Hölder continuity of the unique solution to quasiequilibrium problems in metric spaces⋆

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Abstract. We extend and sharpen the result in [1] on the Hölder continuity of the solution sets of quasivariational inequalities in Hilbert spaces to the case of quasiequilibrium problems in metric spaces. In particular, we show that under the assumptions ensuring the local Hölder continuity of the solution set, this set is a singleton. Applications in some important problems are also provided.

Key Words. Quasiequilibrium problems, Quasivariational inequalities, Traffic network problems, Hölder continuity, Hölder monotonicity, the solution uniqueness.

1. Introduction

In the theory of stability and sensitivity analysis for optimization - related problems Hölder continuity of solutions plays an important role although there may be

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less works in the literature devoted to this property than to semicontinuity, continuity, Lipschitz continuity and (generalized) differentiability. Hölder continuity is in many aspects a high - level regularity, since it is stronger than semicontinuity and continuity. The well-known Rademacher theorem says that a locally Lipschitz function on finite dimensional space is Fréchet differentiable almost everywhere. On the other hand, Lipschitz continuity is a special case of Hölder continuity, where the Hölder degree is one. So Hölder continuity is more general than Lipschitz continuity and in a sense close to differentiability. For variational inequalities, [33 - 35] establish sufficient conditions for the solution to be unique and Hölder continuous in Hilbert spaces. The subtle technique used there is with a heavy recourse to properties of metric projections in Hilbert spaces and linearity of the canonical pair $\langle ., . \rangle$ involved in the variational inequality setting. Subsequently, this result is successfully generalized to various extends for equilibrium problems in metric spaces [2, 4, 5, 11]. These works constitute also a considerable contribution to the stability study for equilibrium problems, since this research field is rather new. Beside them we observe only [3, 6, 7, 8, 11] which are devoted to various kinds of semicontinuity of solution sets. It is known that equilibrium problems were proposed in [13] as a generalization of variational inequalities and optimization problems and include also many optimization - related problems like the fixed - point and coincidence - point problems, the complementarity problem, the traffic equilibria, the Nash equilibrium. However, in variational inequalities and equilibrium problems, the constraint sets are fixed and hence these mathematical models cannot be employed for problem settings in a number of practical situations. This was first observed in [10] where the authors considered random impulse control problems and needed to use constraint sets depending on the state variables. Formulating these problems similarly as variational inequalities led to quasivariational inequalities. [1] is devoted to extend the Hölder continuity of solutions in [33] for variational inequalities to a corresponding result for quasivariational inequalities in Hilbert spaces, raised from traffic network problems. Since the constraint set here is moving, i.e. it depends on the state variable, the authors have to avoid geometric properties of metric projections used in [33]. Nevertheless, the constraint set of the considered quasivariational inequality expresses the fulfillment of the travel demands in the traffic network and hence bears intrinsic linearity. Also, the quasivariational inequality possesses a linear nature due to the canonical pair $\langle ., . \rangle$ involved in the problem setting. The sophisticated reasoning in [1], based on these specific features of the quasivariational inequality under consideration, cannot be adapted when dealing with the generalized problem which is the quasiequilibrium problem. This motivates our aim of this note: to have recourse to other techniques in order to establish Hölder continuity of the solution sets of quasiequilibrium problems in metric spaces. To illustrate applications of our results we supply their consequences in many important problems in Section 3. In particular, we explain advantages of our theorems, when applied to traffic network problems, over that of [1].

Let, throughout the paper if not otherwise specified, X, Λ and M be metric spaces. Let $A \subseteq X$ be nonempty. The problem under our investigation is as follows. Let $K : A \times \Lambda \to 2^X$ be a multifunction with nonempty values and $f : X \times X \times M \to R$ be a function. For each parameter pair $(\lambda, \mu) \in \Lambda \times M$ consider the following quasiequilibrium problem

(QEP) Find $\bar{x} \in K(\bar{x}, \lambda)$ such that, $\forall y \in K(\bar{x}, \lambda)$,

$$f(\bar{x}, y, \mu) \geq 0.$$

Since the solution existence of quasiequilibrium problems has been investigated intensively and widely so far, see e.g. recent papers [20, 21, 22, 23, 26, 36] and the bibliography therein, we focalize our attention only on stability properties assuming always the solution existence in a neighborhood of the considered pair (λ_0, μ_0).

The layout of the remainder of the paper is as follows. The rest of this section is devoted to recalling notions needed in the sequel. The main results on the Hölder continuity of the solution sets are provided in Section 2. In the last Section 3, applications of our results to various situations are discussed.

Our notations are almost standard. $d(.,.)$ stands for the distance in any metric space (the context makes it clear what space is encountered). $d(x, A)$ is the distance from x to subset A in X. R is the space of real numbers and $R_+ = \{r \in R | r \geq 0\}$. $B(x, r)$ denotes the closed ball of radius $r \geq 0$ and centered at x in a metric space X. int$C$ stands for the interior of a subset C. For a normed space X, X^* is the topological dual and $\langle ., . \rangle$ is the canonical pair.

The following Hölder-related notions are employed in the sequel.

Definition 1.1 (Hölder continuity)

(i) (Classical) For $h > 0$ and $\gamma > 0$, a function $p : X \to R$ is called $h.\gamma$-Hölder in $A \subseteq X$ if, $\forall x_1, x_2 \in A$,

$$|p(x_1) - p(x_2)| \leq h d^{\gamma}(x_1, x_2).$$

(ii) (Classical) For l_1, l_2, α_1, $\alpha_2 > 0$, a multifunction $K : X \times \Lambda \to 2^X$ is said to be $(l_1.\alpha_1, l_2.\alpha_2)$-Hölder in $B \subseteq X \times \Lambda$ if, $\forall (x_1, \lambda_1), (x_2, \lambda_2) \in B$,

$$K(x_1, \lambda_1) \subseteq \big\{x \in X | \, \exists z \in K(x_2, \lambda_2), d(x, z) \leq l_1 d^{\alpha_1}(x_1, x_2) + l_2 d^{\alpha_2}(\lambda_1, \lambda_2)\big\}.$$

(iii) (Cf. [5]) For m, β, $\theta > 0$ and $f : X \times X \times M \to R$, f is termed $m.\beta$-Hölder at $\mu_0 \in M$ θ-relative to $A \subseteq X$ if there is a neighborhood V of μ_0 such that $\forall \mu_1, \mu_2 \in V$, $\forall x, y \in A : x \neq y$,

$$|f(x, y, \mu_1) - f(x, y, \mu_2)| \leq m d^{\beta}(\mu_1, \mu_2) d^{\theta}(x, y).$$

Definition 1.2 (Hölder-related monotonicity). Let $g : X \times X \to R$ be a function.

