \in B,$$
$$(u, v)\,\mathrm{s}\,A \times B \quad \text{means} \quad \forall u \in A, \forall v \in B,$$
$$\alpha_1(A, B) \quad \text{means} \quad A \cap B \neq \emptyset,$$
$$\alpha_2(A, B) \quad \text{means} \quad A \subseteq B,$$
$$(u, v)\,\bar{\mathrm{w}}\,A \times B \quad \text{means } \exists u \in A, \forall v \in B \text{ and similarly for } \bar{\mathrm{s}},$$
$$\bar{\alpha_1}(A, B) \quad \text{means } A \cap B = \emptyset \text{ and similarly for } \bar{\alpha}_2.$$

Let $\mathrm{r} \in \{\mathrm{w}, \mathrm{s}\}$ and $\alpha \in \{\alpha_1, \alpha_2\}$. Our general parametric quasivariational inclusion problem is the following, for $(\lambda, \mu, \eta) \in \Lambda \times M \times N$,

($P_{\mathrm{r}\alpha}$) Find $\bar{x} \in S_1(\bar{x}, \lambda)$ such that, $(x, y)\,\mathrm{r}\,S_2(\bar{x}, \lambda) \times T(x, \bar{x}, \eta)$,

$$\alpha\big(F(y, x, \bar{x}, \mu), G(y, \bar{x}, \bar{x}, \mu)\big). \tag{1.1}$$

Note that ($\mathrm{P}_{\mathrm{r}\alpha}$) represents four problems. This statement is not explicit but helps to shorten much the presentation in the sequel. We denote the set of the solutions of ($\mathrm{P}_{\mathrm{r}\alpha}$) corresponding to (λ, μ, η) by $S_{\mathrm{r}\alpha}(\lambda, \mu, \eta)$. The conditions for the solution sets of these four problems to be nonempty are studied in [20, 21]. To ensure that this problem setting is general we discuss several special cases.

(a) If $G(y, x, \bar{x}, \mu) = F(y, x, \bar{x}, \mu) + C$, where $\emptyset \neq C \subseteq Y$, then ($\mathrm{P}_{\mathrm{s}\alpha_2}$) becomes the following quasivariational inclusion of the Minty type studied in [33, 34]:

(QIP) Find $\bar{x} \in S_1(\bar{x}, \lambda)$ such that, $\forall x \in S_2(\bar{x}, \lambda)$, $\forall y \in T(x, \bar{x}, \eta)$,

$$F(y, x, \bar{x}, \mu) \subseteq F(y, \bar{x}, \bar{x}, \mu) + C.$$

(b) If $X = Y, K = D$, $S_2(x,\lambda) := S(x,\lambda), S_1(x,\lambda) = \mathrm{cl}S(x,\lambda)$, $T(x,\bar{x},\eta) = T(\bar{x},\eta)$ (not depending on x), $F(y,x,\bar{x},\mu) = F(y,x,\mu)$ (not depending on $\bar{x}$) and $G(y,x,\bar{x},\mu) = Z \setminus -\mathrm{int}C$, then $(\mathrm{P}_{\mathrm{r}\alpha})$ collapses to the following quasiequilibrium problem considered by many authors:

$(\mathrm{QEP}_{\mathrm{r}\alpha})$ Find $\bar{x} \in \mathrm{cl}S(\bar{x},\lambda)$ such that, $(y,\bar{x}^*)\,\mathrm{r}\,S(\bar{x},\lambda) \times T(\bar{x},\eta)$,

$$\alpha\big(F(\bar{x}^*,y,\mu), Z \setminus -\mathrm{int}C\big).$$

(c) If $X = Y$, $Z = R$, $K = D = X$, $S_1(x,\lambda) \equiv S_2(x,\lambda) := S(x,\lambda)$, $T(x,\bar{x},\eta) \equiv X$, $F(y,x,\bar{x},\mu) = f(x,\bar{x},\mu)$ and $G(y,x,\bar{x},\mu) = [a,b]$, where $f : X \times X \times M \to R$ is a function, then all four problems $(\mathrm{P}_{\mathrm{r}\alpha})$ becomes the following lower and upper bound quasiequilibrium problem investigated in [10, 13]:

(BQEP) Find $\bar{x} \in S(\bar{x},\lambda)$ such that, $\forall y \in S(\bar{x},\lambda)$,

$$a \leq f(y,\bar{x},\mu) \leq b.$$

(d) If X, Z are normed spaces, $Y = X^*$, $M = N$, $S_1(x,\lambda) \equiv S_2(x,\lambda) := S(x,\lambda)$, $F(y,x,\bar{x},\mu) = \langle y, x - \bar{x}\rangle$ and $G(y,x,\bar{x},\mu) = Z \setminus -\mathrm{int}C$, where C is a convex cone in Z with nonempty interior, then four problems $(\mathrm{P}_{\mathrm{r}\alpha})$ are reduced to the following two quasivariational inequalities considered by many authors

(QVI) Find $\bar{x} \in S(\bar{x},\lambda)$ such that $(x,y)\,\mathrm{r}\,S(\bar{x},\lambda) \times T(x,\bar{x},\mu)$,

$$\langle y, x - \bar{x}\rangle \in Z \setminus -\mathrm{int}C.$$

(e) If $X = Y = K$, $S_1(x,\lambda) \equiv S_2(x,\lambda) := S(x,\lambda)$, $T(x,\bar{x},\eta) \equiv Y$, $F(y,x,\bar{x},\mu) = f(x,\mu) - f(\bar{x},\mu)$ and $G(y,x,\bar{x},\mu) \equiv Z \setminus -\mathrm{int}C$, where $f : D \times M \to Z$ is a mapping and $C \subseteq Z$ is a convex cone with nonempty interior, then four problems $(\mathrm{P}_{\mathrm{r}\alpha})$ coincide and become the following problem of finding weak minimizer in quasioptimization:

$(\mathrm{QOP}_{\mathrm{w}})$ Find $\bar{x} \in S(\bar{x},\lambda)$ such that, $\forall x \in S(\bar{x},\lambda)$,

$$f(x,\mu) - f(\bar{x},\mu) \in Z \setminus -\mathrm{int}C.$$

Note that here the prefix "quasi" means that the constraint set $S(x,\lambda)$ depends on x.

(f) If in (e) we replace only G by setting $G(y,x,\bar{x},\mu) = (Z \setminus -C) \cup l(C)$, where $l(C) = C \cap (-C)$, then we have the problem of finding Pareto minimizer in quasioptimization:

$(\mathrm{QOP}_{\mathrm{P}})$ Find $\bar{x} \in S(\bar{x},\lambda)$ such that, $\forall x \in S(\bar{x},\lambda)$,

$$f(x,\mu) - f(\bar{x},\mu) \in (Z \setminus -C) \cup l(C).$$

Recall now some notions. Let X and Y be as above and $Q : X \to 2^Y$ be a multifunction. Q is called lower semicontinuous (lsc) at x_0 if: $Q(x_0) \cap U \neq \emptyset$ for some open subset $U \subseteq Y$ implies the existence of a neighborhood N of x_0 such that, $\forall x \in N, Q(x) \cap U \neq \emptyset$. Q is upper semicontinuous (usc) at x_0 if for each open subset $U \supseteq Q(x_0)$, there is a neighborhood N of x_0 such that

$U \supseteq Q(N)$. Q is said to be Hausdorff lower semicontinuous (H-lsc) at x_0 if for each neighborhood B of the origin in Y, there is a neighborhood N of x_0 such that $Q(x_0) \subseteq Q(x)+B, \forall x \in N$. Q is termed Hausdorff upper semicontinuous (H-usc) at x_0 if the last inclusion replaced by $Q(x) \subseteq Q(x_0)+B, \forall x \in N$. Q is called closed at x_0 if, for each net $(x_\gamma, y_\gamma) \in \text{graph}Q := \{(x, y) \mid y \in Q(x)\} : (x_\gamma, y_\gamma) \to (x_0, y_0)$, $y_0 \in Q(x_0)$. We say that Q satisfies a certain property in a subset $A \subseteq X$ if Q satisfies it at every point of A. If $A = \text{dom}Q := \{x \mid Q(x) \neq \emptyset\}$ we omit "in dom$Q$" in the saying. The following assertions are known and we give a reference is given only for nonpopular statements.

PROPOSITION 1.1.

(i) *Q is lsc at x_0 if and only if $\forall x_\gamma \to x_0$. $\forall y \in Q(x_0), \exists y_\gamma \in Q(x_\gamma), y_\gamma \to y$.*

(ii) *Q is closed if and only if* graph$Q$ *is closed.*

(iii) *Q is closed at x_0 if Q is H-usc at x_0 and $Q(x_0)$ is closed ([3]).*

(iv) *Q is H-usc at x_0 if Q is usc at x_0. Conversely, Q is usc at x_0 if Q is H-usc at x_0 and $Q(x_0)$ is compact.*

(v) *Q is usc at x_0 if $Q(A)$ is compact for any compact subset A of dom$Q$ and Q is closed at x_0.*

(vi) *Q is usc at x_0 if Y is compact and Q is closed at x_0.*

(vii) *Q is lsc at x_0 if Q is H-lsc at x_0. The converse is true if $Q(x_0)$ is compact ([22]).*

2. Semicontinuity of multivalued mappings.

We introduce some definitions related to semicontinuity to have a better insight as follows.

DEFINITION 2.1. Let X be a Hausdorff topological space, Y be a topological vector space, $Q : X \to 2^Y$ and $\emptyset \neq U \subseteq Y$.

(i) Q is called U-lower-level closed at x_0 if

$$[x_\gamma \to x_0, Q(x_\gamma) \subseteq \text{cl}U] \Longrightarrow [Q(x_0) \subseteq \text{cl}U].$$

(ii) Q is said to be U-Hausdorff-lower-level closed at x_0 if

$$[x_\gamma \to x_0, B \text{ is a neighborhood of 0 in} Y] \Longrightarrow [\exists \bar{\gamma}, Q(x_0) \setminus \text{cl}U \subseteq Q(x_{\bar{\gamma}}) + B].$$

(iii) Q is said to be U-upper-level closed at x_0 if

$$[x_\gamma \to x_0, Q(x_\gamma) \cap \text{cl}U \neq \emptyset] \Longrightarrow [Q(x_0) \cap \text{cl}U \neq \emptyset].$$

(iv) Q is termed U-Hausdorff-upper-level closed at x_0 if

$$[x_\gamma \to x_0, Q(x_\gamma) \cap \text{cl}U \neq \emptyset] \Longrightarrow [\forall B, (Q(x_0) + B) \cap \text{cl}U \neq \emptyset].$$

REMARK 2.1. Q is U-lower-level closed if and only if Q is clU-lower-level closed, (while intU $\neq \emptyset$) if and only if Q is intU-lower-level closed. The same assertion is valid for the notions in (ii) - (iv).

Definition 2.2. Let X, Y, Q and U be as in Definition 2.1.

(i) Q is said to be U-lower semicontinuous (U-lsc) at x_0 if

$$[x_\gamma \to x_0, Q(x_0) \cap \text{int}U \neq \emptyset] \Longrightarrow [\exists \bar{\gamma}, Q(x_{\bar{\gamma}}) \cap \text{int}U \neq \emptyset].$$

(ii) Q is said to be U-Hausdorff-lower semicontinuous (U-Hlsc) at x_0 if

$$[x_\gamma \to x_0, B \text{ is a neighborhood of 0 in } Y] \Longrightarrow [\exists \bar{\gamma}, Q(x_0) \cap \text{int}U \subseteq Q(x_{\bar{\gamma}}) + B].$$

(iii) Q is called U-upper semicontinuous (U-usc) at x_0 if

$$[x_\gamma \to x_0, Q(x_0) \subseteq \text{int}U] \Longrightarrow [\exists \bar{\gamma}, Q(x_{\bar{\gamma}}) \subseteq \text{int}U].$$

(iv) Q is termed U-Hausdorff-upper semicontinuous (U-Husc) at x_0 if

$$[x_\gamma \to x_0, Q(x_0) + B \subseteq \text{int}U \text{ for some neighborhood } B \text{ of } 0]$$

$$\Longrightarrow [\exists \bar{\gamma}, Q(x_{\bar{\gamma}}) \subseteq \text{int}U].$$

(v) Q is called lower semicontinuous with respect to U at x_0 if, $\forall x_\gamma \to x_0$, $\forall y \in Q(x_0) \setminus U$, $\exists\ y_\gamma \in Q(x_\gamma), y_\gamma \to y$.

