First and Second-Order Approximations as Derivatives of Mappings in Optimality Conditions for Nonsmooth Vector Optimization*

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Abstract. First and second-order approximations are used to establish both necessary and sufficient optimality conditions for local weakly efficiency and local firm efficiency in nonsmooth vector set-constrained problems. Even continuity and relaxed convexity assumptions are not imposed. Compactness conditions are also relaxed. Examples are provided to show advantages of the presented results over recent existing ones.

Key Words. First and second-order approximations, weak efficiency, firm efficiency, asymptotical pointwise compactness, optimality conditions.

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1. Introduction

Let X and Y be normed spaces and $S \subseteq X$ be nonempty. Let Y be ordered by a convex cone C with nonempty interior. Let $f : X \to Y$ be a mapping. Our goal in this paper is to derive necessary optimality conditions for local weakly efficient points (then also for local efficient and local firm efficient points) and sufficient optimality conditions for local firm efficient points (then for local efficient and local weakly efficient points as well) of the following set-constrained vector optimization problem:

$$\text{Min } f(x), \text{s.t. } x \in S. \tag{1.1}$$

We are interested in both first and second-order conditions. To avoid smoothness assumptions via generalized differentiability, we apply the first-order approximation introduced in [1] and the second-order approximation proposed in [2]. As far as we know, [1]-[4] are the only papers in the literature to employ approximations for considering optimality conditions. [1] uses first-order approximations to study metric regularity and applies approximate subdifferentials proposed by Mordukhovich in [5] and [6] for finite dimensions and by Ioffe in [7] for Banach spaces, to establish optimality conditions. [2] proves only second-order necessary conditions in terms of compact second-order approximations. Second-order approximations of scalar functions and a scalarization technique using support functions are the tools in [3] to establish second-order optimality conditions under strict differentiability and compactness assumptions. In the previous [4], using first and second-order approximations we derive both necessary and sufficient conditions for ideal, weak and Pareto efficiencies in unconstrained and constrained problems of the form (1.1). In this work we improve and sharpen the necessary conditions obtained in [4] and provide new sufficient conditions for firm efficiency (see definition in [8]).

It should be noted that a vast range of generalized differentiability constructions have been developed for studying nonsmooth optimization-related problems in general, and optimality conditions in particular. the reader is referred to [9]-[11], three recent excellent books, for systematic expositions of the subject, and also to [12], [13] for deep and detailed discussions of the issue. The reason for us to utilize first and second-order approximations is that their definitions are very simple and even discontinuous mappings may have second-order approximations. We do not think that these approximations are ones of the most powerful stools for nonsmooth analysis but we can motivate our choice by showing advantages of our results over several recent papers in many situations provided by examples.

The rest of the paper is organized as follows. Section 2 presents basic definitions and preliminaries for the later use. In Section 3 we derive first-order optimality conditions. The final Section 4 is devoted to second-order optimality conditions.

Our notations is basically standard. $\|.\|$ is used for norms in any normed spaces (because of the context no confusion occurs). $\mathbb{N} = \{1, 2, ...n, ...\}$. For normed spaces X and Y, X^* is the topological dual of X; $\langle ., . \rangle$ means the canonical pairing; $B_X(x, r)$ is the open ball in X of radius r and centered at x; $L(X, Y)$ denotes the space of the continuous linear mappings of X into Y and $B(X, X, Y)$ denotes the space of the continuous bilinear mappings of $X \times X$ into Y. For a cone $C \subseteq X$, C^* stands for the positive polar cone of C. For $A \subseteq L(X, Y)$ and $x \in X$ ($B \subseteq B(X, X, Y)$ and $x, z \in X$), $A(x) := \{M(x) : M \in A\}$

($B(x,z) := \{N(x,z) : N \in B\}$, respectively). $o(t^k)$ for $t > 0$ and $k \in \mathbb{N}$ is used to denote a moving point such that $o(t^k)/t^k \to 0$ as $t \to 0^+$. $C^{0,1}$ stands for the space of the locally Lipschitz mappings and $C^{1,1}$ for the space of the Fréchet differentiable mappings whose Fréchet derivatives are locally Lipschitz. For a subset $A \subseteq X$, int A, cl A, bd A and co$A$ denote the interior, closure, boundary and convex hull of A, respetively; cone$A$ and spand$A$ are the cone generated by A and linear hull of A, respectively, i.e.

$$\text{cone}A = \{\lambda a : \lambda \geq 0, a \in A\},$$
$$\text{spand}A = \{\textstyle\sum_{i=1}^n \lambda_i a_i : \lambda_i \in \mathbb{R}, a_i \in A, n \in \mathbb{N}\}.$$

2. Preliminaries

Throughout the paper X and Y are normed spaces if not otherwise stated.

Definition 2.1 [1, 2]. Let $x_0 \in X$ and $g : X \to Y$.

(i) The set $A_g(x_0) \subseteq L(X,Y)$ is said to be a first-order approximation of g at x_0 if there exists a neighborhood U of x_0 such that, for all $x \in U$,

$$g(x) - g(x_0) \in A_g(x_0)(x - x_0) + o(\|x - x_0\|).$$

(ii) A set $(A_g(x_0), B_g(x_0)) \subseteq L(X,Y) \times B(X,X,Y)$ is called a second-order approximation of g at x_0 if

(a) $A_g(x_0)$ is a first-order approximation of g at x_0;

(b) $g(x) - g(x_0) \in A_g(x_0)(x - x_0) + B_g(x_0)(x - x_0, x - x_0) + o(\|x - x_0\|^2)$.

Assume now that $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $g \in C^{0,1}$ and $\partial_C g(x_0)$ ($g \in C^{1,1}$ and $\partial_C^2 g(x_0)$) be the Clarke (generalized) Jacobian of g at x_0, see [14] (the Clarke (generalized) Hessian of g at x_0, see [15], respectively). For $g : \mathbb{R}^n \to \mathbb{R}^m$ being continuous, let $\partial g(x_0)$ be an approximate Jacobian of g at x_0, see [16] and for g being continuously Fréchet differentiable, let $\partial^2 g(x_0)$ be an approximate Hessian of g at x_0, see [17]. We have the following relations between the above-mentioned generalized derivatives.

Proposition 2.2 [1, 2].

(i) *If $g : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz at x_0 then $\partial_C g(x_0)$ is a first-order approximation of g at x_0.*

(ii) *If $g : \mathbb{R}^n \to \mathbb{R}^m$ is in $C^{1,1}$ at x_0 then $(g'(x_0), \frac{1}{2}\partial_C^2 g(x_0))$ is a second-order approximation of g at x_0.*

Proposition 2.3 [4].

(i) *If $g : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and has an approximate Jacobian mapping $\partial g(.)$ which is upper semicontinuous at x_0, then $\text{co}\partial g(x_0)$ is a first-order approximation of g at x_0.*

(ii) *If $g : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable in a neighborhood U of x_0 and has an approximate Hessian mapping $\partial^2 g(.)$ which is upper semicontinuous at x_0, then $(g'(x_0), \frac{1}{2}\text{co}\partial^2 g(x_0))$ is a second-order approximation of g at x_0.*

As shown by Examples 2.1-2.5 in [4], the converse of Proposition 2.3 does not hold and under the assumptions of this proposition one still has other approximations in addition to the mentioned one.

In the sequel the following relaxed compactness notions will be employed.

Definition 2.4

(i) Let f_α and f be in $L(X,Y)$. The net f_α is said to pointwisely converge to f and written as $f_\alpha \xrightarrow{\text{p}} f$ or $f = \text{p-lim}\, f_\alpha$ if $\lim f_\alpha(x) = f(x)$ for all $x \in X$. A similar definition is adopted for $f_\alpha, f \in B(X,X,Y)$.