(i) (See [12]) g is called quasimonotone in $A \subseteq X$ if, $\forall x, y \in A : x \neq y$,

$$[g(x, y) > 0] \Longrightarrow [g(y, x) \leq 0].$$

(ii) (See [4]) For $n, \alpha > 0$, g is termed $n.\alpha-$Hölder strongly pseudomonotone in $A \subseteq X$ if $\forall x, y \in A : x \neq y$,

$$[g(x, y) \geq 0] \Longrightarrow [g(y, x) + n d^{\alpha}(x, y) \leq 0].$$

(iii) (See [4]) For n, $\alpha > 0$, g is said to be $n.\alpha-$Hölder strongly monotone in $A \subseteq X$ if, $\forall x, y \in A : x \neq y$,

$$g(x, y) + g(y, x) + n d^{\alpha}(x, y) \leq 0.$$

Definition 1.3 (Hölder-related monotonicity). Let X be a normed space, $A \subseteq X$ be nonempty, $b : A \to X^*$ is a mapping. The following terminology may be considered a special case of Definition 1.2.

(a) b is said to be quasimonotone in A if, $\forall x, y \in A$,

$$[\langle b(x), y - x\rangle > 0] \Longrightarrow [\langle b(y), x - y\rangle \leq 0].$$

(b) For $n, \alpha > 0$, b is called $n.\alpha$-Hölder strongly pseudomonotone in A if, $\forall x, y \in A$,

$$[\langle b(x), y - x\rangle \geq 0] \Longrightarrow [\langle b(y), x - y\rangle + n\|x - y\|^{\alpha} \leq 0].$$

(c) For $n, \alpha > 0$, b is termed $n.\alpha$-Hölder strongly monotone in A if, $\forall x, y \in A$,

$$\langle b(x) - b(y), y - x\rangle + n\|x - y\|^{\alpha} \leq 0.$$

In this paper we will establish a Hölder continuity of solution sets using the following largest distance notion. For A, $B \subseteq X$ we define

$$\rho(A, B) = \sup_{x\in A,\, y\in B} d(x, y).$$

If A or B is unbounded, then $\rho(A, B) = +\infty$. It is known [29] that solution sets of quasicomplementarity problems are in general unbounded. Hence so are solution sets of quasiequilibrium problems, which are more general problems. To compare with other kinds of distance, recall that the Hausdroff distance of A, B is defined as

$$H(A, B) = \max\big\{\sup_{x\in A} d(x, B); \sup_{y\in B} d(y, A)\big\},$$

and the $r-$Hausdroff distance (see [9]) of A, B is defined by

$$H_r(A, B) = \max\big\{\sup_{x\in A\cap B(0,r)} d(x, B); \sup_{y\in B\cap B(0,r)} d(y, A)\big\},$$

for fixed $r > 0$. It is clear that $H_r(A, B)$ is increasing as r increases and, $\forall r > 0$,

$$\rho(A, B) \geq H(A, B) \geq H_r(A, B).$$

2. Hölder continuity of the solution sets

In the sequel let, for $\lambda \in \Lambda$,

$$E(\lambda) = \big\{x \in X \mid x \in K(x, \lambda)\big\}$$

and, for $(\lambda, \mu) \in \Lambda \times M$, $S(\lambda, \mu)$ be the solution set the corresponding problem (QEP).

Theorem 2.1. *For problem* (QEP) *assume that solutions exist in a neighborhood of the considered point* $(\lambda_0, \mu_0) \in \Lambda \times M$*. Assume further that*

(i) *there are neighborhoods* $U(\lambda_0)$ *of* λ_0 *and* $V(\mu_0)$ *of* μ_0 *such that* f *is* $n_1\delta_1$*-Hölder at* μ_0 θ*-relative to* $E(U(\lambda_0))$ *and,* $\forall x \in E(U(\lambda_0))$, $\forall \mu \in V(\mu_0)$, $f(x, ., \mu)$ *is* $n_2.\delta_2$*-Hölder in* $E(U(\lambda_0))$;

(ii) $\forall \mu \in V(\mu_0)$, $f(., ., \mu)$ *is* $h.\beta$*-Hölder strongly monotone in* $E(U(\lambda_0))$;

(iii) $K(., .)$ *is* $(l_1.\alpha_1, l_2.\alpha_2)-$*Hölder in* $E(U(\lambda_0)) \times \{\lambda_0\}$;

(iv) $\alpha_1\delta_2 = \beta > \theta$ *and* $h > 2n_2 l_1^{\delta_2}$.

Then, for each pair (λ, μ)*, in a neighborhood of* (λ_0, μ_0)*,* (QEP) *has a unique solution* $x(\lambda, \mu)$ *which satisfies the Hölder condition*

$$d\big(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)\big) \leq k_1 d^{\alpha_2\delta_2/\beta}(\lambda_1, \lambda_2) + k_2 d^{\delta_1/(\beta-\theta)}(\mu_1, \mu_2),$$

where k_1 *and* k_2 *are positive constants depending on* $h, \beta, n_1, n_2, \delta_1, \delta_2, \theta$*, etc.*

Proof. Let $\lambda_1, \lambda_2 \in U(\lambda_0)$ and $\mu_1, \mu_2 \in V(\mu_0)$.

Step 1. We prove that, $\forall x(\lambda_1, \mu_1) \in S(\lambda_1, \mu_1)$, $\forall x(\lambda_1, \mu_2) \in S(\lambda_1, \mu_2)$,

$$d_1 := d\big(x(\lambda_1, \mu_1), x(\lambda_1, \mu_2)\big) \leq \left(\frac{n_1}{h - 2n_2 l_1^{\delta_2}}\right)^{1/(\beta-\theta)} d^{\delta_1/(\beta-\theta)}(\mu_1, \mu_2). \tag{2.1}$$

Let $x(\lambda_1, \mu_1) \neq x(\lambda_1, \mu_2)$ (if the equality holds then we are done). Since $x(\lambda_1, \mu_1) \in K(x(\lambda_1, \mu_1), \lambda_1)$, $x(\lambda_1, \mu_2) \in K(x(\lambda_1, \mu_2), \lambda_1)$ and by the Hölder continuity of $K(., \lambda_1)$ there are $x_1 \in K(x(\lambda_1, \mu_1), \lambda_1)$ and $x_2 \in K(x(\lambda_1, \mu_2), \lambda_1)$ such that

$$d\big(x(\lambda_1, \mu_1), x_2\big) \leq l_1 d^{\alpha_1}\big(x(\lambda_1, \mu_1), x(\lambda_1, \mu_2)\big), \tag{2.2}$$

$$d\big(x(\lambda_1,\mu_2),x_1\big) \le l_1 d^{\alpha_1}\big(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2)\big). \tag{2.3}$$

As $x(\lambda_1,\mu_1)$ and $x(\lambda_1,\mu_2)$ are solutions of (QEP), we have

$$f(x(\lambda_1,\mu_1),x_1,\mu_1) \ge 0, \tag{2.4}$$

$$f(x(\lambda_1,\mu_2),x_2,\mu_2) \ge 0. \tag{2.5}$$

On the other hand, assumption (ii) implies that

$$-f(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1) - f(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_1) \ge h d_1^{\beta}.$$

Hence, by (2.4) and (2.5),

$$| \, f(x(\lambda_1,\mu_1),x_1,\mu_1) - f(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1)|$$

$$+ | \, f(x(\lambda_1,\mu_2),x_2,\mu_2) - f(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2)|$$

$$+ | \, f(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2) - f(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_1)| \ge h d_1^{\beta}.$$