Remark 2.2. If intU $= \emptyset$, any $Q : X \to 2^Y$ satisfies (i) - (iv). If intU $\neq \emptyset$, Q is U-lsc if and only if Q is intU-lsc, if and only if Q is clU-lsc. The same is true for the other three notions.

Proposition 2.1. *Let X, Y and Q be as in Definition 2.1. Then Q is lsc (Hlsc, usc and Husc) at x_0 if and only if for all $U \subseteq Y$ Q is U-lsc (U-Hlsc, U-usc and U-Husc, respectively) at x_0.*

Proof. By the similarity we prove only the assertions about U-Hausdorff semicontinuity. Assume that Q is Hlsc at x_0. If $x_\gamma \to x_0$ and B is a neighborhood of 0 in Y, then there is $\bar{\gamma}$ such that $Q(x_0) \subseteq Q(x_{\bar{\gamma}}) + B$ and hence $Q(x_0) \cap \text{int}U \subseteq Q(x_{\bar{\gamma}}) + B$, i.e. Q is U-Hlsc at x_0. To see the converse we simply take $U = Y$.

Now let Q be Husc at x_0. If $x_\gamma \to x_0$ and B is a neighborhood of 0 in Y with $Q(x_0) + B \subseteq \text{int}U$, then since Q is Husc at x_0, there is $\bar{\gamma}$ such that $Q(x_{\bar{\gamma}}) \subseteq Q(x_0) + B \subseteq \text{int}U$, i.e. Q is U-Husc at x_0. Conversely, suppose that Q is U-Husc at x_0, $\forall U \subseteq Y$, but Q is not Husc at x_0, i.e., there are B (neighborhood of 0 in Y) and $x_\gamma \to x_0$, $\forall \gamma, Q(x_\gamma) \nsubseteq Q(x_0) + B$. Since $Q(x_0) + B$ is an open subset, taking $U = Q(x_0) + B$, there is $\bar{\gamma}$, $Q(x_{\bar{\gamma}}) \subseteq Q(x_0) + B$, which is a contradiction. □

Proposition 2.2. *Let X, Y, Q and U be as in Definition 2.1. Then Q is U-lsc (U-Hlsc, U-usc and U-Husc) at x_0 if and only if Q is $Y \setminus U$-lower-level closed ($Y \setminus U$-Hausdorff-lower-level closed, $Y \setminus U$-upper-level closed and $Y \setminus U$-Hausdorff-upper-level closed, respectively) at x_0.*

Proof. We prove only the assertions for "U-lsc" and "U-usc", since the other two assertions can be checked similarly. Suppose that Q is U-lsc at x_0, but there is $x_\gamma \to x_0$ with $Q(x_\gamma) \subseteq \mathrm{cl}(Y \setminus U) = Y \setminus \mathrm{int}U$ and $Q(x_0) \not\subseteq Y \setminus \mathrm{int}U$, i.e., $Q(x_0) \cap \mathrm{int}U \neq \emptyset$. Since Q is U-lsc at x_0, there is $\bar{\gamma}, Q(x_{\bar{\gamma}}) \cap \mathrm{int}U \neq \emptyset$. So $Q(x_{\bar{\gamma}}) \not\subseteq Y \setminus \mathrm{int}U$, which is a contradiction. Conversely, suppose Q is $Y \setminus U$-lower-level closed at x_0, but there is $x_\gamma \to x_0$ with $Q(x_0) \cap \mathrm{int}U \neq \emptyset$ and, $\forall \gamma, Q(x_\gamma) \cap \mathrm{int}U = \emptyset$, i.e., $Q(x_\gamma) \subseteq Y \setminus \mathrm{int}U = \mathrm{cl}(Y \setminus U)$. Since Q is $Y \setminus U$-lower-level closed at x_0, we have $Q(x_0) \subseteq Y \setminus \mathrm{int}U$ and hence $Q(x_0) \cap \mathrm{int}U = \emptyset$, which is impossible.

Now suppose Q is U-usc at x_0, but there is $x_\gamma \to x_0$ such that $Q(x_\gamma) \cap \mathrm{cl}(Y \setminus U) \neq \emptyset$ and $Q(x_0) \cap \mathrm{cl}(Y \setminus U) = \emptyset$, i.e., $Q(x_0) \subseteq Y \setminus \mathrm{cl}(Y \setminus U) = Y \setminus (Y \setminus \mathrm{int}U) = \mathrm{int}U$. Then there exists $\bar{\gamma}$, $Q(x_{\bar{\gamma}}) \subseteq \mathrm{int}U = Y \setminus \mathrm{cl}(Y \setminus U)$, which is a contradiction. Conversely, assume that Q is not U-usc, i.e. there is $x_\gamma \to x_0$ with $Q(x_0) \subseteq \mathrm{int}U$ and, $\forall \gamma$, $Q(x_\gamma) \not\subseteq \mathrm{int}U$. Then $Q(x_\gamma) \cap (Y \setminus \mathrm{int}U) \neq \emptyset$ and hence $Q(x_\gamma) \cap \mathrm{cl}(Y \setminus U) \neq \emptyset$. This together with $Q(x_0) \subseteq \mathrm{int}U$, i.e. $Q(x_0) \cap \mathrm{cl}(Y \setminus U) = \emptyset$, mean that Q is not $Y \setminus U$-upper-level closed. $\square$

PROPOSITION 2.3. *Let X, Y, Q and U be as in Definition 2.1.*

(i) *U-Hausdorff-lower semicontinuity implies U-lower semicontinuity. The converse is not true even under compactness assumptions.*

(ii) *U-upper semicontinuity implies U-Hausdorff-upper semicontinuity. If $Q(x_0)$ is compact then the converse is true at x_0.*

(iii) *Q is lsc with respect to U at x_0 if and only if, for all $V \supseteq U$, Q is lsc with respect to V at x_0. Hence Q is lsc at x_0 if and only if, for all $U \subseteq Y$, Q is lsc with respect to U at x_0.*

(iv) *Q is lsc with respect to U at x_0 if $Q(.) \setminus U$ is lsc at x_0. The converse is true if U is closed.*

Proof.

(i) Suppose Q is U-Hlsc at x_0 and $x_\gamma \to x_0$ such that $Q(x_0) \cap \mathrm{int}U \neq \emptyset, \forall \gamma, Q(x_\gamma) \cap \mathrm{int}U = \emptyset$. Then for each B (neighborhood of 0 in Y), there is γ_B such that $Q(x_0) \cap \mathrm{int}U \subseteq Q(x_{\gamma_B}) + B$. Take $y_0 \in Q(x_0) \cap \mathrm{int}U$. There are $y_{\gamma_B} \in Q(x_{\gamma_B})$ and $b_{\gamma_B} \in B$ such that $y_0 = y_{\gamma_B} + b_{\gamma_B}$. As B is arbitrary, $y_{\gamma_B} \to y_0 \in \mathrm{int}U$. So there is $\bar{\gamma}_B$ such that $y_{\bar{\gamma}_B} \in \mathrm{int}U$, contradicting the fact that, $\forall \gamma, Q(x_\gamma) \cap \mathrm{int}U = \emptyset$. Example 2.1 shows that the converse is not true even under compactness assumptions.

(ii) From Definition 2.1, we see that the first implication is clear. Assume that Q is U-Husc and $Q(x_0)$ is compact. If $Q(x_0) \subseteq \mathrm{int}U$, to see that Q is U-usc we show that there is B (neighborhood of 0 in Y) such that $Q(x_0) + B \subseteq \mathrm{int}U$. If for each B, $Q(x_0) + B \not\subseteq \mathrm{int}U$, there are $y^0_{\gamma_B} \in Q(x_0)$ and $b_{\gamma_B} \in B$ such that $y^0_{\gamma_B} + b_{\gamma_B} \notin \mathrm{int}U$. By the compactness of $Q(x_0)$ we can assume that $y^0_{\gamma_B} \to y_0$ for some $y_0 \in Q(x_0)$. As B is arbitrary $y^0_{\gamma_B} + b_{\gamma_B} \to y_0$. Since $y^0_{\gamma_B} + b_{\gamma_B} \notin \mathrm{int}U, \forall B$, we get a contradiction that $y_0 \notin \mathrm{int}U$.

(iii) The first statement is directly derived from Definition 2.2. It is clear that if Q is lsc at x_0 then Q is lsc with respect to U at x_0, for all $U \subseteq Y$. For the converse, take $U \subset Y$ such that $Q(x_0) \cap U = \emptyset$ to see that Q is lsc at x_0.

(iv) The first assertion is clear. Assume that $Q(.)$ is lsc with respect to U at x_0 and U is closed. For all $y_0 \in Q(x_0) \setminus U$, there exists some net $y_\gamma \in Q(x_\gamma)$ such that $y_\gamma \to y_0$. Since U is closed and $y_0 \notin U$ we have a subnet y_β such that $y_\beta \notin U, \forall \beta$. □

The following example illustrates Proposition 2.3 (ii).

Example 2.1. Let $X = Y = R$ and $Q : R \to 2^R$ is defined by $Q(0) = [0, 4]$, $Q(x) = [0, 1]$, for all $x \neq 0$. Then, Q is R_+-lsc at 0 (since $Q(x) \cap (0, +\infty) \neq \emptyset, \forall x \in R$). But Q is not R_+-Hlsc at 0 since $Q(0) \cap (0, +\infty) = (0, 4] \not\subseteq Q(x) + (-1, 1) = (-1, 2), \forall x \neq 0$.

The following definition extends Definition 2.2 (i), (iii).

Definition 2.3. Let $f : D \to 2^Y$ and $g : K \to 2^Y$.

(i) f is called to be g-lsc at $(x_0, y_0) \in D \times K$ if

$$[(x_\gamma, y_\gamma) \to (x_0, y_0), f(x_0) \cap \text{int}g(y_0) \neq \emptyset] \Longrightarrow [\exists \hat{\gamma}, f(x_{\hat{\gamma}}) \cap \text{int}g(y_{\hat{\gamma}}) \neq \emptyset].$$

(ii) f is said to be g-usc at $(x_0, y_0) \in D \times K$ if

$$[(x_\gamma, y_\gamma) \to (x_0, y_0), f(x_0) \subseteq \text{int}g(y_0)] \Longrightarrow [\exists \hat{\gamma}, f(x_{\hat{\gamma}}) \subseteq \text{int}g(y_{\hat{\gamma}})].$$

We propose the following notation: for $\alpha \in \{\alpha_1, \alpha_2\}$, we write $f \alpha g$ at $(x_0, y_0) \in D \times K$ if

$$[(x_\gamma, y_\gamma) \to (x_0, y_0), \alpha\big(f(x_0), \text{int}g(y_0)\big)] \Longrightarrow [\exists \hat{\gamma}, \alpha\big(f(x_{\hat{\gamma}}), \text{int}g(y_{\hat{\gamma}})\big)].$$

So $f \alpha_1 g$ means that f is g-lsc at (x_0, y_0) and $f \alpha_2 g$ means that f is g-usc at (x_0, y_0). When $g(y) \equiv U$, where $U \subseteq Y$, being g-lsc or g-usc collapses to being U-lsc or U-usc, respectively.

The following definition extends the inclusion property proposed in [3].

Definition 2.4. Let f and g be as in Definition 2.3.