(ii) A subset $A \subseteq L(X,Y)$ ($B \subseteq B(X,X,Y)$, respectively) is called asymptotically pointwisely compact, or asymptotically p-compact if

(a) each bounded net $(f_\alpha) \subseteq A$ ($\subseteq B$, respectively) has a subnet (f_β) and $f \in L(X,Y)$ ($f \in B(X,X,Y)$, respectively) such that $f = \text{p-lim}\, f_\beta$,

(b) each net $(f_\alpha) \subseteq A$ ($\subseteq B$, respectively) with $\lim \|f_\alpha\| = \infty$, the net $(f_\alpha/\|f_\alpha\|)$ has a subnet which pointwisely converges to some $f \in L(X,Y) \setminus \{0\}$ ($f \in B(X,X,Y) \setminus \{0\}$, respectively).

(iii) If in (ii), pointwise convergence, i.e. p-lim, is replaced by convergence, i.e. lim, a subset $A \subseteq L(X,Y)$ (or $B \subseteq B(X,X,Y)$) is called asymptotically compact.

Note that the asymptotical p-compactness in Definition 2.4 is equivalent to the relative p-compactness and the asymptotical p-compactness together defined in [4]. Note also that the asymptotical compactness, corresponding to asymptotical p-compactness in [4], was introduced in [18].

For $A \subseteq L(X,Y)$ and $B \subseteq B(X,X,Y)$ we adopt the notations:

$$\text{p-cl}\, A = \{f \in L(X,Y) : \exists (f_\alpha) \subseteq A, f = \text{p-lim}\, f_\alpha\}, \tag{2.1}$$

$$\text{p-cl}\, B = \{g \in B(X,X,Y) : \exists (g_\alpha) \subseteq B, g = \text{p-lim}\, g_\alpha\}, \tag{2.2}$$

$$A_\infty = \{f \in L(X,Y) : \exists (f_\alpha) \subseteq A, \exists t_\alpha \to 0^+, f = \lim t_\alpha f_\alpha\}, \tag{2.3}$$

$$\text{p-}A_\infty = \{f \in L(X,Y) : \exists (f_\alpha) \subseteq A, \exists t_\alpha \to 0^+, f = \text{p-lim}\, t_\alpha f_\alpha\}, \tag{2.4}$$

$$\text{p-}B_\infty = \{g \in B(X,X,Y) : \exists (g_\alpha) \subseteq B, \exists t_\alpha \to 0^+, g = \text{p-lim}\, t_\alpha g_\alpha\}. \tag{2.5}$$

The sets (2.1), (2.2) are pointwise closures; (2.3) is a recession cone; (2.4), (2.5) are pointwise recession cones.

Remark 2.5

(i) In finite dimensions a convergence occurs if and only if the corresponding pointwise convergence does, but in infinite dimensions the "if" does not hold, see [4, Example 3.1].

(ii) In finite dimensions every subset is asymptotically p-compact and asymptotically compact but in infinite dimensions the asymptotical compactness is stronger, as shown by [4, Example 3.2].

(iii) Assume that X is a Banach space. If $x_\alpha \to x$ in X and $A_\alpha \xrightarrow{\text{p}} A$ in $L(X,Y)$, then $A_\alpha x_\alpha \to Ax$ in Y. Similarly, if $x_\alpha \to x$, $y_\alpha \to y$ in X and $B_\alpha \xrightarrow{\text{p}} B$ in $B(X,X,Y)$, then

$B_\alpha(x_\alpha, y_\alpha) \to B(x, y)$ in Y.

In the sequel we will use the following tangent sets.

Definition 2.6. Let $x_0, v \in X$ and $S \subseteq X$.

(a) The contingent (or Bouligand) cone of S at x_0 [19] is

$$T(S, x_0) = \{v \in X : \exists t_n \to 0^+, \exists v_n \to v, \forall n \in \mathbb{N}, x_0 + t_n v_n \in S\}.$$

(b) The interior tangent (or Dubovitskii-Milyutin) cone of S at x_0 [20] is

$$IT(S, x_0) = \{v \in X : \exists \delta > 0, \forall t \in (0, \delta), \forall u \in B_X(v, \delta), x_0 + tu \in S\}.$$

(c) The second-order contingent set of S at (x_0, v) [21, 22] is

$$T^2(S, x_0, v) = \{w \in X : \exists t_n \to 0^+, \exists w_n \to w, \forall n \in \mathbb{N}, x_0 + t_n v + \tfrac{1}{2} t_n^2 w_n \in S\}.$$

(d) The asymptotic second-order tangent cone of S at (x_0, v) [22, 23] (the name is proposed by Penot in [23]) is

$$T^{''}(S, x_0, v) = \{w \in X : \exists (t_n, r_n) \to (0^+, 0^+) : \tfrac{t_n}{r_n} \to 0, \exists w_n \to w, \\ \forall n \in \mathbb{N}, x_0 + t_n v + \tfrac{1}{2} t_n r_n w_n \in S\}.$$

Proposition 2.7 [24, 25]. *Let $S \subseteq X, x_0 \in \mathrm{cl}\, S$ and $v \in X$. If S is convex and* $\mathrm{int}\, S \neq \emptyset$, *then*

$$\mathrm{int\ cone}(S - x_0) = IT(\mathrm{int} S, x_0).$$

Note from this that if $x_0 \in \mathrm{bd} S$, then

$$0 \notin \mathrm{int\ cone}(S - x_0).$$

Proposition 2.8 [24]. *Assume that $X = \mathbb{R}^m$ and $x_0 \in S \subseteq X$. If $x_n \in S \setminus \{x_0\}$ tends to x_0, then there exists $u \in T(S, x_0) \setminus \{0\}$ and a subsequence, denoted again by x_n, such that*

(i) $\frac{1}{t_n}(x_n - x_0) \to u$, *where* $t_n = \|x_n - x_0\|$;

(ii) *either $z \in T^2(S, x_0, u) \cap u^\perp$ exists such that $(x_n - x_0 - t_n u)/\frac{1}{2} t_n^2 \to z$ or $z \in T^{''}(S, x_0, u) \cap u^\perp \setminus \{0\}$ and $r_n \to 0^+$ exist such that $\frac{t_n}{r_n} \to 0^+$ and $(x_n - x_0 - t_n u)/\frac{1}{2} t_n r_n \to z$.*

Definition 2.9 [26]. Let $y \in Y$ and $A \subseteq Y$. The signed distance from y to A, denoted by $D(y, A)$, is defined by

$$D(y, A) = d(y, A) - d(y, Y \setminus A),$$

where $d(y, A) = \inf\{\|y - a\| : a \in A\}$ is the usual distance.

$D(y, A)$ is called also oriented distance or directed distance in the literature. This concept is widely used in [27] to study second-order optimality conditions.

Proposition 2.10 [28]. *If $Y = \mathbb{R}^m$ and A is a convex cone then*

$$D(y,-A) = \sup_{v^* \in A^*, \|v^*\|=1} \langle v^*, y \rangle.$$

3. First-order optimality conditions for problem (1.1)

Consider problem (1.1) stated in Section 1. Recall first notions of vector optimization.

Definition 3.1

(i) (Classical notion). A point $x_0 \in S$ is called a local weakly efficient solution (local efficient solution) of (1.1) if there is a neighborhood U of x_0 such that, $\forall x \in U \cap S$,

$$f(x) - f(x_0) \notin (-C) \setminus C$$

$$(f(x) - f(x_0) \notin -\text{int } C, \text{ respectively}).$$

The set of all local weakly efficient solutions of (1.1) is denoted by LWE(f, S) and that of local efficient ones by LE(f, S).