Therefore, because of the assumption (i), one has

$$n_2 d^{\delta_2}(x_1,x(\lambda_1,\mu_2)) + n_2 d^{\delta_2}(x_2,x(\lambda_1,\mu_1)) + n_1 d_1^{\theta} d^{\delta_1}(\mu_1,\mu_2) \ge h d_1^{\beta}.$$

This, by (2.2) and (2.3), implies that

$$n_2 l_1^{\delta_2} d_1^{\alpha_1\delta_2} + n_2 l_1^{\delta_2} d_1^{\alpha_1\delta_2} + n_1 d_1^{\theta} d^{\delta_1}(\mu_1,\mu_2) \ge h d_1^{\beta}.$$

Assumption (iv) now yields that

$$d_1^{\beta-\theta} \le \left(\frac{n_1}{h - 2n_2 l_1^{\delta_2}}\right) d^{\delta_1}(\mu_1,\mu_2)$$

and hence (2.1).

Step 2. Now we show that, $\forall x(\lambda_1,\mu_2) \in S(\lambda_1,\mu_2), \forall x(\lambda_2,\mu_2) \in S(\lambda_2,\mu_2)$,

$$d_2 := d\big(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\big) \le \left(\frac{2n_2 l_2^{\delta_2}}{h - 2n_2 l_1^{\delta_2}}\right)^{1/\beta} d^{\alpha_2\delta_2/\beta}(\lambda_1,\lambda_2). \tag{2.6}$$

Let $x(\lambda_1, \mu_2) \neq x(\lambda_2, \mu_2)$. Thanks to (iii) we have $x'_1 \in K(x(\lambda_2, \mu_2), \lambda_1)$ and $x'_2 \in K(x(\lambda_1, \mu_2), \lambda_2)$ such that

$$d\big(x(\lambda_1, \mu_2), x'_2\big) \leq l_2 d^{\alpha_2}(\lambda_1, \lambda_2), \tag{2.7}$$

$$d\big(x(\lambda_2, \mu_2), x'_1\big) \leq l_2 d^{\alpha_2}(\lambda_1, \lambda_2). \tag{2.8}$$

By the Hölder continuity of $K(.,.)$ there are $x''_1 \in K(x(\lambda_1, \mu_2), \lambda_1)$ and $x''_2 \in K(x(\lambda_2, \mu_2), \lambda_2)$, such that

$$d(x'_1, x''_1) \leq l_1 d^{\alpha_1}\big(x(\lambda_1, \mu_2), x(\lambda_2, \mu_2)\big), \tag{2.9}$$

$$d(x'_2, x''_2) \leq l_1 d^{\alpha_1}\big(x(\lambda_1, \mu_2), x(\lambda_2, \mu_2)\big). \tag{2.10}$$

By the definition of (QEP), we have

$$f(x(\lambda_1, \mu_2), x''_1, \mu_2) \geq 0, \tag{2.11}$$

$$f(x(\lambda_2, \mu_2), x''_2, \mu_2) \geq 0. \tag{2.12}$$

It follows from assumption (ii) that

$$-f(x(\lambda_1, \mu_2), x(\lambda_2, \mu_2), \mu_2) - f(x(\lambda_2, \mu_2), x(\lambda_1, \mu_2), \mu_2) \geq h d_2^{\beta}.$$

Due to (2.11) and (2.12), one has

$$\begin{aligned} &| f(x(\lambda_1, \mu_2), x''_1, \mu_2) - f(x(\lambda_1, \mu_2), x'_1, \mu_2)| \\ &+ | f(x(\lambda_1, \mu_2), x'_1, \mu_2) - f(x(\lambda_1, \mu_2), x(\lambda_2, \mu_2), \mu_2)| \\ &+ | f(x(\lambda_2, \mu_2), x''_2, \mu_2) - f(x(\lambda_2, \mu_2), x'_2, \mu_2)| \\ &+ | f(x(\lambda_2, \mu_2), x'_2, \mu_2) - f(x(\lambda_2, \mu_2), x(\lambda_1, \mu_2), \mu_2)| \geq h d_2^{\beta}. \end{aligned}$$

Hence, the Hölder continuity assumptions in (i) of f imply that

$$n_2 d^{\delta_2}(x''_1, x'_1) + n_2 d^{\delta_2}(x'_1, x(\lambda_2, \mu_2)) + n_2 d^{\delta_2}(x''_2, x'_2) + n_2 d^{\delta_2}(x'_2, x(\lambda_1, \mu_2)) \geq h d_2^{\beta}.$$

From (2.7), (2.8), (2.9) and (2.10) we have

$$n_2 l_1^{\delta_2} d_2^{\alpha_1\delta_2} + n_2 l_2^{\delta_2} d^{\alpha_2\delta_2}(\lambda_1,\lambda_2) + n_2 l_1^{\delta_2} d_2^{\alpha_1\delta_2} + n_2 l_2^{\delta_2} d^{\alpha_2\delta_2}(\lambda_1,\lambda_2) \geq h d_2^{\beta}.$$

It follows from assumption (iv) that

$$d_2^{\beta} \leq \left(\frac{2n_2 l_2^{\delta_2}}{h - 2n_2 l_1^{\delta_2}}\right) d^{\alpha_2\delta_2}(\lambda_1,\lambda_2),$$

and then also (2.6).

Step 3. Finally since, $\forall x(\lambda_1,\mu_1) \in S(\lambda_1,\mu_1)$, $\forall x(\lambda_2,\mu_2) \in S(\lambda_2,\mu_2)$,

$$d\big(x(\lambda_1,\mu_1), x(\lambda_2,\mu_2)\big) \leq d_1 + d_2,$$

from (2.1) and (2.6) we get, with $k_1 = \left(\frac{2n_2 l_2^{\delta_2}}{h-2n_2 l_1^{\delta_2}}\right)^{\frac{1}{\beta}}$ and $k_2 = \left(\frac{n_1}{h-2n_2 l_1^{\delta_2}}\right)^{\frac{1}{\beta-\theta}}$,

$$\rho\big(S(\lambda_1,\mu_1), S(\lambda_2,\mu_2)\big) \leq k_1 d^{\alpha_2\delta_2/\beta}(\lambda_1,\lambda_2) + k_2 d^{\delta_1/(\beta-\theta)}(\mu_1,\mu_2). \tag{2.13}$$

Taking $\lambda_2 = \lambda_1$ and $\mu_2 = \mu_1$ we see that the diameter of $S(\lambda_1,\mu_1)$ is 0, i.e. this set is a singleton $\{x(\lambda_1,\mu_1)\}$. $S(\lambda_2,\mu_2)$ is similar. Thus (QEP) has a unique solution in a neighborhood of (λ_0,μ_0) and the Hölder condition in the conclusion of the theorem is satisfied. □

Remark 2.1. In [1], when considering a quasivariational inequality in Hilbert spaces, a special case of our problem (QEP), the main result (Theorem 5.1) is a Hölder property similar to (2.13), but with a r-Hausdorff distance $H_r(A,B)$ replacing our $\rho(A,B)$ (and so weaker than our result). Hence it cannot imply the uniqueness and the Hölder continuity like in our Theorem 2.1.