(i) f is called to have the g-inclusion property at (x_0, y_0) if

$$[(x_\gamma, y_\gamma) \to (x_0, y_0), f(x_0) \cap g(y_0) \neq \emptyset] \Longrightarrow [\exists \hat{\gamma}, f(x_{\hat{\gamma}}) \cap g(y_{\hat{\gamma}}) \neq \emptyset].$$

(ii) f is said to have the strict g-inclusion property at (x_0, y_0) if

$$[(x_\gamma, y_\gamma) \to (x_0, y_0), f(x_0) \subseteq g(y_0)] \Longrightarrow [\exists \hat{\gamma}, f(x_{\hat{\gamma}}) \subseteq g(y_{\hat{\gamma}})].$$

We adopt the following convention: for $\alpha \in \{\alpha_1, \alpha_2\}$, f is called to have the $\alpha - g$-inclusion property at (x_0, y_0) if

$$[(x_\gamma, y_\gamma) \to (x_0, y_0), \alpha\big(f(x_0), g(y_0)\big)] \Longrightarrow [\exists \hat{\gamma}, \alpha\big(f(x_{\hat{\gamma}}), g(y_{\hat{\gamma}})\big)].$$

We also use the similar convention: for $\alpha \in \{\alpha_1, \alpha_2\}$, f is said to have the $\bar{\alpha} - g$-inclusion property at (x_0, y_0) if

$$[(x_\gamma, y_\gamma) \to (x_0, y_0), \bar{\alpha}\big(f(x_0), g(y_0)\big)] \Longrightarrow [\exists \hat{\gamma}, \bar{\alpha}\big(f(x_{\hat{\gamma}}), g(y_{\hat{\gamma}})\big)].$$

Note that when $g(x) \equiv U$, where U is a closed subset of Y, being $\bar{\alpha}_1 - g$ inclusion property or $\bar{\alpha}_2 - g$ inclusion property collapses to being $Y \setminus U$- usc or $Y \setminus U$-lsc, respectively.

3. Lower semicontinuity of the solution sets. For $\lambda \in \Lambda$, $\mu \in M$ and $\eta \in N$. Let $E(\lambda) := \{x \in D \mid x \in S_1(x, \lambda)\}$. Throughout the paper, assume that $S_{\mathrm{r}\alpha}(\lambda, \mu, \eta) \neq \emptyset$ for all (λ, μ, η) in a neighborhood of $(\lambda_0, \mu_0, \eta_0) \in \Lambda \times M \times N$. (About conditions for the solution existence in quasivariational inclusion problems see [20, 21].)

THEOREM 3.1. *Assume for problem* $(\mathrm{P}_{\mathrm{r}\alpha})$ *that, for* $\emptyset \neq U \subseteq X$,

(i) $E(.) \setminus \mathrm{cl}U$ *is lsc at* λ_0; S_2 *is usc and has compact values in* $(E(\lambda_0) \setminus \mathrm{cl}U) \times \{\lambda_0\}$;

(ii) *in* $S_2(D \setminus \mathrm{cl}U, \lambda_0) \times (D \setminus \mathrm{cl}U) \times \{\eta_0\}$, T *is lsc if* $\mathrm{r} = \mathrm{w}$, *and usc and has compact values if* $\mathrm{r} = \mathrm{s}$;

(iii_α) $F\alpha G$ *in* $\big((T(S_2(D \setminus \mathrm{cl}U, \lambda_0), D \setminus \mathrm{cl}U, \eta_0), S_2(D \setminus \mathrm{cl}U, \lambda_0), D \setminus \mathrm{cl}U, \mu_0),$ $(T(S_2(D \setminus \mathrm{cl}U, \lambda_0), D \setminus \mathrm{cl}U, \eta_0), D \setminus \mathrm{cl}U, D \setminus \mathrm{cl}U, \mu_0)\big)$;

($\mathrm{iv}_{\mathrm{r}\alpha}$) $\forall x \in S_{\mathrm{r}\alpha}(\lambda_0, \mu_0, \eta_0), (\hat{x}, y) \, \mathrm{r} \, S_2(x, \lambda_0) \times T(\hat{x}, x, \eta_0)$, $\alpha\big(F(y, \hat{x}, x, \mu_0),$ $\mathrm{int}G(y, x, x, \mu_0)\big)$.

Then $S_{\mathrm{r}\alpha}$ *is* U*-lower-level closed at* $(\lambda_0, \mu_0, \eta_0)$.

Proof. Since $\mathrm{r} \in \{\mathrm{w}, \mathrm{s}\}$ and $\alpha \in \{\alpha_1, \alpha_2\}$ we have in fact four cases corresponding to four different combinations of values of r and α. However, the proof techniques are similar. We consider only the case $\mathrm{r} = \mathrm{w}$ and $\alpha = \alpha_1$. Suppose that $S_{\mathrm{w}\alpha_1}(.,.,.)$ is not U-lower-level closed at $(\lambda_0, \mu_0, \eta_0)$, i.e., $\exists (\lambda_\gamma, \mu_\gamma, \eta_\gamma) \to (\lambda_0, \mu_0, \eta_0)$ such that $S_{\mathrm{w}\alpha_1}(\lambda_\gamma, \mu_\gamma, \eta_\gamma) \subseteq \mathrm{cl}U, \forall \gamma$, but $x_0 \in S_{\mathrm{w}\alpha_1}(\lambda_0, \mu_0, \eta_0) \setminus \mathrm{cl}U$ exists. Then $\forall x_\gamma \in S_{\mathrm{w}\alpha_1}(\lambda_\gamma, \mu_\gamma, \eta_\gamma), x_\gamma \not\to x_0$. Since $E(.) \setminus \mathrm{cl}U$ is lsc at λ_0, there is $\bar{x}_\gamma \in E(\lambda_\gamma) \setminus \mathrm{cl}U$, $\bar{x}_\gamma \to x_0$. By the contradiction assumption, there exists a subnet $\bar{x}_\beta \notin S_{\mathrm{w}\alpha_1}(\lambda_\beta, \mu_\beta), \forall \beta$. This means the existence of $\hat{x}_\beta \in S_2(\bar{x}_\beta, \lambda_\beta)$, $\forall y_\beta \in T(\hat{x}_\beta, \bar{x}_\beta, \eta_\beta)$,

$$F(y_\beta, \hat{x}_\beta, \bar{x}_\beta, \mu_\beta) \cap G(y_\beta, \bar{x}_\beta, \bar{x}_\beta, \mu_\beta) = \emptyset. \tag{2.1}$$

Since $S_2(.,.)$ is usc and has compact values at (x_0, λ_0), one can assume that $\hat{x}_\beta \to \hat{x}_0$ for some $\hat{x}_0 \in S_2(x_0, \lambda_0)$ (taking a subnet if necessary). By assumption

($\mathrm{iv}_{\mathrm{w}\alpha_1}$), there is some $y_0 \in T(\hat{x}_0, x_0, \eta_0)$ such that,

$$F(y_0, \hat{x}_0, x_0, \mu_0) \cap \mathrm{int}G(y_0, x_0, x_0, \mu_0) \neq \emptyset. \tag{2.2}$$

Because of the lower semicontinuity of $T(.,.,.)$ at $(\hat{x}_0, x_0, \eta_0)$, there exists a net $y_\beta \in T(\hat{x}_\beta, \bar{x}_\beta, \eta_\beta)$, $y_\beta \to y_0$. Since F is G-lsc at $\big((y_0, \hat{x}_0, x_0, \mu_0), (y_0, x_0, x_0, \mu_0)\big)$ by (iii_α), we see a contradiction between (2.1) and (2.2). □

Remark 3.1. In assumptions (ii) and (iii_α) we can clearly replace the complicated - looking regions by simpler but bigger regions. This results in weakening the theorem. This replacement can also be applied in the theorems and corollaries in the sequel.

Taking into account Propositions 2.1 and 2.2 we obtain the following immediate consequence of Theorem 3.1.

Corollary 3.1. *Assume for problem* ($\mathrm{P}_{\mathrm{r}\alpha}$) *assumption* ($\mathrm{iv}_{\mathrm{r}\alpha}$) *of Theorem 3.1. Assume further that*

(i′) *E is lsc at λ_0; S_2 is usc and has compact values in $E(\lambda_0) \times \{\lambda_0\}$;*

(ii′) *in $S_2(D, \lambda_0) \times D \times \{\eta_0\}$, T is lsc if* $\mathrm{r} = \mathrm{w}$, *and usc and has compact values if* $\mathrm{r} = \mathrm{s}$;

(iii'_α) *$F\alpha G$ in $\big((T(S_2(D, \lambda_0), D, \eta_0), S_2(D, \lambda_0), D, \mu_0), (T(S_2(D, \lambda_0), D, \eta_0), D, D, \mu_0)\big)$.*

Then $S_{\mathrm{r}\alpha}$ is lsc at $(\lambda_0, \mu_0, \eta_0)$.

Note that assumption ($\mathrm{iv}_{\mathrm{r}\alpha}$) is essential (see Example 2.1 in [3] for a special case). The following example shows a case where Corollary 3.1 is more advantageous than recent ones in the literature.

Example 3.1. Let $X = Y = Z = R$, $\Lambda \equiv M \equiv N = [0, 1]$, $D = K = R$, $S_1(x, \lambda) = S_2(x, \lambda) = [0, \lambda]$, $T(x, \bar{x}, \lambda) = \{\bar{x}\}$, $G(y, x, \bar{x}, \lambda) = [0, +\infty)$, $\lambda_0 = 0$ and

$$F(y, \hat{x}, x, \lambda) = \begin{cases} [1, 2], & \text{if } \lambda = 0, \\ [2, 3], & \text{otherwise.} \end{cases}$$

Then, all assumptions of Corollary 3.1 are fulfilled. (In this case, our four problems are reduced to a quasiequilibrium problem, and $S(\lambda)$ is easily computed as $[0, \lambda]$.) But Theorems 2.1 and 2.3 in [3] and Theorem 2.2 in [4] cannot be applied since F is neither lsc nor usc at $\lambda = 0$ as required in the mentioned theorems.

Although assumption ($\mathrm{iv}_{\mathrm{r}\alpha}$) is essential, it together with (iii_α) can be replaced by using Definition 2.4 as follows.

Theorem 3.2. *Assume* (i) *and* (ii) *as in Theorem 3.1 and replace* (iii_α) *and* ($\mathrm{iv}_{\mathrm{r}\alpha}$) *by*

(v_α) *F has the $\alpha - G$-inclusion property in $\big((T(S_2(D \setminus \mathrm{cl}U, \lambda_0), D \setminus \mathrm{cl}U, \eta_0), S_2(D \setminus \mathrm{cl}U, \lambda_0), D \setminus \mathrm{cl}U, \mu_0), (T(S_2(D \setminus \mathrm{cl}U, \lambda_0), D \setminus \mathrm{cl}U, \eta_0), D \setminus \mathrm{cl}U, D \setminus \mathrm{cl}U, \mu_0)\big)$.*

Then $S_{\mathrm{r}\alpha}$ is U-lower-level closed at $(\lambda_0, \mu_0, \eta_0)$.

Proof. We prove the case where r = w and $\alpha = \alpha_1$ as an example. The other cases are proved similarly. We can repeat the first part of the proof of Theorem 3.1 to have (2.1), $\hat{x}_0 \in S_2(x_0, \lambda_0)$ and $y_0 \in T(\hat{x}_0, x_0, \lambda_0)$ such that $(\hat{x}_\beta, y_\beta) \to (\hat{x}_0, y_0)$ and

$$F(y_0, \hat{x}_0, x_0, \mu_0) \cap G(y_0, x_0, x_0, \mu_0) \neq \emptyset.$$

Assumption (v_α) implies the existence of an index $\bar{\beta}$ such that

$$F(y_{\bar{\beta}}, \hat{x}_{\bar{\beta}}, \bar{x}_{\bar{\beta}}, \mu_{\bar{\beta}}) \cap G(y_{\bar{\beta}}, \bar{x}_{\bar{\beta}}, \bar{x}_{\bar{\beta}}, \mu_{\bar{\beta}}) \neq \emptyset,$$

which contradicts (2.1). □

Similarly one can obtain the following result from Propositions 2.1, 2.2 and Theorem 3.2.