(ii) See e.g. [8]. For $m \in \mathbb{N}$, $x_0 \in S$ is said to be a local firm efficient solution of order m, denoted by $x_0 \in$ LFE(m, f, S) if there are $\alpha > 0$ and a neighborhood U of x_0 such that, $\forall x \in U \cap S \setminus \{x_0\}$,

$$(f(x)+C) \cap B_Y(f(x_0), \alpha\|x-x_0\|^m) = \emptyset. \tag{3.1}$$

Remark 3.2

(i) For $p \geq m$ it is clear that

$$\text{LFE}(m, f, S) \subseteq \text{LFE}(p, f, S) \subseteq \text{LE}(f, S) \subseteq \text{LWE}(f, S).$$

Therefore, necessary conditions for the right-most term hold true also for the others and sufficient conditions for the left-most term are valid for the others as well.

(ii) Instead of "firm efficient" other terms like "strict efficient", "isolated efficient" are also used in the literature. Definition (3.1) is equivalent to each of the following two assertions

(a) $d(f(x) - f(x_0), -C) \geq \alpha\|x - x_0\|^m$, $\forall x \in U \cap S \setminus \{x_0\}$;

(b) x_0 is a local firm optimal solution of order m of the scalar optimization problem

$$\text{Min } \varphi(x), \text{ s.t. } x \in S,$$

where $\varphi(x) = D(f(x) - f(x_0), -C), \forall x \in X$, or in other words, $\forall x \in U \cap S \setminus \{x_0\}$,

$$\varphi(x) \geq \varphi(x_0) + \alpha\|x - x_0\|^m.$$

(iii) If $X = \mathbb{R}^n, Y = \mathbb{R}^m$ and $C \subseteq Y$ is a closed convex cone, then $x_0 \in$ LWE(f, S) if and only if x_0 is a local optimal solution of (SP).

Theorem 3.3. *Assume that $x_0 \in LWE(f, S)$.*

(i) *If $A_f(x_0)$ is a first-order approximation of f at x_0 and $A_f(x_0)$ is asymptotically* *compact, then*

$$\forall v \in T(S, x_0), \exists M \in \text{cl}A_f(x_0) \bigcup (A_f(x_0)_\infty \setminus \{0\}),\ Mv \notin -\text{int}\ C.$$

(ii) *If X is a Banach space, $A_f(x_0)$ is a first-order approximation of f at x_0 and $A_f(x_0)$ is asymptotically p-compact, then*

$$\forall v \in T(S, x_0), \exists M \in \text{p-cl}A_f(x_0) \bigcup (\text{p-}A_f(x_0)_\infty \setminus \{0\}),\ Mv \notin -\text{int}\ C.$$

Proof. The two parts are proved similarly. We consider the more complicated part (ii). For any $v \in T(S, x_0)$, there are $(t_n, v_n) \to (0^+, v)$ such that $x_0 + t_n v_n \in S$, $\forall n \in \mathbb{N}$. By Definition 2.1(i), for large n, $M_n \in A_f(x_0)$ exists such that

$$f(x_0 + t_n v_n) - f(x_0) = t_n M_n v_n + o(t_n). \tag{3.2}$$

Since $x_0 \in \text{LWE}(f, S)$, for n large enough, one has

$$f(x_0 + t_n v_n) - f(x_0) \notin -\text{int}\ C. \tag{3.3}$$

If $\{M_n\}$ is bounded, by extracting a subsequence if necessary we assume $M_n \xrightarrow{\text{p}} M \in \text{p-cl}A_f(x_0)$. Dividing (3.2) by t_n, using (3.2) and (3.3) and passing to the limit we obtain $Mv \notin -\text{int}\ C$.

If $\{M_n\}$ is unbounded, we can assume that $\|M_n\| \to \infty$ and $\frac{M_n}{\|M_n\|} \xrightarrow{\text{p}} M \in \text{p-}A_f(x_0)_\infty \setminus\{0\}$. Dividing (3.2) by $\|M_n\| t_n$ and passing to the limit we have $Mv \notin -\text{int}\ C$. □

Theorem 3.4. *Assume that $X = \mathbb{R}^m$ and $x_0 \in S \subseteq X$, then $x_0 \in LFE(1, f, S)$ if one of the following conditions is satisfied.*

(i) *For an asymptotically compact first-order approximation $A_f(x_0)$ of f at x_0,*

$$\forall v \in T(S, x_0) \setminus \{0\}, \forall M \in \text{cl}A_f(x_0) \bigcup (A_f(x_0)_\infty \setminus \{0\}),\ Mv \notin -\text{cl}\ C.$$

(ii) *For an asymptotically p-compact first-order approximation $A_f(x_0)$ of f at x_0,*

$$\forall v \in T(S, x_0) \setminus \{0\}, \forall M \in \text{p-cl}A_f(x_0) \bigcup (\text{p-}A_f(x_0)_\infty \setminus \{0\}),\ Mv \notin -\text{cl}\ C.$$

Proof. We prove (ii) since (i) can be demonstrated similarly. Suppose to the contrary the existence of $x_n \in B_X(x_0, \frac{1}{n}) \setminus \{x_0\}$ and $c_n \in C$ such that

$$f(x_n) - f(x_0) + c_n \in B_Y(0, \tfrac{1}{n}\|x_n - x_0\|).$$

We can assume that $(x_n - x_0)/\|x_n - x_0\| \to v$ for some $v \in T(S, x_0) \setminus \{0\}$. Then by the definition of $A_f(x_0)$, for large n, there is $M_n \in A_f(x_0)$ such that

$$M_n(x_n - x_0) + o(\|x_n - x_0\|) + c_n \in B_Y(0, \tfrac{1}{n}\|x_n - x_0\|). \tag{3.4}$$

Dividing (3.4) by $\|x_n - x_0\|$ if $\{M_n\}$ is bounded and pointwisely converges to M and by $\|M_n\|\|x_n - x_0\|$ if $\|M_n\|$ tends to infinity and $\frac{M_n}{\|M_n\|}$ pointwisely converges to M, from (3.4) we arrive at a contradiction that $Mv \in -\text{cl}\ C$. □

Remark 3.5. $Mv \notin -\text{int}\ C$ is equivalent to the existence of a Lagrange multiplier $c^* \in C^*$ such that $\langle c^*, Mv\rangle \geq 0$ and $Mv \notin -\text{cl}\ C$ is equivalent to $\langle c^*, Mv\rangle > 0$. Therefore, we have the following two observations from Theorems 3.3 and 3.4.

(a) The gap between the above necessary and sufficient conditions is rather minimal (besides the fact that $\forall M$ replaces $\exists M$): the strict inequality replaces the inequality (in other words, the gap is only the boundary of $-C$). Note that here we do not explicitly need any convexity assumption for the sufficient condition.

(b) Here the multiplier c^* depends on the given feasible direction v. In [29] such directional Lagrange multipliers are clearly stated and studied for the first time. This kind of multipliers is considered in [30] for quasidifferentiable (see [31]) optimization and in [32] for problems with data which are directionally differentiable. The following example shows a directionally nondifferentiable case (so the results in [30, 32] are not applicable but Theorem 3.3 is).

Example 3.6. Let $X = Y = \mathbb{R}, S = [0, +\infty), C = \mathbb{R}_+, x_0 = 0$ and

$$f(x) = \begin{cases} -1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then $T(S, x_0) = S$. For $\alpha < 0$ fixed we can take $A_f(x_0) = (-\infty, \alpha)$ as a first-order approximation of f at x_0. Then

$$\mathrm{cl}A_f(x_0) = (-\infty, \alpha],\ A_f(x_0)_\infty = (-\infty, 0].$$

Choosing $v = 1 \in T(S, x_0)$, one sees that $Mv = M \in -\mathrm{int}\ C$, $\forall M \in \mathrm{cl}A_f(x_0) \bigcup (A_f(x_0)_\infty \setminus\{0\}) = (-\infty, 0)$. By Theorem 3.3, $x_0 \notin \mathrm{LWE}(f, S)$. Since f is directional nondifferentiable, not locally Lipschitz at x_0 and $Df(0, u) = \emptyset, \forall u \in \mathbb{R}$, (where $Df(0, u)$ is the upper Hadamard directional derivative of f at 0 in the direction u, see [37]), the results in [30], [32]-[37] cannot be employed.