Now we use other kinds of monotonicity of f to derive the same conclusion as in Theorem 2.1. This result is more suitable than Theorem 2.1 while applied in some cases like quasioptimization problems (see Subsection 3.4).

Theorem 2.2. *Theorem 2.1 is still valid if assumption* (ii) *is replaced by*

(ii') $\forall \mu \in V(\mu_0)$, $-f(.,.,\mu)$ *is quasimonotone in* $E(U(\lambda_0))$ *and* $f(.,.,\mu)$ *is h.β-Hölder strongly pseudomonotone in* $E(U(\lambda_0))$.

Proof. *Step 1.* We can retain (2.2), (2.3), (2.4) and (2.5), which are not related to (ii).

If $f(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1) \geq 0$ then (ii') implies that

$$-f(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_1) \geq hd_1^{\beta}.$$

By (2.5), we have

$$\begin{aligned}&|\, f(x(\lambda_1,\mu_2),x_2,\mu_2) - f(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2)|\\ &+|\, f(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_2) - f(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_1)| \geq hd_1^{\beta}.\\ &n_2 d^{\delta_2}(x_2,x(\lambda_1,\mu_1)) + n_1 d_1^{\theta} d^{\delta_1}(\mu_1,\mu_2) \geq hd_1^{\beta}.\end{aligned}$$

Then (i) and (2.2) imply that

$$n_2 l_1^{\delta_2} d_1^{\alpha_1\delta_2} + n_1 d_1^{\theta} d^{\delta_1}(\mu_1,\mu_2) \geq hd_1^{\beta}$$

and hence, by (iv),

$$d_1 \leq \left(\frac{n_1}{h - n_2 l_1^{\delta_2}}\right)^{\frac{1}{\beta-\theta}} d^{\frac{\delta_1}{\beta-\theta}}(\mu_1,\mu_2). \tag{2.14}$$

If $f(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1) < 0$ then the quasimonotonicity in (ii') implies that

$$f(x(\lambda_1,\mu_2),x(\lambda_1,\mu_1),\mu_1) \geq 0,$$

$$-f(x(\lambda_1,\mu_1),x(\lambda_1,\mu_2),\mu_1) \geq hd_1^{\beta}.$$

Then, similarly as above, (2.4), (i) and (2.3) together imply that

$$n_2 l_1^{\delta_2} d_1^{\alpha_1\delta_2} \geq hd_1^{\beta}.$$

By assumption (iv), one has $n_2 l_1^{\delta_2} d_1^{\beta} \geq h d_1^{\beta}$. So $d_1 = 0$, since $h > n_2 l_1^{\delta_2}$. Hence we also have (2.14).

Step 2. We can repeat the first part of Step 2 in the proof of Theorem 2.1 to have (2.7) - (2.12).

If $f(x(\lambda_1, \mu_2), x(\lambda_2, \mu_2), \mu_2) \geq 0$, it follows from assumption (ii') that

$$-f(x(\lambda_2, \mu_2), x(\lambda_1, \mu_2), \mu_2) \geq h d_2^{\beta}.$$

Hence, (2.12), (i), (2.7) and (2.10) together yield

$$n_2 l_1^{\delta_2} d_2^{\alpha_1 \delta_2} + n_2 l_2^{\delta_2} d^{\alpha_2 \delta_2}(\lambda_1, \lambda_2) \geq h d_2^{\beta}.$$

Now it follows from assumption (iv) that

$$d_2 \leq \left(\frac{n_2 l_2^{\delta_2}}{h - n_2 l_1^{\delta_2}} \right)^{\frac{1}{\beta}} d^{\frac{\alpha_2 \delta_2}{\beta}}(\lambda_1, \lambda_2). \tag{2.15}$$

Similarly if $f(x(\lambda_1, \mu_2), x(\lambda_2, \mu_2), \mu_2) < 0$, we also have (2.15).

Step 3 is the same as for Theorem 2.1. □

As mentioned in Section 1, the solution set of a quasiequilibrium problem is in general unbounded. As opposed to the case of an equilibrium problem, see [4, 5], where monotonicity assumptions imply directly the uniqueness of the solution, here this uniqueness is established due to all assumptions together.

The following two examples show that assumptions (ii) (or (ii')) and (iii) in Theorems 2.1 and 2.2 are essential.

Example 2.1. Let $X = A = R, \mathit{\Lambda} \equiv M = [0,1], K(x,\lambda) = [\lambda, 1], \lambda_0 = 0$ and

$$f(x, y, \lambda) = (\lambda + 1)x(x - y).$$

Then, (i) is fulfilled with $n_1 = \delta_1 = \theta = 1, n_2 = 2$ and $\delta_2 = 1$; (iii) is satisfied with $l_1 = 0, l_2 = \alpha_2 = 1$ and α_1 is arbitrary. Since $l_1 = 0$ and α_1 is arbitrary, (iv) holds. But $S(0) = \{0; 1\}$ and $S(\lambda) = \{1\}, \forall\lambda \in (0, 1]$. So $S(.)$ is even not lsc at 0. The reason is that assumptions (ii) and (ii') in Theorems 2.1 and 2.2 are violated. Indeed, taking $x = 1, y = 0$ we see that, $\forall\lambda \in [0, 1]$, $f(1, 0, \lambda) = \lambda + 1 > 0$ and $f(0, 1, \lambda) = 0$, and hence f is neither strongly monotone nor strongly pseudomonotone.

Example 2.2. Let X, A, Λ, M and λ_0 be as in Example 2.1, $f(x, y, \lambda) = y - x$ and

$$K(x, \lambda) = \begin{cases} \{0\}, & \text{if } \lambda = 0, \\ \{-1, 0, 1\}, & \text{otherwise.} \end{cases}$$

Then, (i) is fulfilled with $n_1 = \theta = 0, \delta_1 = 1$ and (ii') holds with $h = \beta = 1$. But $S(0) = \{0\}$ and $S(\lambda) = \{-1\}, \forall\lambda \in (0, 1]$. Thus $S(.)$ is neither usc nor lsc at 0. The reason is that (iii) is violated. (Although K is pseudoLipschitz. Indeed picking $P = U(0) = (-\frac{1}{2}, \frac{1}{2})$ we have $K(\lambda_1) \cap P \subseteq K(\lambda_2) + B(0, \mid \lambda_1 - \lambda_2 \mid)$.) Hence, assumption (iii) cannot be relaxed to the pseudoHölder property.

Remark 2.2. If $E(U(\lambda_0))$ is bounded we can take, in assumption (i), $\theta = 0$ since $d(x, y) \leq M, \forall x, y \in E(U(\lambda_0))$, for some $M > 0$. Hence the condition $\beta > \theta$ in (iv) can be omitted.