COROLLARY 3.2. *Assume, for problem* ($\mathrm{P}_{\mathrm{r}\alpha}$), ($\mathrm{v}_\alpha$) *as in Theorem 3.2 and replace* (i) *and* (ii) *by* (i') *and* (ii') *as in Corollary 3.1. Then $S_{\mathrm{r}\alpha}$ is lsc at* $(\lambda_0, \mu_0, \eta_0)$.

The following example explains some advantages of Corollary 3.2.

EXAMPLE 3.2. Let $X, Y, Z, D, K, \Lambda, M, N, T, G$ and λ_0 be as in Example 3.1. Let

$$S_1(x,\lambda) = S_2(x,\lambda) = \begin{cases} [0,\lambda], & \text{if} \quad x \geq 0, \\ [0,-x], & \text{otherwise.} \end{cases}$$

and $F(y, \bar{x}, x, \lambda) \equiv \{1\}$. Then, $E(\lambda) = [0, \lambda]$ and all assumptions of Corollary 3.2 hold. So the solution set $S(.)$ of the quasiequilibrium problem is lsc at $\lambda_0 = 0$ (in this case our four problems are reduced to a quasiequilibrium problem and in fact $S(\lambda) = [0, \lambda]$). While Theorems 2.2 and 2.4 in [3] and Theorem 2.1 in [4] do not work since S_2 is not usc in $R \times \{0\}$.

We now proceed to Hausdorff lower semicontinuity.

THEOREM 3.3. *For* ($\mathrm{P}_{\mathrm{r}\alpha}$) and $\emptyset \neq U \subseteq X$, assume (i), (ii), ($\mathrm{iii}_\alpha$) *and* ($\mathrm{iv}_{\mathrm{r}\alpha}$) *of Theorem 3.1. Assume further that*

($\mathrm{i_h}$) *E is lsc with respect to* int$U$ *at λ_0, $E(\lambda_0) \setminus$* int$U$ *is compact; $S_1(., \lambda_0)$ is closed in $E(\lambda_0)$ and $S_2(., \lambda_0)$ is lsc in $E(\lambda_0)$;*

($\mathrm{ii_h}$) *in $S_2(E(\lambda_0), \lambda_0) \times E(\lambda_0)$, $T(., ., \eta_0)$ is usc and has compact values if* r = w, *and lsc if* r = s;

($\mathrm{iv}'_{\mathrm{r}\alpha}$) *$F(., ., ., \mu_0)$ has the $\bar{\alpha}{-}G(., ., ., \mu_0)$-inclusion property in* $\big(\big(T(S_2(E(\lambda_0), \lambda_0), S_2(E(\lambda_0), \lambda_0), E(\lambda_0), \eta_0) \times S_2(E(\lambda_0), \lambda_0) \times E(\lambda_0)\big), \big(T(S_2(E(\lambda_0), \lambda_0), S_2(E(\lambda_0), \lambda_0), E(\lambda_0), \eta_0) \times E(\lambda_0) \times E(\lambda_0)\big)\big)$.

Then $S_{\mathrm{r}\alpha}$ is U-Hausdorff-lower-level closed at $(\lambda_0, \mu_0, \eta_0)$.

Proof. As an example we demonstrate only for $S_{s\alpha_2}$. We first show that $S_{s\alpha_2}(\lambda_0, \mu_0, \eta_0)$ is closed in D. Suppose that $x_\gamma \in S_{s\alpha_2}(\lambda_0, \mu_0, \eta_0), x_\gamma \to x_0$. Since $S_1(., \lambda_0)$ is closed, $x_0 \in S_1(x_0, \lambda_0)$. If $x_0 \notin S_{s\alpha_2}(\lambda_0, \mu_0, \eta_0)$, there exist $\hat{x}_0 \in S_2(x_0, \lambda_0)$ and $y_0 \in T(\hat{x}_0, x_0, \eta_0)$ such that

$$F(y_0, \hat{x}_0, x_0, \mu_0) \not\subseteq G(y_0, x_0, x_0, \mu_0). \tag{2.3}$$

Since $S_2(., \lambda_0)$ and $T(., ., \eta_0)$ are lsc at x_0 and $(\hat{x}_0, x_0)$, respectively, there are $\hat{x}_\gamma \in S_2(x_\gamma, \lambda_0)$ and $y_{0\gamma} \in T(\hat{x}_\gamma, x_\gamma, \eta_0)$ such that $(\hat{x}_\gamma, y_{0\gamma}) \to (\hat{x}_0, y_0)$. As $x_\gamma \in S_{s\alpha_2}(\lambda_0, \mu_0, \eta_0)$, we have

$$F(y_{0\gamma}, \hat{x}_\gamma, x_\gamma, \mu_0) \subseteq G(y_{0\gamma}, x_\gamma, x_\gamma, \mu_0). \tag{2.4}$$

By assumption ($\text{iv}'_{s\alpha_2}$), we see a contradiction between (2.3) and (2.4). Hence, $S_{s\alpha_2}(\lambda_0, \mu_0, \eta_0)$ is closed and $S_{s\alpha_2}(\lambda_0, \mu_0, \eta_0) \setminus \text{int}U$ is compact, by (i_h). We show that $\forall(\lambda_\gamma, \mu_\gamma, \eta_\gamma) \to (\lambda_0, \mu_0, \eta_0)$, $\forall \bar{x}_0 \in S_{s\alpha_2}(\lambda_0, \mu_0, \eta_0)\setminus\text{int}U$, $\exists \bar{x}_\gamma \in S_{s\alpha_2}(\lambda_\gamma, \mu_\gamma, \eta_\gamma)$, $\bar{x}_\gamma \to \bar{x}_0$. Suppose to the contrary that there exist $(\lambda_\gamma, \mu_\gamma, \eta_\gamma) \to (\lambda_0, \mu_0, \eta_0)$ and $\bar{x}_0 \in S_{s\alpha_2}(\lambda_0, \mu_0, \eta_0) \setminus \text{int}U$ such that $\forall x_\gamma \in S_{s\alpha_2}(\lambda_\gamma, \mu_\gamma, \eta_\gamma)$, $x_\gamma \not\to \bar{x}_0$. Since E is lsc with respect to int$U$ at λ_0, there is $\bar{x}_\gamma \in E(\lambda_\gamma)$, $\bar{x}_\gamma \to \bar{x}_0$. By the contradiction assumption, there exists a subnet $\bar{x}_\beta \notin S_{s\alpha_2}(\lambda_\beta, \mu_\beta, \eta_\beta), \forall \beta$. This means the existence of $\hat{x}_\beta \in S_2(\bar{x}_\beta, \lambda_\beta)$, and $y_\beta \in T(\hat{x}_\beta, \bar{x}_\beta, \eta_\beta)$,

$$F(y_\beta, \hat{x}_\beta, \bar{x}_\beta, \mu_\beta) \not\subseteq G(y_\beta, \bar{x}_\beta, \bar{x}_\beta, \mu_\beta). \tag{2.5}$$

Since S_2 is usc and $S_2(x_0, \lambda_0)$ is compact, one can assume that $\hat{x}_\beta \to \hat{x}_0$ for some $\hat{x}_0 \in S_2(\bar{x}_0, \lambda_0)$ (taking a subnet if necessary). By the upper semicontinuity of T at $(\hat{x}_0, \bar{x}_0, \eta_0)$ and the compactness of $T(\hat{x}_0, \bar{x}_0, \eta_0)$ we can suppose that $y_\beta \to \bar{y}_0 \in T(\hat{x}_0, \bar{x}_0, \eta_0)$. By assumption ($\text{iv}_{s\alpha_2}$),

$$F(\bar{y}_0, \hat{x}_0, \bar{x}_0, \mu_0) \subseteq \text{int}G(\bar{y}_0, \bar{x}_0, \bar{x}_0, \mu_0). \tag{2.6}$$

Since F is G-usc at $\big((\bar{y}_0, \hat{x}_0, \bar{x}_0, \mu_0), (\bar{y}_0, \bar{x}_0, \bar{x}_0, \mu_0)\big)$ by (iii_{α_2}), we see a contradiction between (2.5) and (2.6).

Now suppose that $S_{s\alpha_2}$ is not U-Hausdorff-lower-level closed at $(\lambda_0, \mu_0, \eta_0)$, i.e. $\exists B$ (a neighborhood of the origin in X), $\exists(\lambda_\gamma, \mu_\gamma, \eta_\gamma) \to (\lambda_0, \mu_0, \eta_0)$ such that $\forall\gamma, \exists x_{0\gamma} \in S_{s\alpha_2}(\lambda_0, \mu_0, \eta_0) \setminus \text{cl}U$, $x_{0\gamma} \notin S_{s\alpha_2}(\lambda_\gamma, \mu_\gamma, \eta_\gamma) + B$. Since $S_{s\alpha_2}(\lambda_0, \mu_0, \eta_0) \setminus \text{int}U$ is compact, we can assume that $x_{0\gamma} \to x_0 \in S_{s\alpha_2}(\lambda_0, \mu_0, \eta_0) \setminus \text{int}U$. Then there are a neighborhood B_1 of 0 in X with $B_1 + B_1 \subseteq B$ and γ_1 such that, $\forall \gamma \geq \gamma_1$, $\exists b_\gamma \in B_1$, $x_{0\gamma} = x_0 + b_\gamma$. By the preceding part of the proof there is $x_\gamma \in S_{s\alpha_2}(\lambda_\gamma, \mu_\gamma, \eta_\gamma), x_\gamma \to x_0$, and hence one can assume that there is $\gamma_2, \forall \gamma \geq \gamma_2$, $x_\gamma \in x_0 - B_1$, i.e., there exists $b'_\gamma \in B_1, x_\gamma = x_0 - b'_\gamma$. Hence, $\forall \gamma \geq \gamma_0 = \max\{\gamma_1, \gamma_2\}$,

$$x_{0\gamma} = x_0 + b_\gamma = x_\gamma + b'_\gamma + b_\gamma \in x_\gamma + B.$$

This is impossible due to the fact that $x_{0\gamma} \notin S_{s\alpha_2}(\lambda_\gamma, \mu_\gamma, \eta_\gamma) + B$. Thus, $S_{s\alpha_2}$ is U-Hausdorff-lower-level closed at $(\lambda_0, \mu_0, \eta_0)$. □

Corollary 3.3. *Assume assumptions* (i'), (ii'), (iii'$_\alpha$) *and* (iv$_{r\alpha}$) *of Corollary 3.1, assumptions* (ii_h) *and* (iv'$_{r\alpha}$) *of Theorem 3.3 and replace* (i_h) *by*

(i'_{h}) *E is lsc at λ_0 and $E(\lambda_0)$ is compact; $S_1(.,\lambda_0)$ is closed in $E(\lambda_0)$ and $S_2(.,\lambda_0)$ is lsc in $E(\lambda_0)$.*

Then S_α is Hlsc at $(\lambda_0.\mu_0,\eta_0)$.

The following example shows that the compactness of $E(\lambda_0)$ is essential.

Example 3.3. Let $X = R^2, Y = Z = R, \Lambda \equiv M \equiv N = [0,1], K = R^2, D = R$, $\lambda_0 = 0$, $\forall x = (x_1,x_2) \in R^2, S_1(x,\lambda) = S_2(x,\lambda) = \{(x_1,\lambda x_1^2)\}, T(x,y,\lambda) = [0.1]$, $F(y,x,\bar{x},\lambda) = \{1+\lambda^2\}, G(y,x,\bar{x},\lambda) = [0,+\infty)$. Then $E(\lambda) = \{x = (x_1,x_2) \in R^2 \mid x_2 = \lambda x_1^2\}$. So all assumptions of Corollary 3.3 but (i') are satisfied. However, $S_{\mathrm{r}\alpha}(\lambda) = \{x = (x_1,x_2) \in R^2 \mid x_2 = \lambda x_1^2\}$ is not Hlsc at 0 (although $S_{\mathrm{r}\alpha}$ is lsc at 0). The reason is that $E(0) = \{(x_1,x_2) \in R^2 \mid x_2 = 0\}$ is not compact (but $E(.)$ and $S_i(.,.)$ are continuous and closed).

