A CONVOLUTION APPROACH TO MULTIVARIATE BESSEL PROCESES

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October 6, 2006

Abstract

In this paper we introduce and study Bessel processes $\{\mathbf{B}_t^{\mathbf{x}}\}$ which take values in a d-dimensional nonnegative cone $\mathcal{R}^{+d}$ of $\mathcal{R}^d$ and are constructed via the multi-dimensional Kingman convolution . We prove that every d-variate Bessel process is a stationary independent increments-type process. Moreover, a stochastic integral with respect to $\{\mathbf{B}_t^{\mathbf{x}}\}$ with the convergence in distribution is defined.

1 Introduction and prelimilaries.

Let $\mathcal{P}$ denote the class of all p.m.'s on the positive half-line $\mathcal{R}^+$ endowed with the weak convergence and $\circ := *_{1,\beta}$ denote the Kingman convolution (Hankel transforms) which was introduced by Kingman [5] in connection with the addition of independent spherically symmetric random vectors in Euclean n-space. Namely, for each continuous bounded function f on $\mathcal{R}^+$ we write :

$$\int_0^\infty f(x)\mu *_{1,\beta} \nu(dx) = \frac{\Gamma(s+1)}{\sqrt{\pi}\Gamma(s+\frac{1}{2})}\int_0^\infty\int_0^\infty\int_{-1}^1 f((x^2+2uxy+y^2)^{1/2})(1-u^2)^{s-1/2}\mu(dx)\nu(dy)du, \quad (1)$$

where $\mu, \nu \in \mathcal{P}$, $\beta = 2(s+1) \geqslant 1$(cf. Kingman [5] and Urbanik [15]. The Kingman convolution algebra $(\mathcal{P}, \circ)$ is the most important example of Urbanik convolution algebras (cf Urbanik [15]. In the language of Urbanik convolution algebras, the characteristic measure, say σ_s, of the Kingman convolution has the Rayleigh density

$$\sigma_s(x) = \frac{2}{\Gamma(s+1)}(s+1)^{s+1}x^{2s+1}exp(-(s+1)x^2)dx \quad (2)$$

with the characteristic exponent $\varkappa = 2$ and the kernel Λ_s

$$\Lambda_s = \Gamma(s+1)J_s(x)/(1/2x)^s. \quad (3)$$

The *radial characteristic function* (rad.ch.f.) of a p.m. $\mu \in \mathcal{P}$, denoted by $\hat{\mu}(u)$, is defined by

$$\hat{\mu}(u) = \int_0^\infty \Lambda_s(ux)\mu(dx), \quad (4)$$

for every $u \in \mathcal{R}^+$. In particular,the rad. ch.f. of σ_s is

$$\hat{\sigma}_s(u) = exp(-u^2), u \in \mathcal{R}^+. \quad (5)$$

It should be noted that, since the rad.ch.f. is defined uniquely up to the delation mapping $x \to ax, a > 0, x \in \mathcal{R}^+$, the representation (5) of the rad.ch.f. of σ_s may differ from that in Urbanik [15]. It is known (cf.Kingman[5],Theorem 1), that the kernel Λ_s itself is an ordinary ch.f. of a p.m., say G_s, defined on the interval [-1,1] as the following

$$dG_s(\lambda) = \frac{\Gamma(s+1)}{\pi^{\frac{1}{2}}\Gamma(s+\frac{1}{2})}(1-\lambda^2)^{s-\frac{1}{2}}d\lambda \qquad (s \in (-\frac{1}{2}, \infty)) \tag{6}$$
$$G_{-\frac{1}{2}} = \frac{1}{2}(\delta_1 + \delta_{-1}) \qquad (s = -\frac{1}{2}),$$
$$G_\infty = \delta_0 \qquad (s = \infty).$$

Thus if θ_s denotes a r.v. with distribution F_s then for each $t \in \mathcal{R}^+$,

$$\Lambda_s(t) = Eexp(it\theta_s) = \int_{-1}^{1} exp(itx)dG_s(x). \tag{7}$$

Now we quote a definition of a Bessel process in Revus-Yor[12]

Definition 1. *A Bessel process is the square root of the following unique strong solution of the SDE*

$$Z_t = x + 2\int_0^t \sqrt{Z_s}d\beta_s + \beta t, \tag{8}$$

for any $\beta \geqslant 0$ *and* $x \geqslant 0$.