Remark 2.3. If $K(x, \lambda) := K(\lambda)$, the quasiequilibrium problem (QEP) is reduced to the corresponding equilibrium problem. In this special case, assumption (ii) derives the uniqueness of the solution in some neighborhood of the considered point. For this case, Theorems 2.1 and 2.2 improve Theorem 2.2.1 in [2] and Theorem 4.2 in [11], since our assumptions (ii) and (ii') are imposed locally (in $K(U(\lambda_0))$, not globally.

The following example gives a case where our Theorems 2.1 - 2.2 derive the Hölder continuity of the solution but the mentioned papers cannot be employed.

Example 2.3. Let X, A, Λ, M and λ_0 be as in Example 2.1, $K(x,\lambda) = [\lambda+1, 2]$ and $f(x,y,\lambda) = (\lambda+1)x(y^2 - x^2)$.
Then, (i) holds with $n_1 = 8, \delta_1 = \theta = 1$, $n_2 = 16$ and $\delta_2 = 1$. (ii) and (ii') are fulfilled with $h = 2, \beta = 2$. K is $(0.\alpha_1, 1.1)$-Lipschitz. Since $l_1 = 0$ and α_1 is arbitrary, we take $\alpha_1 = 2$ to see that (iv) is satisfied. Hence Theorem 2.1 (or Theorem 2.2) derives the Hölder continuity of the solution around λ_0 (in fact, $S(\lambda) = \{\lambda+1\}, \forall\lambda \in [0,1]$). But f is neither global strongly monotone nor global strongly pseudomonotone. Indeed, let $x = 1, y = -1$, then we see that, $\forall\lambda \in [0,1]$, $f(1,-1,\lambda) = f(-1,1,\lambda) = 0$. Hence the results in [2] and [11] do not work in this case.

3. Applications

Since quasiequilibrium problems contain as special cases many optimization - related problems, including quasivariational inequalities, quasioptimization problems, fixed - point and coincidence - point problems, complementarity problems, vector optimization, Nash equilibria, etc, we can derive from Theorems 2.1 and 2.2 direct consequences for such special cases. We discuss now only some applications of our results.

3.1. Quasivariational inequallities

Let X be a normed space, Λ and M be metric spaces and $A \subseteq X$ be nonempty. Let $K : A \times \Lambda \to 2^X$ be a multifunction and $T : X \times M \to X^*$ be a mapping, with $K(x,\lambda)$ being closed and convex, $\forall(x,\lambda) \in X \times \Lambda$. For each $(\lambda,\mu) \in \Lambda \times M$

consider the quasivariational inequality problem

(QVI) Find $\bar{x} \in K(\bar{x}, \lambda)$ such that, $\forall y \in K(\bar{x}, \lambda)$,

$$\langle T(\bar{x}, \mu), y - \bar{x} \rangle \geq 0.$$

Setting $f(x, y, \mu) = \langle T(x, \mu), y - x \rangle$, (QVI) becomes a case of (QEP).

Corollary 3.1. *For problem* (QVI) *assume that solutions exist in a neighborhood of* (λ_0, μ_0) *and that*

(a) *there are neighborhoods* $U(\lambda_0)$ *of* λ_0 *and* $V(\mu_0)$ *of* μ_0 *such that,* $\forall x \in E(U(\lambda_0))$*,* $T(x, .)$ *is* $n_3.\delta_3$*-Hölder at* μ_0 *and* $T(., .)$ *is bounded in* $E\big(U(\lambda_0)\big) \times V(\mu_0)$*, and* $E\big(U(\lambda_0)\big)$ *is bounded;*

(b) $\forall \mu \in V(\mu_0)$*,* $T(., \mu)$ *is* $h.\beta$*-Hölder strongly monotone in* $E(U(\lambda_0))$*;*

(c) K *is* $(l_1.\alpha_1, l_2.\alpha_2)$*-Hölder in* $E\big(U(\lambda_0)\big) \times \{\lambda_0\}$*;*

(d) $\alpha_1 = \beta$ *and* $h > 2n_2 l_1$*.*

Then, in a neighborhood of (λ_0, μ_0)*, the solution* $x(\lambda, \mu)$ *of* (QVI) *is unique and satisfies the following Hölder condition*

$$d\big(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)\big) \leq k_1 d^{\alpha_2/\beta}(\lambda_1, \lambda_2) + k_2 d^{\delta_3/\beta}(\mu_1, \mu_2),$$

where k_1 *and* k_2 *are positive constants depending on* h, β, n_3, δ_3*, etc.*

Proof. We verify the assumptions of Theorem 2.1. (i) is fulfilled with $n_1 = Nn_3, \delta_1 = \delta_3, \theta = 0, n_2 = M$ and $\delta_2 = 1$, where $N, M > 0$ are such that $\|T(x, \mu)\| \leq M, \forall (x, \mu) \in E(U(\lambda_0)) \times V(\mu_0)$ and $\|x - y\| \leq N, \forall x, y \in E(U(\lambda_0))$.

For (ii) one have, by (b),

$$0 \geq \langle T(x, \mu) - T(y, \mu), y - x \rangle + h\|y - x\|^\beta$$

$$= f(x, y, \mu) + f(y, x, \mu) + h\|y - x\|^{\beta}.$$

(iii) is the same as (c). Finally, for (iv) from (d) one has $\alpha_1\delta_2 = \beta > \theta$ and $h > 2n_2 l_1$, as $\delta_2 = 1$ and $\theta = 0$. □

The following example gives a case where all the assumptions and conclusion of Corollary 3.1 hold with T being not globally bounded.

Example 3.1. Let $X, \Lambda, M,\ \lambda_0$ be as in Example 2.1, $A = [-2, 2]$, $K(x, \lambda) = [\lambda, 1]$ and $T(x, \lambda) = \frac{(1+\lambda)x}{1+x}$.

Then, T is locally bounded in $[0, 1] \times [0, 1]$. (b) holds with $h = \frac{1}{4}$ and $\beta = 2$. (c) is fulfilled with $l_1 = 0, l_2 = \alpha_2 = 1$ and α_1 is arbitrary and hence (d) is also satisfied. So Corollary 3.1 implies the Hölder continuity of the solution at 0 (in fact $x(\lambda) = \lambda$). But T is not globally bounded.

The following result is derived from Theorem 2.2.

Corollary 3.2. *Corollary 3.1 is still valid if assumption* (b) *is replaced by*

(b') $\forall \mu \in V(\mu_0)$, $-T(., \mu)$ *is quasimonotone in* $E(U(\lambda_0))$ *and* $T(., \mu)$ *is* $h.\beta$-*Hölder-strongly pseudomonotone in* $E(U(\lambda_0))$.

The following example yields a case where Corollary 3.2 can be applied but Corollary 3.1 cannot.

Example 3.2. Let X, Λ, M, λ_0 be as Example 3.1, $A = [0, 1]$, $K(x, \lambda) = [\frac{x+\lambda}{16}, \frac{2x+\lambda}{16}]$ and $T(x, \lambda) = \frac{\lambda+1}{1+x}$.