Similarly, we obtain the following result corresponding to Theorems 3.2 and Corollaries 3.2.

Theorem 3.4. *Assume the assumptions of Theorem 3.3 but* (iii), ($\mathrm{iv}_{\mathrm{r}\alpha}$) *and replace* (iii) and ($\mathrm{iv}_{\mathrm{r}\alpha}$) *by* ($\mathrm{v}_\alpha$) *as in Theorem 3.2. Then $S_{\mathrm{r}\alpha}$ is U-Hausdorff-lower-level closed at (λ_0,μ_0,η_0).*

Corollary 3.4. *Assume assumptions* (i'), (ii') *and* (v_α) *as in Corollary 3.2, assumptions* (ii_{h}) *and* (iv'$_{\mathrm{r}\alpha}$) *as in Theorem 3.3 and replace* (i_{h}) *by*

(i'_{h}) *E is lsc at λ_0 and $E(\lambda_0)$ is compact; $S_1(.,\lambda_0)$ is closed in $E(\lambda_0)$ and $S_2(.,\lambda_0)$ is lsc in $E(\lambda_0)$.*

Then $S_{\mathrm{r}\alpha}$ is Hlsc at (λ_0,μ_0,η_0).

Proof. Due to assumptions (i'), (ii') and (v_α) of Corollary 3.2 we see that $S_{\mathrm{r}\alpha}$ is lsc at (λ_0,μ_0,η_0). On the other hand, by assumptions (ii_{h}), (iv'$_{\mathrm{r}\alpha}$) and (i'$_{\mathrm{h}}$) $S_{\mathrm{r}\alpha}(\lambda_0,\mu_0,\eta_0)$ is compact. By virtue of Proposition 2.1 we imply that $S_{\mathrm{r}\alpha}$ is Hlsc at (λ_0,μ_0,η_0). □

Example 3.3 shows also that the compactness of $E(\lambda_0)$ in Corollary 3.4 cannot be dropped. Even for the special case, where our problems are quasiequilibrium problems, Theorems 3.3 and 3.4 and Corollaries 3.3 and 3.4 are new.

4. Upper semicontinuity of the solution sets

Theorem 4.1. *Assume for problem* ($\mathrm{P}_{\mathrm{r}\alpha}$) *that, for $\emptyset \neq U \subseteq X$,*

(i) *$E(.)\cap \mathrm{cl}U$ is usc and $E(\lambda_0)\cap\mathrm{cl}U$ is compact; S_2 is lsc in $(E(\lambda_0)\cap\mathrm{cl}U)\times\{\lambda_0\}$;*

(ii) *in $S_2(E(\lambda_0)\cap\mathrm{cl}U,\lambda_0)\times(E(\lambda_0)\cap\mathrm{cl}U)\times\{\eta_0\}$, T is usc and has compact values if* $\mathrm{r}=\mathrm{w}$*, and lsc if* $\mathrm{r}=\mathrm{s}$;

($\mathrm{iii}_{\mathrm{u}\alpha}$) *$F$ has the $\bar{\alpha}-G$-inclusion property in $\Big(\big(T(S_2(E(\lambda_0)\cap\mathrm{cl}U,\lambda_0),E(\lambda_0)\cap\mathrm{cl}U,\eta_0)\big)\times S_2(E(\lambda_0)\cap\mathrm{cl}U,\lambda_0)\times(E(\lambda_0)\cap\mathrm{cl}U)\times\{\mu_0\}\big),\big(T(S_2(E(\lambda_0)\cap\mathrm{cl}U,\lambda_0),E(\lambda_0)\cap\mathrm{cl}U,\eta_0)\big)\times(E(\lambda_0)\cap\mathrm{cl}U)\times(E(\lambda_0)\cap\mathrm{cl}U)\times\{\mu_0\}\big)\Big)$.*

Then $S_{\mathrm{r}\alpha}$ is U-upper-level closed at (λ_0,μ_0,η_0).

Proof. Similar arguments can be applied to prove the four cases. We present only the proof for the case where r = s and $\alpha = \alpha_1$. Reasoning ad absurdum, suppose the existence of $(\lambda_\gamma, \mu_\gamma, \eta_\gamma) \to (\lambda_0, \mu_0, \eta_0)$ such that, for each γ, $x_\gamma \in S_{s\alpha_1}(\lambda_\gamma, \mu_\gamma, \eta_\gamma) \cap \mathrm{cl}U$ exists but $S_{s\alpha_1}(\lambda_0, \mu_0, \eta_0) \cap \mathrm{cl}U = \emptyset$. Since $E(.) \cap \mathrm{cl}U$ is usc and $E(\lambda_0) \cap \mathrm{cl}U$ is compact one can assume that x_γ tends to some $x_0 \in E(\lambda_0) \cap \mathrm{cl}U$. If $x_0 \notin S_{s\alpha_1}(\lambda_0, \mu_0, \eta_0)$, there exist $\hat{x}_0 \in S_2(x_0, \lambda_0)$ and $y_0 \in T(\hat{x}_0, x_0, \eta_0)$ such that

$$F(y_0, \hat{x}_0, x_0, \mu_0) \cap G(y_0, x_0, x_0, \mu_0) = \emptyset. \tag{2.7}$$

Since $S_2(.,.)$ is lsc at (x_0, λ_0), there is a net $\hat{x}_\gamma \in S_2(x_\gamma, \lambda_\gamma)$, $\hat{x}_\gamma \to \hat{x}_0$. As $x_\gamma \in S_{s\alpha_1}(\lambda_\gamma, \mu_\gamma, \eta_\gamma)$, one has, $\forall y_\gamma \in T(\hat{x}_\gamma, x_\gamma, \eta_\gamma)$,

$$F(y_\gamma, \hat{x}_\gamma, x_\gamma, \mu_\gamma) \cap G(y_\gamma, x_\gamma, x_\gamma, \mu_\gamma) \neq \emptyset. \tag{2.8}$$

Taking the lower semicontinuity of T at $(\hat{x}_0, x_0, \eta_0)$ into account, one has a net $y'_\beta \in T(\hat{x}_\beta, x_\beta, \eta_\beta)$ such that $y'_\beta \to y_0$. By assumption ($\mathrm{iii}_{u\alpha}$) we see a contradiction between (2.7) and (2.8). Thus, $x_0 \in S_{s\alpha_1}(\lambda_0, \mu_0, \eta_0)$. This in turn is also a contradiction, since $x_\beta \in \mathrm{cl}U, \forall \beta$. □

COROLLARY 4.1. *Assume for* ($\mathrm{P}_{r\alpha}$) *that*

(i′) *E is usc and $E(\lambda_0)$ is compact; S_2 is lsc in $E(\lambda_0) \times \{\lambda_0\}$;*

(ii′) *in $S_2(E(\lambda_0), \lambda_0) \times E(\lambda_0) \times \{\eta_0\}$, T is usc and has compact values if* r = w, *and lsc if* r = s;

($\mathrm{iii}'_{u\alpha}$) *F has the $\bar{\alpha}$−G-inclusion property in* $\Big(\big(T(S_2(E(\lambda_0), \lambda_0), E(\lambda_0), \eta_0)\big) \times S_2(E(\lambda_0), \lambda_0) \times E(\lambda_0) \times \{\mu_0\}\big), \big(T(S_2(E(\lambda_0), \lambda_0), E(\lambda_0), \eta_0)) \times E(\lambda_0) \times E(\lambda_0) \times \{\mu_0\}\big)\Big)$.

Then $S_{r\alpha}$ is both usc and closed at $(\lambda_0, \mu_0, \eta_0)$.

Proof. The upper semicontinuity follows immediately from Theorem 4.1 and Propositions 2.1 and 2.2. Arguments similar to that of the second part of the proof of Theorem 4.1 can be used to demonstrate the closedness of $S_{r\alpha}$. □

EXAMPLE 4.1. Let $X = Y = Z = R, K = D = R, \Lambda \equiv M \equiv N = [0, 1]$, $\lambda_0 = 0$, $S_1(x, \lambda) = S_2(x, \lambda) = [0, \lambda], T(x, \bar{x}, \lambda) = [0, e^{x + \sin \lambda}]$, $G(y, x, \bar{x}, \lambda) \equiv R_+$ and

$$F(y, x, \bar{x}, \lambda) = \begin{cases} \{0\}, & \text{if } \lambda = 0, \\ \{\cos^2(x + \lambda)\}, & \text{otherwise.} \end{cases}$$

Then all assumptions of Corollary 4.1 are satisfied and hence this corollary implies the upper semicontinuity of $S_{r\alpha}(.)$ at 0 (in fact $S_{r\alpha} = [0, \lambda], \forall \lambda \in [0, 1]$). But Theorems 3.2 and 3.3 in [3] cannot be applied since F is neither lsc nor usc. Furthermore, in this case assumptions of Corollary 4.1 are checked easier than that of Theorem 3.1 in [4].

EXAMPLE 4.2. Let $X, Y, Z, K, D, \Lambda, M, N, G, \lambda_0$ be as in Example 4.1, $S_1(x,\lambda) = S_2(x,\lambda) = [\lambda, \lambda+1]$, $T(x,\bar{x},\lambda) = \{\bar{x}\}$ and

$$F(y,x,\bar{x},\lambda) = \begin{cases} \{1\}, & \text{if} \quad \lambda = 0, \\ \{2+\sin^2(x+\lambda)\}, & \text{otherwise.} \end{cases}$$

Then all assumptions of Corollary 4.1 are fulfilled. So this corollary derives the upper semicontinuity at 0 of $S_{\mathrm{r}\alpha}$ (in fact $S_{\mathrm{r}\alpha} = [\lambda, \lambda+1], \forall \alpha \in \Lambda$). But Theorems 3.2 and 3.4 in [3] cannot be applied since F is neither lsc nor usc. Furthermore, Theorems 3.1 and 4.1 in [8] cannot either, because F is neither pseudomonotone nor θ-upper-level closed, $\forall \theta > 1$.

Passing to Hausdorff upper-level closedness we see that the assumptions can be weakened correspondingly as follows.

THEOREM 4.2. *Assume for problem* $(\mathrm{P}_{\mathrm{r}\alpha_1})$ *that, for* $\emptyset \neq U \subseteq X$,

$(\mathrm{i_h})$ *$E(.) \cap \mathrm{cl}U$ is Husc and $E(\lambda_0) \cap \mathrm{cl}U$ is compact; S_2 is lsc in $E(\lambda_0) \cap \mathrm{cl}U \times \{\lambda_0\}$;*

$(\mathrm{ii_h})$ *in $S_2(E(\lambda_0) \cap \mathrm{cl}U, \lambda_0) \times (E(\lambda_0) \cap \mathrm{cl}U) \times \{\eta_0\}$, T is Husc and has compact values if* $\mathrm{r} = \mathrm{w}$, *and lsc if* $\mathrm{r} = \mathrm{s}$;

$(\mathrm{iii_{hu\alpha_1}})$ $\forall (y_0, \hat{x}_0, x_0) \in \big(T(S_2(E(\lambda_0) \cap \mathrm{cl}U, \lambda_0), E(\lambda_0) \cap \mathrm{cl}U, \eta_0)\big) \times S_2(E(\lambda_0) \cap \mathrm{cl}U, \lambda_0) \times E(\lambda_0) \cap \mathrm{cl}U$, $\big[\forall (y_\gamma, \hat{x}_\gamma, x_\gamma, \mu_\gamma) \to (y_0, \hat{x}_0, x_0, \mu_0), \forall \gamma,\ \alpha_1\big(F(y_\gamma, \hat{x}_\gamma, x_\gamma, \mu_\gamma), G(y_\gamma, x_\gamma, x_\gamma, \mu_\gamma)\big)\big] \implies \big[\forall B_Y$ *(open neighborhood of* 0 *in* Y*)*, $\alpha_1\big(F(y_0, \hat{x}_0, x_0, \mu_0) + B_Y, G(y_0, x_0, x_0, \mu_0)\big)\big]$;

$(\mathrm{iv_{hu\alpha_1}})$ $\forall B_X$ *(open neighborhood of* 0 *in* X*)*, $\forall x \notin S_{\mathrm{r}\alpha}(\lambda_0, \mu_0, \eta_0) + B_X$, $\exists B_Y$, $(\hat{x}, y)\, \bar{\mathrm{r}}\, S_2(x, \lambda_0) \times T(\hat{x}, x, \eta_0)$, $\bar{\alpha}_1\big(F(y, \hat{x}, x, \mu_0) + B_Y, G(y, x, x, \mu_0)\big)$.