Example 3.7. Let $X = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, S = [0, \infty), x_0 = 0$ and $f(x) = (|x|, \sqrt[3]{x})$. Then $T(S, x_0) = [0, \infty)$, and for any $\alpha > 0$ and fixed, we have a first-order approximation and related sets:

$$A_f(x_0) = \{(x, y) \in \mathbb{R}^2 : x = \pm 1, y > \alpha\},$$
$$\mathrm{cl}A_f(x_0) = \{(x, y) \in \mathbb{R}^2 : x = \pm 1, y \geq \alpha\},$$
$$A_f(x_0)_\infty = \{(0, y) \in \mathbb{R}^2 : y \geq 0\}.$$

Then the compactness assumption in Theorem 3.4(i) is satisfied. $\forall v \in (0, \infty)$, $\forall M \in \mathrm{cl}A_f(x_0)$, $Mv = (\pm v, yv) \notin -C$ since $y \geq \alpha > 0$. $\forall M \in A_f(x_0)_\infty \setminus \{0\}$, $Mv = (0, yv) \notin -C$. Following Theorem 3.4, $x_0 \in \mathrm{LFE}(1, f, S)$. Observe that f is not locally Lipschitz at x_0. So many results based on local Lipschitz property cannot be employed, e.g. that in [33]-[36]. f is even not calm (see [9]) at x_0 and hence Theorem 3.2 in [37] cannot be applied either.

Note also that Theorem 3.3(i) includes Theorem 3.3(i) and Theorem 4.1(i) in [4]; Theorem 3.4(ii) sharpens Theorem 3.4(i) and Theorem 4.2(i) in [4].

4. Second-order optimality conditions for problem (1.1)

4(a). Differentiable case

In this Subsection 4(a), we consider (1.1) with the assumption that f is Fréchet differentiable at $x_0 \in S$.

Theorem 4.1. *Assume that* $x_0 \in LWE(f, S)$.

(i) *Assume that* $(f'(x_0), B_f(x_0))$ *is an asymptotically compact second-order approximation of* f *at* x_0. *Then, for any* $v \in T(S, x_0)$, $f'(x_0)v \notin -\text{int } C$. *If* $f'(x_0)v \in -\text{bd } C$, *then*

(a) $\forall w \in T^2(S, x_0, v)$, *either* $\exists N \in \text{cl}B_f(x_0)$, $f'(x_0)w + 2N(v, v) \notin -E$,

or $\exists N \in B_f(x_0)_\infty \setminus \{0\}$, $N(v, v) \notin -E$;

(b) $\forall w \in T''(S, x_0, v)$, *either* $\exists N \in B_f(x_0)_\infty$, $f'(x_0)w + N(v, v) \notin -E$,

or $\exists N \in B_f(x_0)_\infty \setminus \{0\}$, $N(v, v) \notin -E$,

where $E = \text{int cone}(C + f'(x_0)v)$.

(ii) *If* X *is a Banach space, the compactness assumption in* (i) *can be reduced to pointwise compactness replacing all the subsets* $\text{cl}B_f(x_0)$ *and* $B_f(x_0)_\infty$ *by* p-$\text{cl}B_f(x_0)$ *and* p-$B_f(x_0)_\infty$, *respectively.*

Proof. (i) (a) For $w \in T^2(S, x_0, v)$, by the definition there are $t_n \to 0^+$ and $x_n \in S$ such that

$$w_n := (x_n - x_0 - t_n v)/\tfrac{1}{2}t_n^2 \to w.$$

Then by the definition of $B_f(x_0)$, for large n, there is $N_n \in B_f(x_0)$ such that

$$f(x_n) - f(x_0) = t_n f'(x_0)(v + \tfrac{1}{2}t_n w_n) + t_n^2 N_n(v + \tfrac{1}{2}t_n w_n, v + \tfrac{1}{2}t_n w_n) + o(t_n^2). \quad (4.1)$$

If $\{N_n\}$ is bounded then we can assume that N_n converges to some $N \in \text{cl}B_f(x_0)$. Hence

$$(f(x_n) - f(x_0) - t_n f'(x_0)v)/\tfrac{1}{2}t_n^2 \to f'(x_0)w + 2N(v, v) =: c. \quad (4.2)$$

Suppose $c \in -E$. By Proposition 2.7, $\alpha > 0$ exists such that, $\forall t \in (0, \alpha)$,

$$f'(x_0)v + t(c + B_Y(0, \alpha)) \subseteq -\text{int } C.$$

Then, for large n,

$$f'(x_0)v + \tfrac{1}{2}t_n(f(x_n) - f(x_0) - t_n f'(x_0)v)/\tfrac{1}{2}t_n^2 \in -\text{int } C.$$

So $\frac{1}{t_n}(f(x_n) - f(x_0)) \in -\text{int } C$, contradicting the local weak efficiency of x_0.

If $\{N_n\}$ is unbounded we can assume that $\|N_n\| \to \infty$ and $N_n/\|N_n\| \to N \in B_f(x_0)_\infty \setminus \{0\}$. Therefore, by (4.1),

$$(f(x_n) - f(x_0) - t_n f'(x_0)v)/t_n^2\|N_n\| \to N(v, v). \quad (4.3)$$

Suppose $N(v, v) \in -E$, i.e. $\beta > 0$ exists such that, $\forall t \in (0, \beta)$,

$$f'(x_0)v + t(N(v, v) + B_Y(0, \beta)) \subseteq -\text{int } C. \quad (4.4)$$

Since $f'(x_0)v \in -\text{bd } C$, by Proposition 2.7, $N(v, v) \neq 0$. Then, it follows from (4.3) that

$t_n\|N_n\| \to 0$. Consequently, (4.4) implies that, for large n,

$$f^{'}(x_0)v + t_n\|N_n\|(f(x_n) - f(x_0) - t_n f^{'}(x_0)v)/t_n^2\|N_n\| \in -\text{int } C,$$

and then $f(x_n) - f(x_0) \in -\text{int } C$, a contradiction.

(b) If $w \in T^{''}(S, x_0, v)$, there are $(t_n, r_n) \to (0^+, 0^+) : \frac{t_n}{r_n} \to 0^+$ and $x_n \in S$ such that

$$w_n := (x_n - x_0 - t_n v)/\tfrac{1}{2}t_n r_n \to w.$$

By definition of second-order approximations, for large n there is $N_n \in B_f(x_0)$ such that

$$\begin{aligned}&(f(x_n) - f(x_0) - t_n f^{'}(x_0)v)/\tfrac{1}{2}t_n r_n \\ &= f^{'}(x_0)w_n + (2t_n/r_n)(N_n(v + \tfrac{1}{2}r_n w_n, v + \tfrac{1}{2}r_n w_n) + o(t_n^2))/\tfrac{1}{2}t_n r_n.\end{aligned} \tag{4.5}$$

By using subsequences if necessary, we have only the following three subcases.

First subcase. If $(2t_n/r_n)N_n \to 0$, (4.5) implies that

$$(f(x_n) - f(x_0) - t_n f^{'}(x_0)v)/\tfrac{1}{2}t_n r_n \to f^{'}(x_0)w. \tag{4.6}$$

We claim that $f^{'}(x_0)w \notin -E$, i.e. we have the first possibility in (i)(a) with $N = 0$. In fact, if there were $\alpha > 0$ such that, $\forall t \in (0, \alpha)$,

$$f^{'}(x_0)v + t(f^{'}(x_0)w + B_Y(0, \alpha)) \subseteq -\text{int } C,$$

then, for large n,

$$f^{'}(x_0)v + \tfrac{1}{2}r_n(f(x_n) - f(x_0) - t_n f^{'}(x_0)v)/\tfrac{1}{2}t_n r_n \in -\text{int } C$$

and hence $f(x_n) - f(x_0) \in -\text{int } C$, a contradiction.