Then all assumptions of Corollary 3.2 are fulfilled with $n_3 = \frac{1}{14}, \delta_3 = 1, \theta = 0, n_2 = 2, \delta_2 = 1, h = \frac{14}{15}, \beta = 1, l_1 = l_2 = \frac{1}{16}, \alpha_1 = \alpha_2 = 1$. Hence Corollary 3.2 implies the Lipschitz continuity of the unique solution (in fact $x(\lambda) = \frac{\lambda}{15}$). While Corollary 3.1 cannot be employed since T is not strongly monotone.

Remark 3.1. Let $\bar{x} = x(\bar{\lambda}, \bar{\mu})$ be the solution of the variational inequality corresponding to (QVI), i.e. when K does not depend on x. Using similar arguments, Corollaries 3.1 and 3.2 can be proved when replacing assumption (c) by the following Aubin property (known also as pseudo-Lipschitz property) of K around $(\bar{x}, \bar{\lambda})$: there exist neighborhoods P of $\bar{x}$, $\mathcal{V}(\bar{\lambda})$ of $\bar{\lambda}$ and $k > 0$ such that

$$K(\lambda_1) \cap P \subseteq K(\lambda_2) + k\|\lambda_1 - \lambda_2\|B,$$

$\forall \lambda_1, \lambda_2 \in \mathcal{V}(\bar{\lambda})$ (see Corollary 3.2 of [5]).

3.2. Traffic network problems

A widely accepted notion of equilibrium flows for transportation network problem was introduced in 1952 by Wardrop [31] together with a basic traffic network principle. Since then, traffic network problems have raised a great interest and much developed in both theory and methodology view points, see e.g. [14, 16, 18, 19, 24, 25, 28, 30, 32]. Several important turning points may be observed as follows. The variational approach to such traffic problems begins with Smith [30], who proved that the Wardrop equilibrium can be expressed in terms of variational inequalities. In [15, 27], because of diverse practical situations, travel demands of transportation networks are proposed to be elastic. Namely, these demands may depend on the equilibrium vector flow. Then Wardrop equilibriums of the network problem are expressed as solutions of the corresponding quasivariational inequality. In [1], the Hölder continuity of the solution sets of such parametric elastic traffic problems is established. However, by a mistake, some assumptions of the main theorem contradict each other and hence they are satisfied in no cases. In this subsection, using Corollaries 3.1 and 3.2 we establish even a stronger Hölder continuity and uniqueness of the solution of the considered elastic traffic problem.

We first describe the elastic traffic netword problem. Let N be the set of nodes, L be that of links (or arcs), $W = (W_1, ..., W_l)$ be the set of origin-destination pairs (O/D pairs for short). Assume that the pair W_j, $j = 1, ..., l$, is connected by a set P_j of paths and P_j contains $r_j \geq 1$ paths. Let $F = (F_1, ..., F_m)$ be the path vector flow, where $m = r_1 + ... + r_l$. Following Giannessi [17] the capacity of these paths must be taken into account in practise. So we assume that the capacity restriction is

$$F \in A := \{F \in R^m : 0 \leq \gamma_s \leq F_s \leq \Gamma_s, s = 1, ..., m\}.$$

Assume further that the travel cost on the path flow $F_s, s = 1, ..., m$, depends on the whole path vector flow F and is $T_s(F) \geq 0$. Then we have the path cost vector $T(F) = (T_1(F), ..., T_m(F))$.

Following Wardrop [31] a path vector flow H is said to be an equilibrium vector flow if $\forall W_j$, if $p \in P_j$ and $s \in P_j$ then

$$[T_p(H) < T_s(H)] \Longrightarrow [H_s = \gamma_s \quad \text{or} \quad H_p = \Gamma_p].$$

Now assume that the perturbation on the traffic expresses by parameter λ of a metric space Λ. Assume further that the travel demand g_j of the O/D pair W_j depends on $\lambda \in \Lambda$ and also on the equilibrium vector flow H as explained in [15, 27]. Denote the travel vector demand by $g = (g_1, ..., g_l)$ and set

$$\phi_{js} = \begin{cases} 1, & \text{if} \quad s \in P_j, \\ 0, & \text{if} \quad s \notin P_j, \end{cases}$$

$$\phi = \{\phi_{js}\}, j = 1, ..., l; s = 1, ..., m.$$

Then the path vector flows meeting the travel demands are called the feasible path vector flows and form the constraint set

$$K(H, \lambda) = \{F \in A \mid \phi F = g(H, \lambda)\}.$$

ϕ is called the O/D pair - path incidence matrix.

Assume further that the path costs are also perturbed, i.e. depend on a perturbation parameter μ of a metric space M: $T_s(F,\mu), s = 1, ..., m$.

Remark 3.2. The above traffic model is formulated in terms of path flow variables. Another way to describe the traffic problem is using link flow variables. But the latter model can be employed only if the travel cost is additive, i.e. any path cost is the sum of the link costs for all the links involved in the path. So the "path model" we use here does not need this additivity.

Our traffic network problem is equivalent to a quasivariational inequality as follows.

Lemma 3.3 (See e.g. [14, 30]). *A path vector flow $H \in K(H,\lambda)$ is an equilibrium flow if and only if it is a solution of the following quasivariational inequality of the form* (QVI) *in Subsection 3.1:*

Find $H \in K(H,\lambda)$ such that, $\forall F \in K(H,\lambda)$,

$$\langle T(H,\mu), F - H\rangle \geq 0,$$

where $X = R^m$.

We need also the following simple assertion

Lemma 3.4 (See Proposition 5.1 of [1]). *Assume that g is $(L_1.\alpha_1, L_2.\alpha_2)$ - Hölder continuous at (x_0,λ_0), i.e., $\exists N(x_0)$ (neighborhood of x_0), $\exists U(\lambda_0)$ (neighborhood of λ_0), $\forall x_1, x_2 \in N(x_0)$, $\forall \lambda_1, \lambda_2 \in U(\lambda_0)$,*

$$\|g(x_1,\lambda_1) - g(x_2,\lambda_2)\| \leq L_1\|x_1 - x_2\|^{\alpha_1} + L_2\|\lambda_1 - \lambda_2\|^{\alpha_2}.$$

Then there exist $l_1, l_2 > 0$ such that $K(.,.)$ is $(l_1.\alpha_1, l_2.\alpha_2)$ - Hölder at (x_0, λ_0).

The following results are implied from Corollaries 3.1 and 3.2.

Corollary 3.5. *Assume that solutions of the parametric elastic traffic network problem exist and that assumptions $(a), (b)$ and (d) of Corollary 3.1 are fulfilled. Replace (c) by*

(c') *g is $(L_1.\alpha_1, L_2.\alpha_2)$ - Hölder continuous in $E\big(U(\lambda_0)\big) \times \{\lambda_0\}$.*

Then, in a neighborhood of (λ_0, μ_0), the solution is unique and satisfies the same Hölder condition as in Corollary 3.1.