It should be noted that Shiga and Watanabe [14] characterized the Bessel family as one-parameter semigroups of distributions on path spaces $\mathbf{W} = C(\mathcal{R}^+, \mathcal{R})$ which stands for a convolution approach to Bessel processes. Our aim in this paper is to study Bessel processes via the Kingman convolution method. Therefore we will assume that the dimension $\beta \geqslant 1$. Moreover, we will consider the Bessel process started at 0 only and will denote it by $B(t), t \geqslant 0$.

2 The Cartersian product of Kingman convolution algebras

This concept was introduced in Nguyen [10]. Namely, let $\mathcal{P}(\mathcal{R}^{k+})$ denote the class of all p.m.'s on $\mathcal{R}^{k+}$ equipped with the weak convergence. Let $\mathbf{F}_1, \mathbf{F}_2 \in \mathcal{P}(\mathcal{R}^{k+})$ be of the product form

$$\mathbf{F}_i = \tau_i^1 \times ... \times \tau_i^k \tag{9}$$

where $\tau_i^j \in \mathcal{P}$, j=1,2,... and i=1,2. We put

$$\mathbf{F}_1 \bigcirc_{\mathbf{k}} \mathbf{F}_2 = (F_1^1 \circ F_1^2) \times ... \times (F_1^k \circ F_2^k). \tag{10}$$

Since convex combinations of p.m.'s of the form (9) are dense in $\mathcal{P}(\mathcal{R}^{k+})$ the relation (10) can be extended to arbitrary p.m.'s on $\mathcal{P}(\mathcal{R}^{k+})$. For every $\mathbf{F} \in \mathcal{P}(\mathcal{R}^{k+})$ the k-dimensional radial ch.f $\hat{\mathbf{F}}$ is defined by

$$\hat{\mathbf{F}}(\mathbf{t}) = \int_{\mathcal{R}^{k+}} \prod_{j=1}^{k} \Lambda_s(t_j x_j)\mathbf{F}(\mathbf{dx}), \tag{11}$$

Let $\lambda, \lambda_1, ..., \lambda_k$ be i.i.d. r.v's with the common distribution G_s. Let $\mathbf{X} = (X_1, ..., X_k)$ be a $\mathcal{R}^{k+}$-valued random vector with distribution $\mathbf{F}$. Further, suppose that r.v's $\mathbf{X}$ and $\mathbf{\Lambda}$, where $\mathbf{\Lambda} = (\lambda_1, ..., \lambda_k)$, are independent. Set

$$\mathbf{\Lambda X} = \{\lambda_1 X_1, ..., \lambda_k X_k\} \tag{12}$$

and

$$G_s \mathbf{F} \stackrel{d}{=} \mathbf{\Lambda X}. \tag{13}$$

Then, we have

$$\widehat{\mathbf{F}}(\mathbf{y}) = E(e^{i<\mathbf{y}, \mathbf{\Lambda X}>}), \tag{14}$$

where $\mathbf{y} = (y_1, ..., y_k) \in \mathcal{R}^{k+}$ and $<,>$ denotes the inner product in $\mathcal{R}^k$. In fact, we have

$$\begin{aligned} E(e^{i<(\lambda_1 y_1, ..., \lambda_k y_k), \mathbf{X}>}) &= \int_{\mathcal{R}^{k+}} E(e^{i\sum_{j=1}^{k}(y_j x_j \lambda_j} F(d\mathbf{x}) \\ &= \int_{\mathcal{R}^{k+}} \Pi_{j=1}^{k} \Lambda_s(t_j x_j) F(d\mathbf{x}) \\ &= \widehat{\mathbf{F}}(\mathbf{y}). \end{aligned}$$

Thus, $\widehat{\mathbf{F}}(\mathbf{y})$ is an ordinary symmetric k-dimensional ch.f., and hence it is uniformly continuous. The following theorem is a simple consequence of (1.3) and (2.2).