Then $S_{\mathrm{r}\alpha_1}$ is U-Hausdorff-upper-level closed at $(\lambda_0, \mu_0, \eta_0)$.

Proof. We demonstrate the assertion only for $S_{\mathrm{w}\alpha_1}$. Suppose $S_{\mathrm{w}\alpha_1}$ is not U-Hausdorff-upper-level closed at $(\lambda_0, \mu_0, \eta_0)$, i.e., there are $(\lambda_\gamma, \mu_\gamma, \eta_\gamma) \to (\lambda_0, \mu_0, \eta_0)$ and B (open neighborhood of 0 in X) such that $x_\gamma \in S_{\mathrm{w}\alpha_1}(\lambda_\gamma, \mu_\gamma, \eta_\gamma) \cap \mathrm{cl}U$ exists for all γ, but $(S_{\mathrm{w}\alpha_1}(\lambda_0, \mu_0, \eta_0) + B) \cap \mathrm{cl}U = \emptyset$. By the compactness of $E(\lambda_0) \cap \mathrm{cl}U$ and the Hausdorff upper semicontinuity of $E(.) \cap \mathrm{cl}U$ at λ_0, we can assume that $x_\gamma \to x_0$ for some $x_0 \in E(\lambda_0) \cap \mathrm{cl}U$. If $x_0 \notin S_{\mathrm{w}\alpha_1}(\lambda_0, \mu_0, \eta_0) + B$, $(\mathrm{iv_{hu\alpha_1}})$ yields some neighborhood B_Y of 0 in Y and some $\hat{x}_0 \in S_2(x_0, \lambda_0)$ such that, $\forall y_0 \in T(\hat{x}_0, x_0, \eta_0)$,

$$\big(F(y_0, \hat{x}_0, x_0, \mu_0) + B_Y\big) \cap G(y_0, x_0, x_0, \mu_0) = \emptyset. \tag{2.9}$$

The lower semicontinuity of S_2 at (x_0, λ_0) yields $\hat{x}_\gamma \in S_2(x_\gamma, \lambda_\gamma)$ such that $\hat{x}_\gamma \to \hat{x}_0$. Since $x_\gamma \in S_{\mathrm{w}\alpha_1}(\lambda_\gamma, \mu_\gamma, \eta_\gamma)$, there is $y_\gamma \in T(\hat{x}_\gamma, x_\gamma, \eta_\gamma)$ such that

$$F(y_\gamma, \hat{x}_\gamma, x_\gamma, \mu_\gamma) \cap G(y_\gamma, x_\gamma, x_\gamma, \mu_\gamma) \neq \emptyset. \tag{2.10}$$

Since T is H-usc and $T(\hat{x}_0, x_0, \eta_0)$ is compact, one has a subnet $y_\beta \in T(\hat{x}_\beta, x_\beta, \mu_\beta)$ such that $y_\beta \to y_0$ for some $y_0 \in T(\hat{x}_0, x_0, \eta_0)$. Assumption $(\mathrm{iii_{hu\alpha_1}})$ shows a contradiction between (2.9) and (2.10). Thus, $x_0 \in S_{\mathrm{w}\alpha_1}(\lambda_0, \mu_0, \eta_0) + B$. Since $x_\beta \in \mathrm{cl}U, \forall \beta$, we have $x_0 \in \mathrm{cl}U$, contradicting the fact that $\big(S_{\mathrm{w}\alpha_1}(\lambda_0, \mu_0, \eta_0) + B\big) \cap \mathrm{cl}U = \emptyset$. □

COROLLARY 4.2. *Assume assumption* ($\text{iv}_{\text{hu}\alpha_1}$) *of Theorem 4.2 and assume further that*

(i'_{h}) *E is Husc at λ_0 and $E(\lambda_0)$ is compact; S_2 is lsc in $E(\lambda_0) \times \{\lambda_0\}$;*

(ii'_{h}) *in $S_2(E(\lambda_0), \lambda_0) \times E(\lambda_0) \times \{\eta_0\}$, T is Husc and have compact values if* $\mathrm{r} = \mathrm{w}$, *and lsc if* $\mathrm{r} = \mathrm{s}$;

($\text{iii}'_{\text{hu}\alpha_1}$) $\forall (y_0, \hat{x}_0, x_0) \in \big(T(S_2(E(\lambda_0), \lambda_0), E(\lambda_0), \eta_0)\big) \times S_2(E(\lambda_0), \lambda_0) \times E(\lambda_0)$, $\big[\forall (y_\gamma, \hat{x}_\gamma, x_\gamma, \mu_\gamma) \to (y_0, \hat{x}_0, x_0, \mu_0), \forall \gamma,\ \alpha_1\big(F(y_\gamma, \hat{x}_\gamma, x_\gamma, \mu_\gamma), G(y_\gamma, x_\gamma, x_\gamma, \mu_\gamma)\big)\big] \Longrightarrow \big[\forall B_Y$ *(open neighborhood of* 0 *in* Y), $\alpha_1\big(F(y_0, \hat{x}_0, x_0, \mu_0) + B_Y, G(y_0, x_0, x_0, \mu_0)\big)\big]$.

Then $S_{\mathrm{r}\alpha_1}$ is Husc at $(\lambda_0, \mu_0, \eta_0)$.

For the particular case of quasiequilibrium problems, Example 3.2 in [3] show also that assumption ($\text{iv}_{\text{hu}\alpha_1}$) is essential.

5. Special cases. Since our quasivariational inclusion problem includes many problems in optimization as mentioned in Section 1, from the main results of Sections 3 and 4 it is not hard to derive consequences for these particular problems. In this section we discuss in details, as an example, first the quasiequilibrium problem ($\text{QEP}_{\mathrm{r}\alpha}$) stated in Section 1(b). Next we investigate a practical problem of traffic network equilibria, which is a special case of the quasivariational inequality (QVI) encountered in Section 1(d).

5.1. Quasiequilibrium problems. Consider ($\text{QEP}_{\mathrm{r}\alpha}$), which represents four quasiequilibrium problems, of interest for a number of authors while dealing with existence conditions, but studied as far as we know only in [3, 4, 8] for semicontinuity of the solution sets. Let $E(\lambda) = \{x \in D \mid x \in \mathrm{cl}S(x, \lambda)\}$ and $S_{\mathrm{r}\alpha}(\lambda, \mu, \eta)$ be the solution set of ($\text{QEP}_{\mathrm{r}\alpha}$) corresponding to (λ, μ, η).

COROLLARY 5.1. *Assume for* ($\text{QEP}_{\mathrm{r}\alpha}$) *that*

(a) *E is lsc at λ_0; S is usc and has compact values in $E(\lambda_0) \times \{\lambda_0\}$;*

(b) *in $D \times \{\eta_0\}$, T is lsc if* $\mathrm{r} = \mathrm{w}$, *and usc with compact values if* $\mathrm{r} = \mathrm{s}$;

(c) *in $D \times D \times \{\mu_0\}$, F is $(Z \setminus -\mathrm{int}C)$-lsc if* $\mathrm{r} = \mathrm{w}$, *and usc if* $\mathrm{r} = \mathrm{s}$;

(d) $\forall x \in S_{\mathrm{r}\alpha}(\lambda_0, \mu_0, \eta_0)$, $(y, \bar{x}^*)\ \mathrm{r}\ S(\bar{x}, \lambda_0) \times T(\bar{x}, \eta_0)$,

$$\alpha\big(F(\bar{x}^*, y, \mu_0), Z \setminus -\mathrm{cl}C\big).$$

Then $S_{\mathrm{r}\alpha}$ is lsc at $(\lambda_0, \mu_0, \eta_0)$.

Proof. The inclusion follows directly from Corollary 3.1, since (a) - (d) being satisfied imply (i') - (iii'_α), (iv_α) of Corollary 3.1 being fulfilled. □

Corollary 5.1 includes properly Theorem 2.2 of [4]. The properness is shown by the following example.

Example 5.1. Let $X = Z = R, C = R_+, D = R$, $\Lambda \equiv M \equiv N = [0,1], \lambda_0 = 0$, $S(x,\lambda) \equiv [0,1], T(x,\lambda) = \{x\}$ and

$$F(x,y,\lambda) = \begin{cases} \{2\}, & \text{if } \lambda = 0, \\ \{1\}, & \text{if } \lambda \neq 0. \end{cases}$$

Then all assumptions of Corollary 5.1 are clearly satisfied. Direct calculations give $S_{r\alpha}(\lambda) = [0,1]$. However, Theorem 2.2 of [4] is not applicable since F is not lsc when $\mathrm{r} = \mathrm{w}$ and not usc when $\mathrm{r} = \mathrm{s}$.

For the special case where $T(x,\eta) = \{x\}$, problem ($\mathrm{P}_{r\alpha}$) becomes (QEP) and (SQEP) investigated in [3]. This example indicates also that Corollary 5.1 is properly stronger than Theorems 2.1 and 2.3 in [3].

The following result is a direct consequence of Corollary 3.2.

Corollary 5.2. *Assume* (a) *and* (b) *of Corollary 5.1 and replace* (c) *and* (d) *by*

(e) *in $D \times D \times \{\mu_0\}$, F has the C-inclusion property if* $\mathrm{r} = \mathrm{w}$, *and the strict C-inclusion property if* $\mathrm{r} = \mathrm{s}$.

Then $S_{r\alpha}$ is lsc at $(\lambda_0, \mu_0, \eta_0)$.

When $T(x,\eta) = \{x\}$, this corollary collapses to Theorems 2.2 and 2.4 of [3]. Although it is slightly weaker than Theorem 2.1 of [4], its assumptions are easier to be checked as the following example makes it clear.

Example 5.2. Let $X, Z, C, \Lambda, M, N, \lambda_0$ and D be as in Example 5.1. Let $S(x,\lambda) = [0,\lambda]$, $T(x,\lambda) = [0, e^{x+\lambda}]$ and

$$F(x,y,\lambda) = \begin{cases} \{0\}, & \text{if } \lambda = 0, \\ \{\sin^2(x+\lambda)\}, & \text{if } \lambda \neq 0. \end{cases}$$

Then the assumptions of Corollary 5.2 are not hard checked as satisfied but computing the set $U_{r\alpha}$ in [4] is rather difficult.

The results below are followed directly from Corollaries 3.3 and 3.4, respectively.

Corollary 5.3. *Assume for* ($\mathrm{QEP}_{r\alpha}$) *assumptions* (a) - (d) *of Corollary 5.1. Assume further that*

(a′) *$E(\lambda_0)$ is compact;* cl$S(.,\lambda_0)$ *is closed in $E(\lambda_0)$; $S(.,\lambda_0)$ is lsc in $E(\lambda_0)$;*

(b′) *in $E(\lambda_0)$, $T(.,\eta_0)$ is usc with compact values if* $\mathrm{r} = \mathrm{w}$, *and lsc if* $\mathrm{r} = \mathrm{s}$;

(c′) *in $D \times D$, $F(.,.,\mu_0)$ is $-C$-usc if* $\mathrm{r} = \mathrm{w}$, *and $-C$-lsc if* $\mathrm{r} = \mathrm{s}$.

Then $S_{r\alpha}$ is Hlsc at $(\lambda_0, \mu_0, \eta_0)$.

Corollary 5.4. *If we replace assumptions* (c) *and* (d) *of Corollary 5.3 by assumption* (e) *of Corollary 5.2, then $S_{r\alpha}$ is still Hlsc at $(\lambda_0, \mu_0, \eta_0)$.*

Passing to upper semicontinuity we easily derive from Corollary 4.1 the following sufficient condition.