Corollary 3.6. *Corollary 3.5 is still valid if assumption* (b) *is replaced by*

(b') *$\forall \mu \in V(\mu_0)$, $-T(.,\mu)$ is quasimonotone in $E(U(\lambda_0))$ and $T(.,\mu)$ is $h.\beta$-Hölder-strongly pseudomonotone in $E(U(\lambda_0))$.*

Remark 3.3. [1] considers the same elastic traffic network, where A is a compact, convex subset containing 0 (this condition is essential in Lemma 5.2). The authors impose assumptions similar to (a), (b) and (c') of Corollary 3.5 (but globally in A, not only in $E(U(\lambda_0))$ and hence the study there cannot be applied in our Example 3.1). Instead of our assumption (d), another technical assumption is imposed, using the lower bound $f_0 > 0$ of the norm $\|F\|$ of all $F \in A \setminus \{0\}$ (see Remark 3.1 in [1]). Then, the following local Hölder continuity of the solution set $S(.,.)$ is proved (see the main result, Theorem 5.1, in [1])

$$H_r(S(\lambda_1, \mu_1), S(\lambda_2, \mu_2)) \leq k_1 d^{\xi}(\lambda_1, \lambda_2) + k_2 d^{\zeta}(\mu_1, \mu_2),$$

where k_1, k_2, ξ and ζ are similar to the corresponding constants of Corollary 3.5. (This continuity cannot imply the solution uniqueness.) However, we can see that there does not exist a case where all the assumptions of this Theorem 5.1 are satisfied, since for the convex set A containing 0, we cannot have $\|F\| \geq f_0 > 0$, $\forall F \in A \setminus \{0\}$.

3.3. A quasioptimization problem

Let X, Λ, M, A and K be as in for (QEP) in Section 1. and $g : A \times M \to R$ be a function. For each $(\lambda, \mu) \in \Lambda \times M$, consider the problem of

(QOP) finding $\bar{x} \in K(\bar{x}, \lambda)$ such that

$$g(\bar{x}, \mu) = \min_{y \in K(\bar{x}, \lambda)} g(y, \mu).$$

Since the constraint set depends on the minimizer $\bar{x}$, this is a quasioptimization problem.

Setting $f(x, y, \mu) = g(y, \mu) - g(x, \mu)$, (QOP) becomes a special case of (QEP).

The following results are derived from Theorem 2.2 (Theorem 2.1 cannot be applied since $f(x, y, \mu) + f(y, x, \mu) = 0, \forall x, y \in A$ and $\mu \in M$).

Corollary 3.7. *Assume for* (QOP) *that solutions exist in a neighborhood of the considered point* $(\lambda_0, \mu_0) \in \Lambda \times M$. *Assume further that*

(a$_2$) *there are neighborhoods* $U(\lambda_0)$ *of* λ_0 *and* $V(\mu_0)$ *of* μ_0 *such that* $g(x, .)$ *is* $N_1.\delta_1$*- Hölder at* μ_0 *in* $E(U(\lambda_0))$ *and,* $\forall \mu \in V(\mu_0)$, $g(., \mu)$ *is* $N_2.\delta_2$*-Hölder in* $E(U(\lambda_0))$;

(b$_2$) $\forall \mu \in V(\mu_0)$, $f(., ., \mu)$ *is* $h.\beta$*-Hölder strongly pseudomonotone in* $E(U(\lambda_0))$, *i.e.* $\forall x, y \in E(U(\lambda_0))$, $\forall \mu \in V(\mu_0)$,

$$[g(y, \mu) - g(x, \mu) \geq 0] \Longrightarrow [g(x, \mu) - g(y, \mu) + h\|x - y\|^{\beta} \leq 0];$$

(c_2) $K(.,.)$ *is* $(l_1.\alpha_1, l_2.\alpha_2)-$*Hölder in* $E(U(\lambda_0)) \times \{\lambda_0\}$;

(d_2) $\alpha_1\delta_2 = \beta$ *and* $h > 2N_2 l_1^{\delta_2}$.

Then, in a neighborhood of (λ_0, μ_0) (QOP) *has a unique solution* $x(\lambda, \mu)$ *which satisfies the Hölder condition*

$$d\big(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)\big) \le k_1 d^{\alpha_2\delta_2/\beta}(\lambda_1, \lambda_2) + k_2 d^{\delta_1/\beta}(\mu_1, \mu_2),$$

where k_1 *and* k_2 *are positive constants depending on* $h, \beta, N_1, N_2, \delta_1, \delta_2,$ *etc.*

Proof. We verify assumptions of Theorem 2.2 with $f(x, y, \mu) = g(y, \mu) - g(x, \mu)$. For assumption (i) one has

$$|f(x, y, \mu_1) - f(x, y, \mu_2)| \le |g(y, \mu_1) - g(y, \mu_2)| + |g(x, \mu_1) - g(x, \mu_2)|$$

$$\le N_1|\mu_1 - \mu_2|^{\delta_1} + N_1|\mu_1 - \mu_2|^{\delta_1} = 2N_1|\mu_1 - \mu_2|^{\delta_1}.$$

$$|f(x, y_1, \mu) - f(x, y_2, \mu)| = |g(y_1, \mu) - g(x, \mu) - g(y_2, \mu) + g(x, \mu)|$$

$$= |g(y_1, \mu) - g(y_2, \mu)| \le N_2|y_1 - y_2|^{\delta_2}.$$

(So (i) is satisfied with $n_1 = 2N_1, n_2 = N_2, \delta_1, \delta_2, \theta = 0$.)

For assumption (ii) we have

$$[f(x, y, \mu) = g(y, \mu) - g(x, \mu) < 0] \Longrightarrow [f(x, y, \mu) = g(x, \mu) - g(y, \mu) > 0],$$

and hence $-f$ is quasimonotone in $E(U(\lambda_0))$.

$$[f(x, y, \mu) = g(y, \mu) - g(x, \mu) \ge 0] \Longrightarrow [g(x, \mu) - g(y, \mu) + h\|x - y\|^\beta \le 0]$$

or, the same, $[f(y, x, \mu) + h\|x - y\|^\beta \le 0]$.

So f is $h.\beta$- Hölder strongly pseudomonotone in $E(U(\lambda_0))$.

Assumptions (iii) and (iv) are the same as (c_2) and (d_2), respectively. □

3.4. A fixed - point problem

Let H, M be Hilbert spaces, $A \subseteq H$ be nonempty convex subset of H and $B : A \times M \to A$ be a mapping. The fixed - point problem under our consideration is, for $\mu \in M$,

(FP) Find $\bar{x} \in A$ such that, $\bar{x} = B(\bar{x}, \mu)$.

This problem is equivalent to the following special case of (QEP)

(EP_1) Find $\bar{x} \in A$ such that, $\forall y \in A$,

$$\langle \bar{x} - B(\bar{x}, \mu), y - \bar{x} \rangle \geq 0.$$

Indeed, if $\bar{x}$ is a solution of (FP), i.e., $\bar{x} = B(\bar{x}, \mu)$ and hence $\langle \bar{x} - B(\bar{x}, \mu), y - \bar{x} \rangle = 0, \forall y \in A$. Thus $\bar{x}$ is a solution of (EP_1). Conversely, if $\bar{x}$ is a solution of (EP_1), taking $y = B(\bar{x}, \mu)$, we see that $\langle \bar{x} - B(\bar{x}, \mu), B(\bar{x}, \mu) - \bar{x} \rangle \geq 0$ and then $\|\bar{x} - B(\bar{x}, \mu)\| = 0$, i.e., $\bar{x}$ is a solution of (FP).