Theorem 2. *The pair $(\mathcal{P}(\mathcal{R}^{k+}, \bigcirc_{\mathbf{k}})$ is a commutative topological semigroup with $\delta_{\mathbf{0}}$ as the unit element. Moreover, the operation $\bigcirc_{\mathbf{k}}$ is distributive w.r.t. convex combinations of p.m.'s $\in \mathcal{P}(\mathcal{R}^{k+})$.*

In the sequel, the pair $(\mathcal{P}(\mathcal{R}^{k+}, \bigcirc_{\mathbf{k}})$ will be called a *k-dimensional Kingman convolution algebra*. For each vector $\mathbf{x} \in \mathcal{R}^{k+}$ the *generalized translation operators* (shortly, g.t.o.'s) $\mathbf{T^x}, \mathbf{x} \in R^{k+}$ acting on the Banach space $C_b(\mathcal{R}^{k+})$ of real bounded continuous functions f are defined, for each $\mathbf{y} \in \mathcal{R}^{k+}$, by

$$\mathbf{T^x} f(\mathbf{y}) = \int_{\mathcal{R}^{k+}} f(\mathbf{u}) \delta_{\mathbf{x}} \bigcirc_{\mathbf{k}} \delta_{\mathbf{y}}(d\mathbf{u}). \tag{15}$$

In terms of these g.t.o.'s the k-dimentional rad. ch.f. of p.m.'s on $\mathcal{R}^{k+}$ can be characterized as the following:

Theorem 3. *A real bounded continuous function f on $\mathcal{R}^{k+}$ is a rad.ch.f. of a p.m., if and only if $f(\mathbf{0}) = 1$ and f is $\{\mathbf{T^x}\}$-nonnegative definite in the sense that for any $\mathbf{x}_1, ..., \mathbf{x}_k \in \mathcal{R}^k$ and $\lambda_1, ..., \lambda_k \in C$*

$$\sum_{i,j=1}^{k} \lambda_i \lambda_j \mathbf{T}^{\mathbf{x}_i} f(\mathbf{x}_j) \geqslant 0. \tag{16}$$

(See [10] for the proof).

Lemma 4. *Every p.m. $\mathbf{F}$ defined on $\mathcal{R}^{k+}$ is uniquely determined by its rad.ch.f. $\widehat{\mathbf{F}}$ and the following formula holds:*

$$\mathbf{F}_1 \widehat{\bigcirc_{\mathbf{k}}} \mathbf{F}_2(\mathbf{t}) = \widehat{\mathbf{F}_1}(\mathbf{y}) \widehat{\mathbf{F}_2}(\mathbf{t}), \tag{17}$$

where $\mathbf{F}_1, \mathbf{F}_2 \in \mathcal{P}(\mathcal{R}^{k+})$ *and* $\mathbf{y} \in \mathcal{R}^{k+}$.

Proof. The formula (17) follows from formulas (1,3,12,14). Now using the formulas (2,3,9) and a theorem of Weber([6],p.394) and integrating the function $\hat{\mathbf{F}}(t_1 u_1, ..., t_k u_k)$,
$t_j, u_j \in \mathcal{R}^+, j = 1, ..., k, k - times$ w.r.t. σ_s, we get

$$\int_{\mathcal{R}^{k+}} \hat{\mathbf{F}}(t_1 u_1, ..., t_k u_k) \sigma_s(du_1) ... \sigma_s(du_k) = \tag{18}$$