Second subcase. If $(2t_n/r_n)\|N_n\| \to a > 0$, then $\|N_n\| \to \infty$ and $t_n\|N_n\| \to 0$. We can assume that $\frac{N_n}{\|N_n\|} \to N \in B_f(x_0)_\infty \setminus \{0\}$. Hence it follows from (4.5) that

$$a(f(x_n) - f(x_0) - t_n f^{'}(x_0)v)/t_n^2\|N_n\| \to f^{'}(x_0)w + aN(v, v) =: c. \tag{4.7}$$

If c were in $-E$, $\beta > 0$ would exist such that, $\forall t \in (0, \beta)$,

$$f^{'}(x_0)v + t(c + B_Y(0, \beta)) \subseteq -\text{int } C.$$

Hence, for large n,

$$f^{'}(x_0)v + t_n\|N_n\|(f(x_n) - f(x_0) - t_n f^{'}(x_0)v)/t_n^2\|N_n\| \in -\text{int } C,$$

and then $f(x_n) - f(x_0) \in -\text{int } C$, impossible.

Third subcase. If $(2t_n/r_n)\|N_n\| \to \infty$, then $\|N_n\| \to \infty$ and we can assume that $\frac{N_n}{\|N_n\|} \to N \in B_f(x_0)_\infty \setminus \{0\}$. Hence we have (4.3). Similarly as for part (a), we arrive at $N(v, v) \notin -E$.

(ii) is proved similarly as (i). □

The first and second-order approximations contain many other generalized differential constructions as particular cases. As instances we derive the following direct consequences of Theorem 4.1, based on Propositions 2.2 and 2.3.

Corollary 4.2. *Assume that* $X = \mathbb{R}^n, Y = \mathbb{R}^m$ *and* $f : X \to Y$ *is of the class* $C^{1,1}$ *at* $x_0 \in S \subseteq X$. *If* $x_0 \in \text{LWE}(f, S)$ *then,* $\forall v \in T(S, x_0)$, $f^{'}(x_0)v \notin -\text{int } C$. *Moreover, if*

$f^{'}(x_0)v \in -\text{bd } C$ *then*

(i) $\forall w \in T^2(S, x_0, v)$, $\exists N \in \partial^2_C f(x_0)$, $f^{'}(x_0)w + N(v, v) \notin -E$;

(ii) $\forall w \in T^{''}(S, x_0, v)$, $f^{'}(x_0)w \notin -E$,

where $E = \text{int cone}(C + f^{'}(x_0)v)$.

Corollary 4.3. *Assume that* $X = \mathbb{R}^n, Y = \mathbb{R}^m$ *and* $f : X \to Y$ *is of the class* C^1 *at* $x_0 \in S \subseteq X$. *Assume further that* f *has an approximate Hessian mapping* $\partial^2 f(.)$ *which is upper semicontinuous at* x_0. *If* $x_0 \in \text{LWE}(f, S)$ *then,* $\forall v \in T(S, x_0)$, $f^{'}(x_0)v \notin -\text{int } C$. *Moreover, if* $f^{'}(x_0)v \in -\text{bd } C$ *then*

(i) $\forall w \in T^2(S, x_0, v)$, *either* $\exists N \in \text{clco}\partial^2 f(x_0)$, $f^{'}(x_0)w + N(v, v) \notin -E$ *or*

$\exists N \in \text{co}\partial^2 f(x_0)_\infty \setminus \{0\}$, $N(v, v) \notin -E$;

(ii) $\forall w \in T^{''}(S, x_0, v)$, *either* $\exists N \in \text{co}\partial^2 f(x_0)_\infty$, $f^{'}(x_0)w + N(v, v) \notin -E$ *or*

$\exists N \in \text{co}\partial^2 f(x_0)_\infty \setminus \{0\}$, $N(v, v) \notin -E$,

where $E = \text{int cone}(C + f^{'}(x_0)v)$.

In the following examples, $T^2(S, x_0, v) = \emptyset$ and hence the results using this second contingent set cannot be applied.

Example 4.4. Let $X = Y = \mathbb{R}^2$, $S = \{(x, y) \in \mathbb{R}^2 : x^3 + y^2 = 0\}$, $x_0 = (0, 0)$, $C = \mathbb{R}^2_+$ and

$$f(x, y) = (\tfrac{1}{2}x|x| - y + \tfrac{1}{2}y|y|, x - \tfrac{1}{2}x|x| + \tfrac{1}{2}y|y|).$$

$T(S, x_0) = \{(v_1, 0) \in \mathbb{R}^2 : v_1 \leq 0\}$, $f \in C^{1,1}$ and $f^{'}(x_0) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Choose $v = (-1, 0) \in T(S, x_0)$ we have $f^{'}(x_0)v = (0, -1) \in -\text{bd } C$. Furthermore $T^2(S, x_0, v) = \emptyset$ and choose $w \in T^{''}(S, x_0, v) = \mathbb{R}^2$, $w = (w_1, w_2)$ with $w_2 > 0$ we have $f^{'}(x_0)w \in -E = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 0\}$. Therefore, Corollary 4.2 shows that $x_0 \notin \text{LWE}(f, S)$. However, since $\forall v \in T(S, x_0) \setminus \{0\}$, $f^{'}(x_0)v \notin -\text{int } C$ and $\text{int } S = \emptyset$, Theorems 3.1 and 4.1 in [37] fail to be applied.

In the next example, $f \notin C^{1,1}$ at the reference point.

Example 4.5. Let $X = \mathbb{R}^2, Y = \mathbb{R}, C = \mathbb{R}_+, x_0 = (0, 0)$,

$$S = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = |x_1|^{5/4}\},$$

$$f(x_1, x_2) = -\tfrac{2}{3}|x_1|^{3/2} + \tfrac{1}{2}x_2^2 - x_2.$$

Then $T(S, x_0) = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$, $f^{'}(x_1, x_2) = (-|x_1|^{1/2}, x_2 - 1)$, $f^{'}(x_0) = (0, -1)$, $f \notin C^{1,1}$ at x_0 and $E = \text{int } C$. We can take

$$B_f(x_0) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1/2 \end{pmatrix} : \alpha < 0 \right\}.$$

Hence

$$\mathrm{cl}B_f(x_0) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1/2 \end{pmatrix} : \alpha \leq 0 \right\},$$

$$B_f(x_0)_\infty = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} : \alpha \leq 0 \right\}.$$

To apply Theorem 4.1 we choose $v = (1, 0) \in T(S, x_0)$ to see that

$f^{'}(x_0)v = 0 \in -\mathrm{bd}\ C$, $T^2(S, x_0, v) = \emptyset$, $T^{''}(S, x_0, v) = \mathbb{R}^2$.

Then taking $w = (0, 1) \in T^{''}(S, x_0, v)$ we have

$f^{'}(x_0)w + N(v, v) = -1 + \alpha \in -E$, $\forall N \in B_f(x_0)_\infty$,

$N(v, v) = \alpha \in -E$, $\forall N \in B_f(x_0)_\infty \setminus \{0\}$.

Consequently, $x_0 \notin \mathrm{LWE}(f, S)$ by Theorem 4.1. Since $f \notin C^{1,1}$ many known results fail to be applied. Because $T^2(S, x_0, v) = \emptyset$, the theorems in [4] are not applicable.

The following example constitutes an infinite dimensional case where Theorem 4.1 can easily reject a suspected point.