The following results are derived from Corollaries 2.1 and 2.2.

Corollary 3.8. *Assume that* (FP) *has solutions in a neighborhood of* μ_0 *and that*

(a_3) $B(x, .)$ *is* $\mathcal{N}_1.\delta_1$*- Hölder at* μ_0 *in* A*, and* $(x, \mu) \mapsto x - B(x, \mu)$ *is bounded in* A : $\|x - B(x, \mu)\| \leq \mathcal{N}_2, \forall x \in A$ *and* $\mu \in V(\mu_0)$ *(neighborhood of* μ_0*);*

(b_3) $\forall \mu \in V(\mu_0)$*,* $(x, \mu) \mapsto x - B(x, \mu)$ *is* $h.\beta$*-Hölder strongly monotone in* A *for some* $h > 0, \beta > 1$*.*

Then, for each μ*, in a neighborhood of* μ_0*,* (FP) *has a unique solution* $x(\mu)$ *which satisfies the Hölder condition*

$$\|x(\mu_1) - x(\mu_2)\| \leq \mathcal{K} \|\mu_1 - \mu_2\|^{\delta_1/(\beta - 1)},$$

for some $\mathcal{K} > 0$*.*

Proof. Setting $f(x,y,\mu) = \langle x - B(x,\mu), y - x\rangle$, $K(x,\lambda) = A$, we check the assumptions of Theorem 2.1. For Assumption (i) we have

$$|f(x,y,\mu_1) - f(x,y,\mu_2)| = |\langle x - B(x,\mu_1), y - x\rangle - \langle x - B(x,\mu_2), y - x\rangle|$$

$$= |\langle B(x,\mu_2) - B(x,\mu_1), y - x\rangle| \leq |B(x,\mu_2) - B(x,\mu_1)|\|y - x\|$$

$$\leq \mathcal{N}_1\|\mu_1 - \mu_2\|^{\delta_1}\|y - x\|.$$

$$|f(x,y_1,\mu) - f(x,y_2,\mu)| = |\langle x - B(x,\mu), y_1 - x\rangle - \langle x - B(x,\mu), y_2 - x\rangle|$$

$$= |\langle x - B(x,\mu), y_1 - y_2\rangle| \leq \|x - B(x,\mu)\|\|y_1 - y_2\| \leq \mathcal{N}_2\|y_1 - y_2\|.$$

(So (i) holds with $\theta = 1, n_1 = \mathcal{N}_1, n_2 = \mathcal{N}_2, \delta_1$ and $\delta_2 = 1$.)

Assumption (ii) is the same as (b_3). Assumption (iii) holds with $l_1 = l_2 = 0$ and α_l, α_2 are arbitrary, and hence assumption (iv) is fulfilled, since $\beta > 1$. □

Similarly, we have

Corollary 3.9. *Corollary 3.8 is still valid if assumption* (b_3) *is replaced by*

(b'_3) $\forall\mu \in V(\mu_0)$, $(x,\mu) \mapsto x - B(x,\mu)$ *is quasimonotone in* A *and* $(x,\mu) \mapsto x - B(x,\mu)$ *is h.β-Hölder-strongly pseudomonotone in* A.

3.5. A coincidence - point problem

Let H, M be Hilbert space, $A \subseteq H$ be nonempty convex subset of H and $g, h : A \times M \to 2^A$. Consider the coincidence - point problem

(C) Find $(\bar{x}_1, \bar{x}_2) \in A \times A$ such that, $\bar{x}_1 = g(\bar{x}_2, \mu)$ and $\bar{x}_2 = h(\bar{x}_1, \mu)$.

To restate (C) as a particular case of (QEP) we set $\mathcal{H} = H \times H$, $\mathcal{A} = A \times A$, $B : \mathcal{A} \times M \to \mathcal{A}$ and $f : \mathcal{A} \times \mathcal{A} \times M \to R$ being defined as follows. For each $x = (x_1, x_2), y = (y_1, y_2) \in A \times A$,

$$B(x,\mu) = g(x_2,\mu) \times h(x_1,\mu),$$

$$f(x,y,\mu) = \langle x - B(x,\mu), y - x\rangle$$
$$= \langle x_1 - g(x_2,\mu), y_1 - x_1\rangle + \langle x_2 - h(x_1,\mu), y_2 - x_2\rangle.$$

We see that (C) is equivalent to the problem

(EP$_2$) Find $\bar{x} = (\bar{x}_1, \bar{x}_2) \in A \times A$ such that, $\forall y = (y_1, y_2) \in A \times A$,

$$\langle \bar{x}_1 - g(\bar{x}_2,\mu), y_1 - \bar{x}_1\rangle + \langle \bar{x}_2 - h(\bar{x}_1,\mu), y_2 - \bar{x}_2\rangle \geq 0. \tag{3.1}$$

Indeed, if $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is a solution of (C), then $\bar{x}_1 = g(\bar{x}_2,\mu)$ and $\bar{x}_2 = h(\bar{x}_1,\mu)$ and hence for each $y = (y_1, y_2) \in A \times A$,

$$\langle \bar{x}_1 - g(\bar{x}_2,\mu), y_1 - \bar{x}_1\rangle + \langle \bar{x}_2 - h(\bar{x}_1,\mu), y_2 - \bar{x}_2\rangle = 0.$$

Thus, $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is a solution of (EP$_2$). Conversely, if $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is a solution of (EP$_2$), putting $y_1 = g(\bar{x}_2,\mu)$ and $y_2 = h(\bar{x}_1,\mu)$ in (3.1) we obtain $\|\bar{x}_1 - g(x_2,\mu)\| = \|\bar{x}_2 - h(\bar{x}_1,\mu)\| = 0$, i.e., $\bar{x} = x(\bar{x}_1, \bar{x}_2)$ is a solution of (C).

The following results are directly derived from Corollaries 3.8 and 3.9.

Corollary 3.10. *Assume for problem* (C) *that all assumption of Corollary 3.8 are fulfilled. Then, in a neighborhood of μ_0, the coincidence point $(x_1(\mu), x_2(\mu))$ of $g(.,\mu)$ and $h(.,\mu)$ is unique and satisfies the Hölder condition*

$$\|x(\mu_1) - x(\mu_2)\| \leq \mathcal{K}\|\mu_1 - \mu_2\|^{\delta_1/(\beta-1)},$$

for some $\mathcal{K} > 0$.

Corollary 3.11. *Corollary 3.10 is still valid if assumption* (b$_3$) *is replaced by*

(b'$_3$) *$\forall \mu \in V(\mu_0)$, $(x,\mu) \mapsto x - B(x,\mu)$ is quasimonotone in $\mathcal{A}$ and $(x,\mu) \mapsto x - B(x,\mu)$ is h.β-Hölder-strongly pseudomonotone in $\mathcal{A}$.*

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