$$\int_{\mathcal{R}^+} ... \int_{\mathcal{R}^+} \prod_{j=1}^{k} \Lambda_s(t_j x_j u_j) \mathbf{F}(\mathbf{dx}) \sigma_s(du_1) ... \sigma_s(du_k)$$

$$= \int_{\mathcal{R}^{k+}} \prod_{j=1}^{k} exp\{-t_j^2 x_j^2\} \mathbf{F}(\mathbf{d}x),$$

which, by change of variables $y_j = x_j^2, j = 1, ..., k$ and by the uniqueness of the k-dimensional Laplace transform, implies that $\mathbf{F}$ is uniquely determined by the left-hand side of (18). □

Definition 5. *A distribution* $\mathbf{F}$ *on* $\mathcal{R}^{k+}$ *is said to be Rayleigh, if the* $G_s\mathbf{F}$ *defined by (12) is a k-dimensional symmetric Gaussian p.m.*

The following theorem is obvious and its proof is omitted.

Theorem 6. *A distribution* $\mathbf{F}$ *of a r.v* $\mathbf{X}$ *on* $\mathcal{R}^{k+}$ *is Rayleigh, if and only if for every* $\mathbf{x} \in \mathcal{R}^{k+}$ *the r.v.* $< \mathbf{x}, \mathbf{X} >$ *is one-dimensional Rayleigh.*

It is the same as in the case k=1, the i.d. elements can be defined as the following:

Definition 7. *A p.m.*$\mu \in \mathcal{P}(\mathcal{R}^{k+}$ *is called i.d.if for every natural m there exists a p.m.*μ_m *such that*

$$\mu = \mu_m \bigcirc_{\mathbf{k}} ... \bigcirc_{\mathbf{k}} \mu_m, (m - terms). \tag{19}$$

Moreover, a nonnegative stochastic process $\xi_t, t \in \mathcal{T}$ *is said to be i.d., if each its finite dimensional distribution is i.d.*

Let $ID(\bigcirc_{\mathbf{k}})$ denote the class of all i.d. elements in $(\mathcal{P}(\mathcal{R}^{k+}, \bigcirc_{\mathbf{k}})$. The following theorem is a slight generalization of Theorem 7 in Kingman [5].

Theorem 8. $\mu \in ID(\bigcirc_{\mathbf{k}})$ *if and only if there exist a finite measure M on* $\mathcal{R}^{k+}$ *with the property that* $M(\{\mathbf{0}\}) = 0$ *and for each* $\mathbf{y} = (t^1, ..., t^k) \in \mathbf{R}^{k+}$

$$-log\hat{\mu}(\mathbf{y}) = \int_{\mathcal{R}^{k+}} (1 - \prod_{j=1}^{k} \Lambda_s(< t_j x_j >) \frac{1 + \|\mathbf{x}\|^2}{\|\mathbf{x}\|^2} M(d\mathbf{x}), \tag{20}$$

where the integrand on the right-hand side of (21) is assumed to be

$$\lim_{\|\mathbf{x}\| \to 0} (1 - \prod_{j=1}^{k} \Lambda_s(< t_j x_j >) \frac{1 + \|\mathbf{x}\|^2}{\|\mathbf{x}\|^2} = \sum_{j=1}^{k} t_j^2. \tag{21}$$

In particular, if $M = 0$ *then* μ *becomes a Rayleigh measure with the rad.ch.f.*

$$-log\hat{\mu}(\mathbf{y}) = \sum_{j=1}^{k} \lambda_j t_j^2, \tag{22}$$

for any $\mathbf{y} \in \mathcal{R}^{k+}$ *and* $\lambda_j \geqslant 0, j = 1, ..., k.$

Proof. The proof of the first part of Theorem is a similar to that of Theorem 7 in Kingman [5]. To prove the remainder part we assume that k=2. The proof for the case $k \geqslant 3$ is similar. For $\mathbf{t}, \mathbf{x} \in \mathcal{R}^{2+}$ we put

$$H = H(t_1, t_2, x_1, x_2) := \frac{1 - \Lambda_s(t_1 x_1)\Lambda(t_2 x_2)}{x_1^2 + x_2^2} \tag{23}$$