Example 4.6. Let $X = l^2, Y = \mathbb{R}, C = \mathbb{R}_+, x_0 = 0,$

$S = \{(x_1, x_2, ...) \in l^2 : |x_2|^3 = x_1^4\}$,

$f(x) = \sum_{i=2}^\infty \frac{1}{i} x_i - (\sum_{i=1}^\infty x_i^2)^{3/4} = \sum_{i=2}^\infty \frac{1}{i} x_i - \|x\|^{3/2}$.

Then $T(S, x_0) = \{x \in l^2 : x_2 = 0\}$, $f^{'}(x_0) = (0, \frac{1}{2}, \frac{1}{3}, ...)$ and

$B_f(x_0) = \{N_\alpha \in B(l^2, l^2, \mathbb{R}) : \alpha < -1\}$,

where, for $x, y \in l^2$,

$N_\alpha(x, y) = \alpha \sum_{i=1}^\infty x_i y_i$.

Since $N_\alpha \in \mathrm{spand}\{N_1\}$, $(f^{'}(x_0), B_f(x_0))$ is an asymptotically p-compact second-order approximation of f at x_0. We have

p-cl$B_f(x_0) = \{N_\alpha \in B(l^2, l^2, \mathbb{R}) : \alpha \leq -1\}$,

p-$B_f(x_0)_\infty = \{N_\alpha \in B(l^2, l^2, \mathbb{R}) : \alpha \leq 0\}$.

Taking $v = (1, 0, 0, ...) \in T(S, x_0)$ we see that $f^{'}(x_0)v = 0 \in -\mathrm{bd}\ C$, $E = \mathrm{int}\ C$ and

$T^2(S, x_0, v) = \emptyset$, $T^{''}(S, x_0, v) = l^2$.

Choosing now $w = (1, -1, 0, 0, ...) \in T^{''}(S, x_0, v)$, we obtain

$f^{'}(x_0)w + N(v, v) = -\frac{1}{2} + \alpha \in -E$, $\forall N \in$ p-$B_f(x_0)_\infty$,

$N(v, v) = \alpha \in -E$, $\forall N \in$ p-$B_f(x_0)_\infty \setminus \{0\}$.

By virtue of Theorem 4.1, $x_0 \notin \mathrm{LWE}(f, S)$. Of course we cannot employ approximate Hessians and Clarke Hessians here as l^2 is infinite dimensional. The results in [4] are also not applicable since $T^2(S, x_0, v) = \emptyset$.

Theorem 4.7. *Assume that $X = \mathbb{R}^m$, f is Fréchet differentiable at $x_0 \in S$ and one of the following conditions is satisfied.*

(i) *$(f'(x_0), B_f(x_0))$ is an asymptotically compact second-order approximation of f at x_0 and $\forall v \in T(S, x_0) \setminus \{0\}$ with $f'(x_0)v \in -\text{cl } C$, $\forall N \in B_f(x_0)_\infty \setminus \{0\}$, $N(v, v) \notin -F$, and*

(a) $\forall w \in T^2(S, x_0, v) \cap v^\perp$, $\forall N \in \text{cl}B_f(x_0)$,

$$f'(x_0)w + 2N(v, v) \notin -F;$$

(b) $\forall w \in T''(S, x_0, v) \cap v^\perp \setminus \{0\}$, $\forall N \in B_f(x_0)_\infty$,

$$f'(x_0)w + N(v, v) \notin -F,$$

where $F = \text{cl cone}(C + f'(x_0)v)$.

(ii) *is assertion* (i) *with "compact" replaced by "p-compact" and* cl$A$ *and* A_∞ *replaced by* p-cl$A$ *and* p-A_∞ *for all involved subsets.*

Then $x_0 \in \text{LFE}(2, f, S)$.

Proof. (i) Suppose to the contrary that there are $x_n \in S \cap B_X(x_0, \frac{1}{n}) \setminus \{x_0\}$ and $c_n \in C$ such that

$$f(x_n) - f(x_0) + c_n \in B_Y(0, \tfrac{1}{n}t_n^2), \tag{4.8}$$

where $t_n = \|x_n - x_0\|$. We can assume that $\frac{1}{t_n}(x_n - x_0) \to v \in T(S, x_0)\setminus\{0\}$. Dividing (4.8) by t_n and passing to the limit one gets $f'(x_0)v \in -\text{cl } C$. On the other hand, by Proposition 2.8, it suffices to consider the following two cases (using subsequences if necessary).

First case. By the definition of $B_f(x_0)$, for large n one has $N_n \in B_f(x_0)$ satisfying (4.1). If $\{N_n\}$ is bounded we have $N \in \text{cl}B_f(x_0)$ satisfying (4.2). On the other hand, it follows from (4.8) that

$$[(f(x_n) - f(x_0) - t_n f'(x_0)v) + (c_n + t_n f'(x_0)v)]/t_n^2 \to 0.$$

Then, from (4.2), $c := f'(x_0)w + 2N(v, v) \in -F$, contradicting assumption (a).

While $\{N_n\}$ is unbounded, we can assume $\{N_n\} \to \infty$, $\frac{N_n}{\|N_n\|} \to N \in B_f(x_0)_\infty \setminus \{0\}$ and (4.3). From (4.8) we see that

$$(f(x_n) - f(x_0) - t_n f'(x_0)v)/t_n^2\|N_n\| + (c_n + t_n f'(x_0)v)/t_n^2\|N_n\| \to 0.$$

Then, (4.3) implies that $N(v, v) \in -F$, a contradiction.

Second case. There is $r_n \to 0^+$ such that $\frac{t_n}{r_n} \to 0^+$ and $w_n := (x_n - x_0 - t_n v)/\frac{1}{2}t_n r_n \to w \in T''(S, x_0, v) \cap v^\perp \setminus \{0\}$. Again by the definition of $B_f(x_0)$, for large n, there exists $N_n \in B_f(x_0)$ satisfying (4.5). It suffices to investigate the following three subcases.

(1) $(2t_n/r_n)N_n \to 0$. Then one has (4.6). On the other hand, (4.8) implies that

$$[(f(x_n) - f(x_0) - t_n f'(x_0)v) + (c_n + t_n f'(x_0)v)]/t_n r_n \to 0.$$

Hence, $f'(x_0)w \in -F$, a contradiction to assumption (b) with $N = 0$.

(2) $(2t_n/r_n)\|N_n\| \to a > 0$. Then, $\|N_n\| \to \infty$ and $t_n\|N_n\| \to 0$. Therefore, $\frac{N_n}{\|N_n\|} \to$

$N \in B_f(x_0)_\infty \backslash \{0\}$ and one gets (4.7). Similarly as above, (4.8) implies also a contradiction.

(3) $(2t_n/r_n)\|N_n\| \to \infty$. Then, $\|N_n\| \to \infty$, $\frac{N_n}{\|N_n\|} \to N \in B_f(x_0)_\infty \setminus \{0\}$ and one has again (4.3). Analogously as before, one arrives at $N(v,v) \in -F$, again a contradiction.

(ii) can be proved similarly as (i). □

Remark 4.8 The gap between second-order necessary and sufficient conditions given in Theorems 4.1 and 4.7 is only the boundary of the "shifted" ordering cone.