COROLLARY 5.5. *For* ($\mathrm{QEP}_{\mathrm{r}\alpha}$) *assume that*

(a) *E is usc with $E(\lambda_0)$ being compact; S is lsc in $E(\lambda_0) \times \{\lambda_0\}$;*

(b) *in $D \times \{\eta_0\}$, T is usc with compact values if* $\mathrm{r} = \mathrm{w}$, *and lsc if* $\mathrm{r} = \mathrm{s}$;

(c) *in $D \times D \times \{\mu_0\}$, F is $-C$-usc if* $\mathrm{r} = \mathrm{w}$, *and $-C$-lsc if* $\mathrm{r} = \mathrm{s}$;

Then, $S_{\mathrm{r}\alpha}$ is usc and closed at $(\lambda_0, \mu_0, \eta_0)$.

Note that this corollary is stronger than Theorems 3.1, 4.1 of [8] and Theorems 3.2, 3.4 of [3] when applied to the special cases studied there. The following example gives a case where Corollary 5.5 can be employed but the mentioned theorems cannot.

EXAMPLE 5.3. Let $X, Z, \Lambda, M, N, C, D$ and λ_0 be as in Example 5.1. Let $S(x, \lambda) = [\lambda, \lambda + 1]$, $T(x, \lambda) = \{x\}$ and

$$F(x, y, \lambda) = \begin{cases} \{1\}, & \text{if} \quad \lambda = 0, \\ \{2\}, & \text{if} \quad \lambda \neq 0. \end{cases}$$

Then the assumptions of Corollary 5.5 are easily seen satisfied and direct computations give $S_{\mathrm{r}\alpha}(\lambda) = [\lambda, \lambda + 1]$. However, the mentioned theorems are not applicable since F is not pseudomonotone and not α-upper-level closed for $\alpha > 1$.

Note further that for more special case of ($\mathrm{QEP}_{\mathrm{r}\alpha}$) of quasivariational inequalities (similar to (QVI) stated in Section 1(d)), Corollary 5.5 includes Theorems 3.1, 3.2 of [32], Theorem 2.1 of [11], Theorems 2.2, 2.3 of [27] and Theorems 4.1, 4.3 of [28].

For Hausdorff upper semicontinuity we restrict ourselves to the case where $\alpha = \alpha_1$. The following corollary is direct consequence of Corollary 4.2.

COROLLARY 5.6. *For problems* ($\mathrm{QEP}_{\mathrm{r}\alpha_1}$) *assume that*

(a) *E is Husc at λ_0 and $E(\lambda_0)$ is compact; S is lsc in $E(\lambda_0) \times \{\lambda_0\}$;*

(b) *in $E(\lambda_0) \times \{\eta_0\}$, T is Husc with compact values if* $\mathrm{r} = \mathrm{w}$, *and lsc if* $\mathrm{r} = \mathrm{s}$;

(c) $\big[\forall (y_\gamma, x_\gamma, \mu_\gamma) \to (y_0, x_0, \mu_0), \alpha_1(F(x_\gamma, y_\gamma, \mu_\gamma), Z \setminus -\mathrm{int}C)\big] \Longrightarrow \big[\forall B_Z$ *(neighborhood of* 0 *in* Z*)*, $\alpha_1(F(x_0, y_0, \mu_0) + B_Z, Z \setminus -\mathrm{int}C)\big]$;

(d) $\forall B_X$ *(neighborhood of* 0 *in* X*)*, $\forall \bar{x} \notin S_{\mathrm{r}\alpha_1}(\lambda_0, \mu_0, \eta_0) + B_X, \exists B_Z, (y, \bar{x}^*)\,\mathrm{r}\; S(\bar{x}, \lambda_0) \times T(\bar{x}, \eta_0), \bar{\alpha}_1\big(F(\bar{x}^*, y, \mu_0) + B_Z, Z \setminus -\mathrm{int}C\big)\big]$.

Then $S_{\mathrm{r}\alpha_1}$ is Husc at $(\lambda_0, \mu_0, \eta_0)$.

This corollary clearly contains Theorem 3.3 of [3] and Theorem of [4].

5.2. Traffic network problems. [46] began the process of the mathematical study of transportation network problems by introducing the later-so-called Wardrop equilibrium flow and proving basic network principles. [41] made a turning point by proving that the Wardrop equilibria of the network are just the solutions of the variational inequality corresponding to the network problem. [14, 35] considered the case where the travel demands of the network problem depended on the equilibrium flows to meet practical situations. Then the problem corresponded to a quasivariational inequality. In these papers the authors also proposed to consider the case where the travel cost was a multifunction of the path flow. [25, 26] extended the Wardrop traffic equilibrium to this case. Up to now efforts have been devoted to the solution existence. We observe only [1, 5, 6] where Hölder continuity of the unique solution to traffic network problems is established. In this subsection we apply the main results in Sections 3 and 4 to establish sufficient conditions for the semicontinuities of the equilibrium flow of the following traffic network problem proposed in [25, 26], which is a slight generalization of the previous models to meet the practice.

Let a traffic network consist of nodes and links (or arcs). Let $W = (W_1, ..., W_l)$ be the set of origin-destination pairs (O/D pairs for short). Assume that the pair W_j, $j = 1, ..., l$, is connected by a set P_j of paths and P_j contains $r_j \geq 1$ paths. Let $F := (F_1, ..., F_m)$, $m := r_1 + ... + r_l$ stand for the path (vector) flow. Assume that the travel cost of the path s, $s = 1, ..., m$ is a set $T(F, \mu) \subseteq R_+$, where $\mu \in M$ is a perturbing parameter. Then we have a multifunction $T : R^m \times M \to 2^{R^m_+}$ with $T(F, \mu) = (T_1(F, \mu), ..., T_m(F, \mu))$. As proposed in [19], the capacity of the paths is taken into account and this results in the constraint

$$F \in A := \{F \in R^m \mid 0 \leq \gamma_s \leq F_s \leq \Gamma_s, s = 1, ..., m\}.$$

For the case of multivalued costs, the following generalized Wardrop equilibrium was proposed in [25].

Definition 5.1.

(i) A path (vector) flow $\bar{F}$ is called a weak equilibrium (vector) flow if, $\forall W_j, \forall q, s \in P_j, \exists t \in T(\bar{F}, \mu)$,

$$[t_q < t_s] \Longrightarrow [\bar{F}_q = \Gamma_q \quad \text{or} \quad \bar{F}_s = \gamma_s].$$

(ii) A path (vector) flow $\bar{F}$ is said to be a strong equilibrium (vector) flow if (i) is satisfy with $\exists t \in T(\bar{F}, \mu)$ being replaced by $\forall t \in T(\bar{F}, \mu)$.

Assume that the travel demand ρ_j of the O/D pair $W_j, j = 1, ..., l$, depends on the (weak or strong) equilibrium (vector) flow. Assume further that the network suffers a perturbation expressed by a perturbing parameter $\lambda \in \Lambda$. So we have a mapping $\rho : R^m \times \Lambda \to R^l_+$. Let the Kronecker numbers be

$$\phi_{js} = \begin{cases} 1, & \text{if} \quad s \in P_j, \\ 0, & \text{if} \quad s \notin P_j, \end{cases}$$

and

$$\phi = \big(\phi_{js}\big), j = 1, ..., l; s = 1, ..., m.$$

Then, the set of the path (vector) flows satisfying exactly the demands is

$$\big\{F \in R^m \mid F \in A, \phi F = \rho(\bar{F}, \lambda)\big\}.$$

However, we are interested in vector flows satisfying the demands with tolerances as follows. Let $\varepsilon : R^m \to R_+$ be a continuous function. We adopt the set of the feasible path flows as

$$K(\bar{F}, \lambda) := \big\{F \in R^m \mid F \in A, \phi F \in B\big(\rho(\bar{F}, \lambda), \varepsilon(\bar{F})\big)\big\},$$

where $B(\rho, \varepsilon)$ is the closed ball of radius ε and centered at ρ.

Similarly as for the classical case, for the case with multivalued costs it is proved in [25] that a feasible path flow $\bar{F}$ is a weak (or strong) equilibrium flow if and only if $\bar{F}$ is a solution of the case r = w (or r = s, respectively) of the quasivariational inequality

(QVI') Find $\bar{F} \in K(\bar{F}, \lambda)$ such that $(F, \bar{t}) \,\mathrm{r}\, K(\bar{F}, \lambda) \times T(\bar{F}, \mu)$,

$$\langle \bar{t}, F - \bar{F} \rangle \geq 0,$$

which is a special case of (QVI) in Section 1(d).

Let $H(U, V)$ stand for the Hausdorff distance between the two sets U, V in a metric space with the metric d, i.e.

$$H(U, V) = \max\big\{\sup_{u \in U} d(u, V), \sup_{v \in V} d(v, U)\big\},$$

where $d(u, V) = \inf_{v \in V} d(u, v)$ is the distance between u and V. We precede the consideration of the solution set by some auxiliary results. The proof of the first lemma is elementarily technical and is omitted.

Lemma 5.1. *For two balls $B(\rho_1, \varepsilon_1)$ and $B(\rho_2, \varepsilon_2)$ in R^l we have*

$$H\big(B(\rho_1, \varepsilon_1), B(\rho_2, \varepsilon_2)\big) = d(\rho_1, \rho_2) + |\varepsilon_1 - \varepsilon_2|.$$

Lemma 5.2 ([1], Lemma 5.1). *Let I be a $l \times m$ matrix, let $e_1, e_2 \in R^l$ and let S_i be the solution set of the equation $Ix = e_i, i = 1, 2$. Then there exists $\theta = \theta(I) > 0$ such that, $\forall x_1 \in S_1, \exists x_2 \in S_2$,*

$$\|x_1 - x_2\| \leq \theta \|e_1 - e_2\|.$$

Lemma 5.3. *$K(F, \lambda)$ is compact for all F and λ. If $\rho(.,.)$ and $\varepsilon(.)$ are continuous mappings at (F_0, λ_0), then the feasible set $K(.,.)$ is continuous (in the Berge sense) at (F_0, λ_0).*

Proof. Set

$$K_1(F, \lambda) = \big\{F \in R^m \mid \phi F \in B(\rho(F, \lambda), \varepsilon(F))\big\}.$$

Then $K_1(F,\lambda)$ is closed and convex and hence $K(F,\lambda) = K_1(F,\lambda) \cap A$ is compact, since A is compact. Now we compute the Hausdorff distance of $K_1(F_1,\lambda_1)$ and $K_1(F_2,\lambda_2)$. For arbitrary $F \in K_1(F_1,\lambda_1)$, there is $e_1 \in B(\rho(F_1,\lambda_1),\varepsilon(F_1))$ such that $\phi F = e_1$. By Lemma 5.1, $e_2 \in B(\rho(F_2,\lambda_2),\varepsilon(F_2))$ exists such that

$$\|e_1 - e_2\| \le \|\rho(F_1,\lambda_1) - \rho(F_2,\lambda_2)\| + |\varepsilon(F_1) - \varepsilon(F_2)|.$$

Let G be in $K_1(F_2,\lambda_2)$ with $\phi G = e_2$. Lemma 5.2 implies the existence of $\theta = \theta(\phi) > 0$ such that

$$\|F - G\| \le \theta\|e_1 - e_2\| \le \theta\big(\|\rho(F_1,\lambda_1) - \rho(F_2,\lambda_2)\| + |\varepsilon(F_1) - \varepsilon(F_2)|\big).$$

Consequently

$$H\big(K_1(F_1,\lambda_1), K_1(F_2,\lambda_2)\big) \le \theta\|\rho(F_1,\lambda_1) - \rho(F_2,\lambda_2)\| + \theta|\varepsilon(F_1) - \varepsilon(F_2)|.$$

By this, the continuity of ρ and ε at (F_0,λ_0) implies the Hausdorff continuity of $K_1(.,.)$ at (F_0,λ_0). Hence, so is $K(.,.) = K_1(.,.) \cap A$. By the compactness of $K(F_0,\lambda_0)$, $K(.,.)$ is also (Berge) continuous at (F_0,λ_0). □

Lemma 5.4. *If $\rho(.,.)$ and $\varepsilon(.)$ are continuous then $E(.) = \{F \in A \mid F \in K(F,.)\}$ is closed, usc and has compact values.*

Proof. We check that $E(.)$ is closed. We have

$$E(\lambda) = A \cap \{F \in R^m \mid \|\phi F - \rho(F,\lambda) \le \varepsilon(F)\} := A \cap A_1(\lambda).$$

Let $F_n \in E(\lambda_n), \lambda_n \to \lambda_0$ and $F_n \to F_0$. Then $F_0 \in A$ as A is compact. On the other hand, as ρ, ε and ϕ are continuous, passing

$$\|\phi F_n - \rho(F_n,\lambda_n)\| \le \varepsilon(F_n)$$

to the limit one obtains

$$\|\phi F_0 - \rho(F_0,\lambda_0)\| \le \varepsilon(F_0),$$

i.e. $F_0 \in E(\lambda_0)$ and hence $E(.)$ is closed. Since A is compact, $E(.)$ is usc. □

As opposed to the upper semicontinuity, the lower semicontinuity of $E(.)$ is not guaranteed by the continuity of $\rho(.,.)$ and $\varepsilon(.)$ as shown by the following example.