By virtue of Kingman([5], Formula (24)) and by the series representation of $\Lambda_s(.)$ (Kingman[5], Formula (4))and by the fact that the measure G_s is symmetric on the interval $[-1, 1]$ we have

$$\begin{aligned} 1 - \Lambda_s(t_1 x_1)\Lambda_s(t_2 x_2) &= \int_{-1}^{1} \{1 - \Lambda_s((t_1^2 x_1^2 + t_2^2 x_2^2 + 2u t_1 t_2 x_1 x_2)^{1/2})\} dG_s(u) \\ &= \int_{-1}^{1} (\frac{1}{2}(t_1^2 x_1^2 + t_2^2 x_2^2 + 2u t_1 t_2 x_1 x_2) dG_s(u) - R \\ &= \frac{1}{2}(t_1^2 x_1^2 + t_2^2 x_2^2) - R \end{aligned} \tag{24}$$

where R is given by

$$R = \int_{-1}^{1} \sum_{r=2}^{\infty} (-\frac{1}{2})^r (t_1^2 x_1^2 + t_2^2 x_2^2 + 2u t_1 t_2 x_1 x_2)^r s!/r!(s+r)! dG_s(u).$$

which implies that for fixed t_1, t_2 we have

$$\lim_{(x_1^2 + x_2^2) \to 0} \frac{R}{x_1^2 + x_2^2} = 0. \tag{25}$$

Consequently, such that for any $t_1, t_2 \geqslant 0$

$$\lim_{(x_1^2 + x_2^2) \to 0} \frac{1 - \Lambda_s(t_1 x_1)\Lambda_s(t_2 x_2)}{x_1^2 + x_2^2} = t_2^2 + t_2^2 \tag{26}$$

which proves (21). Now, letting M in (20) tend to measure zero and integrating both sides of (22) w.r.t. $\frac{1 + \|\mathbf{x}\|^2}{\|\mathbf{x}\|^2} M(d\mathbf{x})$ we conclude, by virtue of (25)and (26), that the formula (20) holds. Finally, since every projection of the limit p.m. is Rayleigh, it follows from Theorem Theorem 7 in Kingman [5]that the limit p.m. with rad.ch.f. of the form (22) must be a k-dimensional Rayleigh p.m. □

It is evident, from (22), that μ is Rayleigh in $\mathcal{R}^{k+}$ if and only if for each $\mathbf{y} \in \mathcal{R}^{k+}$ the image of μ under the projection $\Pi_{\mathbf{y}}\mathbf{x} =< \mathbf{x}, \mathbf{y} >$ from $\mathcal{R}^{k+}$ onto $\mathcal{R}^{+}$ is Rayleigh on $\mathcal{R}^{+}$. Hence and by the Cramér property of the Kingman convolution (cf. Urbanik [16]) we have the following theorem:

Theorem 9. *Suppose that $\mu, \nu \in \mathcal{P}(\mathcal{R}^{k+})$ and $\mu \bigcirc_k \nu$ is Rayleigh. Then both of them are Rayleigh.*

3 Multivariate symmetric random walks

Given a p.m. $\mu \in \mathcal{P}$ and n=1,2,... we put, for any $x \in \mathcal{R}^{+}$ and $B \in \mathcal{B}(\mathcal{R}^{+})$, the Borel σ-field of R^{+},

$$P_n(x, E) = \delta_x \circ \mu^{\circ n}(E), \tag{27}$$

here the power is taken in the convolution $\circ$ sense. Using the rad.ch.f. one can show that $\{P_n(x, E)\}$ satisfies the Chapman-Kolmogorov equation and therefore, there exists a homogenuous Markov sequence, say $\{S_n^x\}$, n=0,1,2,..., with $\{P_n(x, E)\}$ as its transition probability. More generaly, we have