Example 4.9. Let $X = \mathbb{R}, Y = \mathbb{R}^2, S = [0, \infty), x_0 = 0, C = \mathbb{R}^2_+$ and $f(x) = (\frac{1}{2}x^2, \frac{3}{4}\sqrt[3]{x^4})$. Then $T(S, x_0) = S, f^{'}(x_0) = (0,0)$. For $\alpha > 0$ and fixed we can take $B_f(x_0) = \{(\frac{1}{2}, y) : y > \alpha\}$, $B_f(x_0)_\infty = \{(0,y) : y \geq 0\}$. $\forall v \in T(S,x_0) \setminus \{0\} = (0,\infty)$, $f^{'}(x_0)v = 0 \in -\text{cl } C$ and $F = C$. So $\forall N \in B_f(x_0)_\infty \setminus \{0\}$, $N(v,v) = (0, yv^2) \notin -F$. $\forall w \in T^2(S, x_0, v) \cap v^\perp = \{0\}$, $\forall N \in \text{cl}B_f(x_0)$, $f^{'}(x_0)w + 2N(v,v) = (v^2, 2yv^2) \notin -F$ (as $y \geq \alpha$). $T^{''}(S, x_0, v) \cap v^\perp \backslash \{0\} = \emptyset$. Following Theorem 4.7(i), $x_0 \in \text{LFE}(2, f, S)$. We see that $f \notin C^{1,1}$ and then second-order conditions using the Clarke generalized Hessian cannot be applied. Since $f^{'}$ is not calm at x_0, the results in [37] cannot be employed either.

4(b). Nondifferentiable case

For general cases, to establish similar second-order necessary optimality conditions we have to use some first-order approximation $A_f(x_0)$ which is rather good in the sense of boundedness as follows.

Theorem 4.10. *Assume for problem (1.1) that* $A_f(x_0)$ *is a bounded first-order approximation of* f *at* $x_0 \in \text{LWE}(f, S)$.

(i) *Assume that* $(A_f(x_0), B_f(x_0))$ *is an asymptotically compact second-order approximation of* f *at* x_0 *and* $\forall v \in T(S, x_0)$, $\exists M \in \text{cl}A_f(x_0)$, $Mv \notin -\text{int } C$. *Assume further that* $\forall v \in T(S, x_0)$, $A_f(x_0)v \subseteq -\text{bd } C$. *Then*

(a) $\forall w \in T^2(S, x_0, v)$, *either* $\exists \overline{M} \in \text{cl}A_f(x_0)$, $\exists N \in \text{cl}B_f(x_0)$, $\overline{M}w + 2N(v,v) \notin -\text{int } C$, *or* $\exists N \in B_f(x_0)_\infty \setminus \{0\}$, $N(v,v) \notin -\text{int } C$;

(b) $\forall w \in T^{''}(S, x_0, v)$, *either* $\exists \overline{M} \in \text{cl}A_f(x_0)$, $\exists N \in B_f(x_0)_\infty$, $\overline{M}w + N(v,v) \notin -\text{int } C$, *or* $\exists N \in B_f(x_0)_\infty \setminus \{0\}$, $N(v,v) \notin -\text{int } C$.

(ii) *If* X *is a Banach space then the compactness assumption in* (i) *can be reduced to the pointwise compactness replacing* $\text{cl}A_f(x_0)$, $\text{cl}B_f(x_0)$ *and* $B_f(x_0)_\infty$ *by* p-$\text{cl}A_f(x_0)$, p-$\text{cl}B_f(x_0)$ *and* p-$B_f(x_0)_\infty$, *respectively.*

Proof. (a) By the similarity of (i) and (ii) we prove only (ii). For any $v \in T(S, x_0)$, $\exists M \in \text{cl}A_f(x_0)$, $Mv \notin -\text{int } C$ by Theorem 3.3. For $w \in T^2(S, x_0, v)$, there are $t_n \to 0^+$ and

$x_n \in S$ such that

$$w_n := (x_n - x_0 - t_n v)/\tfrac{1}{2}t_n^2 \to w.$$

As $x_0 \in \mathrm{LWE}(f, S)$, for large n, there are $M_n \in A_f(x_0)$ and $N_n \in B_f(x_0)$ such that

$$f(x_n) - f(x_0) = t_n M_n(v + \tfrac{1}{2}t_n w_n) + t_n^2 N_n(v + \tfrac{1}{2}t_n w_n, v + \tfrac{1}{2}t_n w_n) + o(t_n^2)$$
$$\notin -\mathrm{int}\, C.$$

Since $A_f(x_0)v \subseteq -\mathrm{bd}\, C$, this implies, for large n, that

$$M_n w_n + 2N_n(v + \tfrac{1}{2}t_n w_n, v + \tfrac{1}{2}t_n w_n) + 2o(t_n^2)/t_n^2 \notin -\mathrm{int}\, C. \tag{4.9}$$

By the boundedness and the assumed compactness of $A_f(x_0)$, we can assume that $M_n \xrightarrow{\text{p}} \overline{M} \in \text{p-cl}A_f(x_0)$. Similarly, if $\{N_n\}$ is bounded, assume that $N_n \xrightarrow{\text{p}} N \in \text{p-cl}B_f(x_0)$. Passing (4.9) to the limit we get

$$\overline{M}w + 2N(v, v) \notin -\mathrm{int}\, C.$$

While $\{N_n\}$ is unbounded, we can assume $\|N_n\| \to \infty$ and $\frac{N_n}{\|N_n\|} \xrightarrow{\text{p}} N \in B_f(x_0)_\infty \setminus \{0\}$. Dividing (4.9) by $\|N_n\|$ and passing to the limit we arrive at $N(v, v) \notin -\mathrm{int}\, C$.

(b) For $w \in T''(S, x_0, v)$, there are $(t_n, r_n) \to (0^+, 0^+) : \frac{t_n}{r_n} \to 0^+$ and $x_n \in S$ such that

$$w_n := (x_n - x_0 - t_n v)/\tfrac{1}{2}t_n r_n \to w.$$

Similarly as for (a) one has, for n sufficiently large,

$$M_n w_n + (2t_n/r_n)N_n(v + \tfrac{1}{2}t_n w_n, v + \tfrac{1}{2}t_n w_n) + 2o(t_n^2)/t_n r_n \notin -\mathrm{int}\, C. \tag{4.10}$$

We can assume that $M_n \xrightarrow{\text{p}} \overline{M} \in \text{p-cl}A_f(x_0)$. Considering subsequences if necessary from (4.10) we have only the following three subcases.

First subcase. $(2t_n/r_n)N_n \to 0$. Passing (4.10) to the limit one gets $\overline{M}w \notin -\mathrm{int}\, C$, i.e. (b) with $N = 0$.

Second subcase. $(2t_n/r_n)\|N_n\| \to a > 0$. Then $\|N_n\| \to \infty$ and we can assume that $\frac{N_n}{\|N_n\|} \xrightarrow{\text{p}} N \in B_f(x_0)_\infty \setminus \{0\}$. Passing (4.10) to the limit one gets (b) (with N as aN).

Third subcase. $(2t_n/r_n)\|N_n\| \to \infty$. Then, similarly, dividing (4.10) by $(2t_n/r_n)\|N_n\|$ and passing to the limit one arrives at $N(v, v) \notin -\mathrm{int}\, C$. □

Note that Theorem 4.10(ii)(a) includes Theorem 4.1 in [4] as a special case where $A_f(x_0) = \{f'(x_0)\}$. The other parts of Theorem 4.10 are new.

Example 4.11. Let $X = Y = \mathbb{R}^2, C = \mathbb{R}^2_+, S = \{x \in \mathbb{R}^2 : x_1 + x_2^{2/3} = 0\}$, $x_0 = (0, 0)$ and $f(x_1, x_2) = (-x_2, x_1 + |x_2|)$. Then

$$T(S, x_0) = \{(v_1, 0) : v_1 \leq 0\},$$

$A_f(x_0) = \{\begin{pmatrix} 0 & -1 \\ 1 & \pm 1 \end{pmatrix}\}$, $B_f(x_0) = \{0\}$.

Choose $v = (-1, 0)$. Then

$A_f(x_0)v = (0, -1) \in -\text{bd } C$, $T^2(S, x_0, v) = \emptyset$, $T^{''}(S, x_0, v) = \mathbb{R}^2$.