Example 5.4. Let $m = l = 1, \phi = 1, \Lambda = [0,1]$, $A = [0.2]$, $\rho(F,\lambda) = F^2 - 2\lambda^2 - \lambda$ and $\varepsilon(F) = F^2$. Then

$$E(\lambda) = A \cap \{F \in R \mid F \in B(F^2 - 2\lambda^2 - \lambda, F^2)\} := A \cap A_1(\lambda).$$

We compute

$$\begin{aligned} A_1(\lambda) &= \{F \in R \mid |F^2 - F - 2\lambda^2 - \lambda| \le F^2\} \\ &= [-2\lambda^2 - \lambda, -\lambda] \cup [\frac{1}{2} + \lambda, \infty). \end{aligned}$$

Consequently,

$$E(\lambda) = \begin{cases} \{0\} \cup [\frac{1}{2}, 2], & \text{if} \quad \lambda = 0, \\ [\frac{1}{2} + \lambda, 2], & \text{if} \quad \lambda \ne 0. \end{cases}$$

Therefore $E(.)$ is not lsc at 0.

Now we study the semicontinuity of the solution set $S_{\mathrm{r}}(\lambda,\mu)$ of (QVI'), i.e. of the traffic network problem, with respect to (λ,μ).

COROLLARY 5.7. *Assume that*

(a) $\rho(.,.)$ *and* $\varepsilon(.)$ *are continuous in* $R^m\times\{\lambda_0\}$ *and* R^m, *respectively;*

(b) *in* $A\times\{\mu_0\}$, T *is usc with compact values if* $\mathrm{r}=\mathrm{w}$, *and lsc if* $\mathrm{r}=\mathrm{s}$.

Then S_{r} *is usc and closed at* (λ_0,μ_0) *and* $S_{\mathrm{r}}(\lambda_0,\mu_0)$ *is compact.*

Proof. To apply Corollary 4.1 we verify its assumptions. (i') is fulfilled by assumption (a) and Lemmas 5.3, 5.4. (ii') is satisfied by (b). Finally, by the continuity of $\langle .,.\rangle$, if $(t_n,F_n,\bar F_n)\to(t_0,F_0,\bar F_0)$ and $\langle t_n,F_n-\bar F_n\rangle\geq 0$ then $\langle t_0,F_0-\bar F_0\rangle\geq 0$. Hence $(\mathrm{iii}'_{\mathrm{u}\alpha})$ is satisfied.

For the compactness of $S_{\mathrm{r}}(\lambda_0,\mu_0)$ it suffices to prove the closedness of $S_{\mathrm{r}}(\lambda_0,\mu_0)$ (as A is compact). By the similarity we do this only for the case $\mathrm{r}=\mathrm{w}$. Let $\bar F_n\in S_{\mathrm{w}}(\lambda_0,\mu_0),\bar F_n\to\bar F_0$. Then, $\forall F_n\in K(\bar F_n,\lambda_0)$, $\exists\bar t_n\in T(\bar F_n,\mu_0)$,

$$\langle\bar t_n,F_n-\bar F_n\rangle\geq 0. \tag{5.1}$$

By the closedness of $E(\lambda_0),\bar F_0\in E(\lambda_0)$. Suppose to the contrary that $\bar F_0\notin S_{\mathrm{w}}(\lambda_0,\mu_0)$, i.e. $\exists F_0\in K(\bar F_0,\lambda_0),\forall\bar t\in T(\bar F_0,\mu_0)$,

$$\langle\bar t,F_0-\bar F_0\rangle<0. \tag{5.2}$$

By the lower semicontinuity of K (see Lemma 5.3), there exists $\bar F_n\in K(\bar F_n,\lambda_0)$, $F_n\to F_0$. Since $T(.,\mu_0)$ is usc with compact values, we can assume that $\bar t_n\to\bar t_0$, for some $\bar t_0\in T(\bar F_0,\mu_0)$, leading to a contradiction between (5.1) and (5.2) □

REMARK 5.1. Note that for the lower semicontinuity of S_{r} we cannot apply the results in Section 3 as in Corollary 5.7. The reason is that assumption $(\mathrm{iv}_{\mathrm{r}\alpha})$ is not satisfied in this special case (since for each $\bar F\in S_{\mathrm{r}}(\lambda_0,\mu_0)$, we take $F=\bar F\in K(\bar F,\lambda_0)$, then for all $t\in T(\bar F,\mu_0)$, $\langle t,F-\bar F\rangle=0$). Now we derive the lower semicontinuity of S_{r} by using a relaxed assumption of $(\mathrm{iv}_{\mathrm{r}\alpha})$ as follows.

In the sequel we use the following condition $(\mathrm{A_r}),\mathrm{r}\in\{\mathrm{w},\mathrm{s}\}$.

$(\mathrm{A_w})$ $\forall F_1,F_2\in S_{\mathrm{w}}(\lambda_0,\mu_0)$,

$$[\exists t_1\in T(F_1,\mu_0),\langle t_1,F_2-F_1\rangle>0]\Rightarrow[\forall t_2\in T(F_2,\mu_0),\langle t_2,F_2-F_1\rangle\geq 0].$$

$(\mathrm{A_s})$ $\forall F_1,F_2\in S_{\mathrm{s}}(\lambda_0,\mu_0)$,

$$[\forall t_1\in T(F_1,\mu_0),\langle t_1,F_2-F_1\rangle>0]\Rightarrow[\exists t_2\in T(F_2,\mu_0),\langle t_2,F_2-F_1\rangle\geq 0].$$

COROLLARY 5.8. *In addition to the assumptions of Corollary 5.7, assume that* $(\mathrm{A_r})$ *is satisfied. Assume further that*

(c) $\forall\bar F\in S_{\mathrm{r}}(\lambda_0,\mu_0)$, $(F,\bar t\,)\,\mathrm{r}\,\big(S_{\mathrm{r}}(\lambda_0,\mu_0)\setminus\{\bar F\}\big)\times T(\bar F,\mu_0)$,

$$\langle\bar t,F-\bar F\rangle>0.$$

Then $S_{\rm r}$ *is continuous at* (λ_0, μ_0).

Proof. By Corollary 5.7, $S_{\rm r}$ is usc at (λ_0, μ_0). By the similarity we show that $S_{\rm r}$ is lsc at (λ_0, μ_0) only for r = s. Suppose there are $(\lambda_n, \mu_n) \to (\lambda_0, \mu_0)$ and $F_0 \in S_{\rm s}(\lambda_0, \mu_0)$ such that $\forall F_n \in S_{\rm s}(\lambda_n, \mu_n), F_n \not\to F_0$. Let $\bar{F}_n \in S_{\rm s}(\lambda_n, \mu_n)$. Since E is usc and $E(\lambda_0)$ is compact, we can assume that $\bar{F}_n \to \bar{F}_0$, for some $\bar{F}_0 \in E(\lambda_0)$. By Corollary 5.7, $S_{\rm s}$ is closed at (λ_0, μ_0), whence $\bar{F}_0 \in S_{\rm s}(\lambda_0, \mu_0)$. Hence $\bar{F}_0 \neq F_0$. Assumption (c) implies that, $\forall \bar{t} \in T(\bar{F}_0, \mu_0), \forall t \in T(F_0, \mu_0)$,

$$\langle t, \bar{F}_0 - F_0 \rangle > 0,$$

$$\langle \bar{t}, F_0 - \bar{F}_0 \rangle > 0,$$

contradicting assumption $(\mathrm{A_s})$. □

The following assumptions can replace assumptions $(\mathrm{A_r})$ and (c) of Corollary 5.8.

$(\mathrm{A'_w})$ $\forall F_1, F_2 \in S_{\rm w}(\lambda_0, \mu_0)$,

$$[\exists t_1 \in T(F_1, \mu_0), \langle t_1, F_2 - F_1 \rangle \geq 0] \Rightarrow [\forall t_2 \in T(F_2, \mu_0), \langle t_2, F_2 - F_1 \rangle \geq 0].$$

$(\mathrm{A'_s})$ $\forall F_1, F_2 \in S_{\rm s}(\lambda_0, \mu_0)$,

$$[\forall t_1 \in T(F_1, \mu_0), \langle t_1, F_2 - F_1 \rangle \geq 0] \Rightarrow [\exists t_2 \in T(F_2, \mu_0), \langle t_2, F_2 - F_1 \rangle \geq 0].$$

Note that $(\mathrm{A_w})$ ($(\mathrm{A_s})$) is a strong (weak, respectively) quasimonotonicity of T, while $(\mathrm{A'_w})$ ($(\mathrm{A'_s})$) is a strong (weak, respectively) pseudomonotonicity of T.

COROLLARY 5.9. *Corollary 5.8 is still valid if assumption* $(\mathrm{A_r})$ *is replaced by* $(\mathrm{A'_r})$ *and assumption* (c) *by*

(c′) $\forall F_1, F_2 \in S_{\rm r}(\lambda_0, \eta_0), \exists t_2 \in T(F_2, \mu_0), \langle t_2, F_2 - F_1 \rangle = 0 \Longrightarrow F_1 = F_2$;

(d′) $\forall F_1 \in S_{\rm r}(\lambda_0, \mu_0)$, $(F_2, \bar{t}) \, {\rm r} \, S_{\rm r}(\lambda_0, \mu_0) \times T(F_1, \mu_0)$,

$$\langle \bar{t}, F_2 - F_1 \rangle \geq 0.$$

Proof. We prove the case where r = w similarly to the first part of the proof for Corollary 5.8, we have $\bar{F}_0 \in S_{\rm w}(\lambda_0, \mu_0)$ and $\bar{F}_0 \neq F_0$. By (d'), $\exists t \in T(F_0, \mu_0)$,

$$\langle t, \bar{F}_0 - F_0 \rangle \geq 0.$$

By assumption $(\mathrm{A'_w})$, $\forall \bar{t} \in T(\bar{F}_0, \mu_0)$,

$$\langle \bar{t}, \bar{F}_0 - F_0 \rangle \geq 0.$$

By (d'), $\exists \bar{t} \in T(\bar{F}_0, \mu_0)$,

$$\langle \bar{t}, F_0 - \bar{F}_0 \rangle \geq 0,$$

and hence $\langle \bar{t}, F_0 - \bar{F}_0 \rangle = 0$. Assumption (c') implies that $\bar{F}_0 = F_0$, which is impossible. □

In the case where T is a single-valued mapping, Corollary 5.9 improves Theorem 4.1 of [32], since here (c') and (d') need to be fulfilled only at $F \in S_{\rm r}(\lambda_0, \mu_0)$. Even for this special case, Corollary 5.8 is a new one.

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