Lemma 10. *Suppose that $\{\mu_k, k = 1, 2, ...\}$ is a sequence of p.m's on $\mathcal{R}^{k+}$. For any $0 \leqslant n < m, \mathbf{x} \in \mathcal{R}^{k+}, E \in \mathcal{B}(\mathcal{R}^{k+})$,*

$$P_{n,m}(\mathbf{x}, E) = \delta_{\mathbf{x}} \bigcirc_k \mu_n \bigcirc_k \mu_{n+1} \bigcirc_k ... \bigcirc_k \mu_{m-1}(E). \tag{28}$$

Then, $\{P_{n,m}(\mathbf{x}, E)\}$ satisfies the Chapman-Kolmogorov equation and therefore there exists a Markov sequence $\{\mathbf{X}_n^{\mathbf{x}}\}, n = 0, 1, 2, ...$ with $P_{n,m}(x, E)$ as its transition probability.

Proof. It can be proved by using the rad.ch.f. □

Since σ_s is i.d. w.r.t. the Kingman convolution the family of p.m.'s

$$q(t, x, E) := \sigma_s^{\circ t} \circ \delta_x(E)$$

where $t, x \in R$, E is a Borel subset of $\mathcal{R}^{+}$ and the power is taken in the Kingman convolution sense, satisfies the Chapman-Kolmogorov equation and stands for a transition probability of a homogenuous Markov process

$B_t^x, t, x \in R^+$, such that , with probability 1, its realizations are continuous (cf. Nguyen [8] and Shiga-Wantanabe [14]).

Let H_s be a k-dimensional Rayleigh measure with rad.ch.f.(20) and

$$P(t, \mathbf{x}, E) := \ H_s^t \bigcirc_s \delta_{\mathbf{x}}(E), \tag{29}$$

where $t \geqslant 0, \mathbf{x} \in \mathcal{R}^{k+}, E$ is a Borel subset of $\mathcal{R}^{k+}$ and the power is taken in the sense of convolution $\bigcirc_s$. Then there exists a homogeneous Markov process, denoted by $\{\mathbf{B}_t^{\mathbf{x}}\}$ with values in $\mathcal{R}^{k+}$ and transition probability (28).

Definition 11. *Every Markov process $\{\mathbf{B}_t^{\mathbf{x}}\}$ with transition probability given by (28) is called a k-dimensional Bessel process.*

From the above definition and by (28) we have:

Theorem 12. *The rad.ch.f. of $\{\mathbf{B}_t^{\mathbf{x}}\}, t \geqslant 0$ is of the form*

$$-log \mathbf{E}\Lambda(< \mathbf{y}, \mathbf{B}_t^{\mathbf{x}} > = < \mathbf{y}, \mathbf{x} > t + t \sum_{j=1}^{k} \lambda_j^2 y_j^2, \tag{30}$$

where $\mathbf{y} \in \mathcal{R}^{k+}, \lambda_j \geqslant 0, j = 1, ..., k$ and $t \geqslant 0$.

Suppose that $\mathbf{X}^j = \{X_1^j, X_2^j, ..., X_k^j\}, j = 1, 2$ are $\mathcal{R}^{k+}$-valued independent r.v.'s with the corresponding distributions $\mathbf{F}_j, j = 1, 2$. Put

$$\mathbf{X}^1 \bigoplus \mathbf{X}^2 = \{X_1^1 \oplus X_1^2, ..., X_k^1 \oplus X_k^2\}. \tag{31}$$

Then we get a *k-dimensional radial sum* of r.v.'s. By induction one can define such an operation for a finite number of r.v.'s. It is evident that the radial sum is defined up to distribution of r.v.'s and that the operation $\bigoplus$ is associative.

It is a natural problem to consider the usual multiplication of a $\mathcal{R}^{k+}$-valued r.v. and a nonnegative scalar. It is easy to see that the multiplication is distributive w.r.t. the radial sums defined by (31) which helps us to introduce the following stochastic integral.