Choose further $w = (-2, 1) \in T^{''}(S, x_0, v)$ we see that, $\forall M \in \text{cl}A_f(x_0)$,

$Mw = (-1, -2 \pm 1) \in -\text{int } C$.

Therefore, Theorem 4.10 rejects x_0 from the expected local weak efficiency. As before, since $T^2(S, x_0, v) = \emptyset$, the results in [4] are not applicable. Moreover, it is easy to see that our Theorems 3.3 and 4.1 and the results in [34]-[37] fail to be applied either, since the necessary conditions there are satisfied.

The following theorem can be proved in the same way as Theorems 3.4 and 4.2 in [4] but the conclusion is stronger.

Theorem 4.12. *Consider problem (1.1) with* $X = \mathbb{R}^m$ *and* C *is closed.* $x_0 \in S$ *belongs to* $\text{LFE}(2, f, S)$ *if one of the following conditions is satisfied.*

(i) $(A_f(x_0), B_f(x_0))$ *is an asymptotically compact second-order approximation of* f *at* x_0 *and*

(a) $\forall u \in X \setminus \{0\}$, $\forall M \in \text{cl}A_f(x_0) \bigcup (A_f(x_0)_\infty \setminus \{0\})$, $Mu \in Y \setminus (-C \setminus C)$;

(b) $\forall v \in T(S, x_0) \setminus \{0\}$, $\exists M \in \text{cl}A_f(x_0) \bigcup (A_f(x_0)_\infty \setminus \{0\})$, $\forall N \in \text{cl}B_f(x_0)$ $\bigcup (B_f(x_0)_\infty \setminus \{0\})$, $Mv \in C \cap (-C)$ *and* $N(v, v) \in \text{int } C$.

(ii) *is assertion* (i) *with "compact" replaced by "p-compact" and* cl A *and* A_∞ *replaced by* p-cl A *and* p-A_∞, *respectively, for all involved subsets.*

(iii) Y *is finite dimensional and* $\forall u \in \text{cone}(S - x_0)$, $\forall v \in T(S, x_0)$, $\forall M \in A_f(x_0), \forall N \in \text{cl}B_f(x_0) \bigcup (B_f(x_0)_\infty \setminus \{0\})$, $Mu \notin -\text{int } C$ *and* $N(v, v) \in \text{int } C$.

The gap between the necessary condition and sufficient one in Theorems 4.10 and 4.12 is rather big. Employing additionally second-order tangent sets we reduce this gap to $-\text{bd}C$, as for the differentiable case, in the following theorem.

Theorem 4.13. *Suppose that* $X = \mathbb{R}^m$ *and* $x_0 \in S$.

(i) *Assume that* $(A_f(x_0), B_f(x_0))$ *is an asymptotically compact second-order approximation of* f *at* x_0 *with* $A_f(x_0)$ *being bounded. Assume further that,* $\forall v \in T(S, x_0) \setminus \{0\}$, $A_f(x_0) \subseteq (-\text{cl } C) \cap \text{cl } C$ *and* $\forall N \in B_f(x_0)_\infty \setminus \{0\}$, $N(v, v) \notin -\text{cl } C$. *Assume for second-order tangent sets that*

(a) $\forall v \in T(S, x_0) \setminus \{0\}$, $\forall w \in T^2(S, x_0, v) \cap v^\perp$, $\forall M \in \text{cl}A_f(x_0)$, $\forall N \in \text{cl}B_f(x_0)$,

$Mw + 2N(v, v) \notin -\text{cl } C$;

(b) $\forall v \in T(S, x_0) \setminus \{0\}$, $\forall w \in T^{''}(S, x_0, v) \cap v^\perp \setminus \{0\}$, $\forall M \in \text{cl}A_f(x_0)$, $\forall N \in B_f(x_0)_\infty$,

$Mw + N(v, v) \notin -\text{cl } C$.

Then $x_0 \in \text{LFE}(2, f, S)$.

(ii) *is* (i) *with "compact" is replaced by "p-compact" and* cl A *and* A_∞ *by* p-cl A *and* p-A_∞, *respectively, for the involved subsets.*

Proof. (i) Suppose that $x_n \in S \cap B_X(x_0, \frac{1}{n}) \setminus \{x_0\}$ and $c_n \in C$ exist such that (4.8) holds. We can assume that $\frac{1}{t_n}(x_n - x_0) \to v \in T(S, x_0) \setminus \{0\}$. By Proposition 2.8 we have only two cases as follows.

First case. $w_n := (x_n - x_0 - t_n v)/\frac{1}{2}t_n^2 \to w \in T^2(S, x_0, v) \cap v^\perp$. On the other hand, for large n, by (4.8) there are $M_n \in A_f(x_0)$ and $N_n \in B_f(x_0)$ satisfying

$$M_n w_n + 2N_n(v + \tfrac{1}{2}t_n w_n, v + \tfrac{1}{2}t_n w_n) + o(t_n^2)/\tfrac{1}{2}t_n^2 = d_n/\tfrac{1}{2}t_n^2 - c_n^{'}, \tag{4.11}$$

where $d_n \in B_Y(0, \frac{1}{t_n^2})$ and $c_n^{'} = (c_n + t_n M_n v)/\frac{1}{2}t_n^2 \in \text{cl } C$.

By the boundedness of $A_f(x_0)$, $M \in \text{cl}A_f(x_0)$ exists such that $M_n \to M$ (using a subsequence if necessary). If $\{N_n\}$ is bounded we can assume that $N_n \to N$ for some $N \in \text{cl}B_f(x_0)$. Letting $n \to \infty$ in (4.11) one gets the contradiction

$$Mw + 2N(v, v) \in -\text{cl } C.$$

$\{N_n\}$ is unbounded one can assume that $\|N_n\| \to \infty$ and $\frac{N_n}{\|N_n\|} \to N$ for some $N \in B_f(x_0)_\infty \setminus \{0\}$. Dividing (4.11) by $\|N_n\|$ and passing to the limit one obtains $N(v, v) \in -\text{cl}C$, contradicting (a) with $M = 0$.

Second case. There is $r_n \to 0^+$ such that $\frac{t_n}{r_n} \to 0^+$ and

$$w_n := (x_n - x_0 - t_n v)/\tfrac{1}{2}t_n r_n \to w \in T^{''}(S, x_0, v) \cap v^\perp \setminus \{0\}.$$

On the other hand, (4.8) implies that, for n sufficiently large,

$$M_n w_n + (2t_n/r_n)N_n(v + \tfrac{1}{2}r_n w_n, v + \tfrac{1}{2}r_n w_n) + o(t_n^2)/\tfrac{1}{2}t_n r_n = d_n/\tfrac{1}{2}t_n r_n - c_n^{'}, \tag{4.12}$$

where $d_n \in B_Y(0, \frac{1}{n}t_n^2)$ and $c_n^{'} = (c_n + t_n M_n v)/\frac{1}{2}t_n r_n \in \text{cl } C$.

As before $M_n \to M \in \text{cl}A_f(x_0)$. We have now three subcases.

(1) $(2t_n/r_n)N_n \to 0$. Passing $n \to \infty$ in (4.12) one sees the contradiction $Mw \in -\text{cl}C$.

(2) $(2t_n/r_n)\|N_n\| \to a > 0$. Then $\|N_n\| \to \infty$ and $\frac{N_n}{\|N_n\|} \to N \in B_f(x_0)_\infty \setminus \{0\}$. Dividing (4.12) by $(2t_n/r_n)\|N_n\|$ and passing to the limit one receives the contradiction

$$Mw + aN(v, v) \in -\text{cl } C.$$

(3) $(2t_n/r_n)\|N_n\| \to \infty$. Then doing as for the subcase (2) one arrives at the contradiction $N(v, v) \in -\text{cl } C$. □

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