Definition 13. *Let $\mathcal{C}$ be a σ-ring of subsets of a set $\mathcal{X}$. A function*

$$\mathbf{M} : \mathcal{C} \to \mathbf{L}^+ := \mathbf{L}^+(\Omega, \mathcal{F}, \mathcal{P}), \tag{32}$$

where $\mathbf{L}^+$ *denotes the class of all nonnegative r.v.'s on* $(\Omega, \mathcal{F}, \mathcal{P})$, *is said to be an* $\bigcirc_k$*-scattered random measure, if*

(i) $\mathbf{M}(\emptyset) = 0$ *(P.1),*

(ii) For any $A, B \in \mathcal{C}, A \cap B = \emptyset$, *then* $\mathbf{M}(A) and \mathbf{M}(B)$ *are independent and*

$$\mathbf{M}(A \cup B) \stackrel{d}{=} \mathbf{M}(A) \bigoplus \mathbf{M}(B)$$

(iii) For any $A_1, A_2, \ldots \in \mathcal{C}$, *the r.v.'s* $\mathbf{M}(A_j), j = 1, 2, \ldots$ *are independent and*

$$\mathbf{M}(\cup_{j=1}^{\infty} A_j) \stackrel{d}{=} \bigoplus_{j=1}^{\infty} \mathbf{M}(A_j), \tag{33}$$

where the series on the right-hand side of (33) is convergent in distribution.

It should be noted that the above definition of $\bigcirc_k$*-scattered random measure* is subject to the equality in probability which, however, can be modified in the same way as Rajput and Rosinski ([11],Lemma 5.1 and Theorem 5.2) so that the new $\bigcirc_k$*-scattered random measure* is defined almost surely. Specificly, we state without proof the above mentioned Lemma used by Rajput and Rosinski.

Lemma 14. *(O. Kallenberg) Let* ξ *and* η' *be random elements defined on the probability space* (Ω, P) *and* (Ω', P'), *and taking values in the spaces S and T, respectively, where S is a separable metric space and T is a Polish space. Assume that* $\xi \stackrel{d}{=} f(\eta')$ *for some Borel measurable function* $f : T \to S$. *Then there exists a random element* $\eta \stackrel{d}{=} \eta'$ *on the ("randomized") probability space* $(\Omega \times [0, 1], P \times Leb)$ *such that* $\eta = f(\eta')$ *a.s.* $P \times Leb$.

It is well known that if $\{W(t)\}, t \in R^+$ is a Wiener process, then there exists a Gaussian stochastic measure $\mathbf{N}(A), A \in \mathcal{B}_0$, where $\mathcal{B}_0$ is the $\sigma-$ring of bounded Borel subsets of $\mathcal{R}^+$ with the property that, for every $t \geqslant 0$, $\mathbf{W}(t) = \mathbf{N}((0, t])$. The same it is also true for Bessel processes. Namely, we get

Theorem 15. *Suppose that* $\{\mathbf{B}_t^{\mathbf{0}}\}$ *is a Bessel process started at* $\mathbf{0}$. *Then there exists a unique* $\bigcirc_k$*-scattered random measure* $\{\mathbf{M}(A)\}, A \in \mathcal{B}_0$, *such that for each* $t \geqslant 0$

$$\mathbf{M}((0, t]) \stackrel{d}{=} \mathbf{B}^{\mathbf{0}}(t). \tag{34}$$

Proof. It is the same as the proof for the case k=1 in Nguyen([10], Theorem 4.2). □

Definition 16. *Let* $\mathbf{M}$ *be a* $\bigcirc_k$*-scattered random measure defined by the equation (33). Then for any* $0 \leqslant s < t$ *the quantities* $\mathbf{M}((s,t])$ *are called* $\bigcirc_k$*-increments of the Bessel process* $\{\mathbf{B}_t^0\}$.

By the same reasoning as in Nguyen ([10],Theorem 4.3) we have

Theorem 17. *Every k-dimensional Bessel process* $\mathbf{B_t^0}, t \geqslant 0$ *is a stationary independent* $\bigcirc_k$*-increments process.*

Now we proceed to construct a new non-linear stochastic integration of a nonnegative function w.r.t. a Bessel process. For simplicity we assume that k=1 and write the Bessel process started at 0 as $B(t), t \geqslant$. Let M denote the ∘-scattered random measure associated with $B(.)$ and let $\mathcal{L}^{2+}[0,T], T > 0$ the Hilbert space of all measurable nonnegative functions f on [0,T] such that

$$\|f\|^2 := \int_0^T f(u)^2 du < \infty. \tag{35}$$

Given a partition $\Pi := \{t_0 = 0 < t_1 < ... < t_N \leqslant T\}$ of an interval $[0,T], T > 0$ we put

$$f_\Pi(t) = \sum_{i=0}^{N} f_{t_i} \chi_{(t_i,t_{i+1}](t)}. \tag{36}$$

Then, the integral $\int_0^T f_\Pi(t) d^\circ B(t)$ is defined as

$$\int_0^1 f_\Pi(t) d^\circ B(t) \stackrel{d}{=} \bigoplus_{i=1}^{N} f_{t_i} B([t_i, t_{i+1}))). \tag{37}$$

The integral $\int_0^T f(t) d^\circ B(t)$ is defined as:

$$\int_0^T \xi(t) d^\circ B(t) = lim_{|\Pi| \to 0} \bigoplus_{i=1}^{N} f_i M(t_i, t_{(i+1)}), \tag{38}$$

where $|\Pi| := max\{t_{i+1} - t_i, i = 0, 1, ...N\}$ and the limit is taken in the distribution sense, provided it exists.

Theorem 18. *For each function* $f \in \mathcal{L}^{2+}[0,T]$ *the integral (36) exists in the convergence in distribution and for any* $\alpha > 0$ *the rad.ch.f. of* $S := \int_0^T \alpha f(u) d^\circ B(u)$

is given by

$$-\log \mathbf{E}\Lambda_s(vS) = v^2 \int_0^T f^2(u)du, \tag{39}$$

$v \geqslant 0.$

Proof. We have

$$\begin{aligned} -\log \mathbf{E}\Lambda_s(v \bigoplus_{i=1}^{N} f_i M(t_i, t_{i+1}) &= v^2 \sum_{i=1}^{N} (t_{i+1} - t_i) f_i^2 \\ &\to v^2 \int_0^T f^2(u)du \end{aligned} \tag{40}$$

which implies the conclusion of the theorem. □

By the above definition and by using the rad.ch.f. we get the following theorem:

Theorem 19. *(i) Let $f_1, f_2 \in \mathcal{L}^{2+}[0,T]$ and $c \geqslant 0$. We have*

$$\int_0^T c d^\circ B(t) = cB(T); \tag{41}$$

(ii) If $supp(f_1) \cap supp(f_2) = \emptyset$, then $\int_0^T f_1(t)d^\circ B(t)$ and $\int_0^T f_2(t)d^\circ B(t)$ are independent and

$$\int_0^T \{f_1(t)(t) + f_2(t)\}d^\circ B(t) = \int_0^T f_1(t)d^\circ B(t) + \int_0^T f_2(t)d^\circ B(t) \tag{42}$$

(iii) (non-linearity) In general

$$\int_0^T \{f_1(t)(t) + f_2(t)\}d^\circ B(t) \neq \int_0^T f_1(t)d^\circ B(t) + \int_0^T f_2(t)d^\circ B(t). \tag{43}$$

iii If $f_n \to f$ in $\mathcal{L}^{2+}[0,T]$, then

$$\int_0^T f_n(t)d^\circ B(t) \to \int_0^T f(t)d^\circ B(t) \tag{44}$$

in distribution.

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