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On the Stability of the Solution Sets of General Multivalued Vector Quasiequilibrium Problems[1]

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Abstract. In this paper we give sufficient conditions for the semicontinuity of solution sets of general multivalued vector quasiequilibrium problems. All kinds of semicontinuity are considered: lower semicontinuity, upper semicontinuity, Hausdorff upper semicontinuity and closedness. Moreover, we investigate all the "weak" and "middle" and "strong" solutions of quasiequilibrium problems. Many examples are provided to give more insights and comparisons with recent existing results.

Key Words. Quasiequilibrium problems, lower semicontinuity, upper semicontinuity, Hausdorff upper semicontinuity, closedness of the solution multifunction, quasivariational inequalities.

1. Introduction

Stability of the solution set of a parametric optimization problem has been studied intensively in the literature, where stability can be understood as semicontinuity, continuity, Lipschitz continuity or (generalized) differentiability. Refs. $1-6$ deal with stability for equilibrium and quasiequilibrium problems. These problems began to be of interest of an increasing number of authors, after Ref. 7, where they were introduced as a generalization of optimization and variational inequality problems. This generalization has proved to be of great importance, since the problems include many problems such as the fixed point and coincidence point problems, the complementarity problem, the Nash equilibria problem, etc and have a wide range of applications in industry and pure and applied sciences. Until now, the generality of the problem settings has been extended to a very high level, but the main efforts have been made for the study of existence of solutions. See e.g. recent Refs $7-15$. For a recent survey see Ref. 16. For quasivariational inclusions, which are related to quasiequilibrium problems, the reader is referred to Refs. 17 and 18 about the Lipschitz continuity of the solution maps.

The aim of the present paper is to investigate various kinds of semicontinuity of the solution sets of quasiequilibrium problems. Our problem settings are general enough to include most of the known quasiequilibrium problems. The motivation for us to choose a weak type of stability as semicontinuity is that, as usually appeared in the literature, to ensure a stability property of the solution set, assumptions of the same property should be imposed on the data of the problem. However, in many practical situations such assumptions are not satisfied. Moreover, for a number of applications, the semicontinuity of the solution sets is enough, see e.g. the argument in Refs. 3, 19 and 20. We extend the results of Refs. $1-3$ under more relaxed assumptions. Applying to variational and quasivariational inequalities, special cases of quasiequilibrium problems, our theorems improve the corresponding results of Refs. $19-22$. The results of the paper are followed by many examples showing their advantages and counterexamples explaining the invalidity of the converse assertions.

The organization of the paper is as follows. In the remaining part of this section we formulate the problems under consideration, discuss some relations and recall definitions needed in the sequel. Section 2 is devoted to lower semicontinuity of the solution sets. In Section 3 three kinds of upper semicontinuity of these sets are studied. We also investigate cases where some or all solution sets of our problems coincide.

The problems under our consideration are as follows. Throughout the paper, unless otherwise specified, let X, M, N and Λ be Hausdorff topological spaces and Y be a topological vector space. Let $K : X \times \Lambda \to 2^X$, $G : X \times N \to 2^X$ and $F : X \times X \times M \to 2^Y$ be multifunctions. Let $C \subseteq Y$ be a closed subset with nonempty interior. As usual a problem involving single-valued mappings will be splitted into many generalized ones while the mappings become multivalued. For the sake of simplicity we adopt the following notations. Letters w, m and s are used for a weak, middle and strong, respectively, kinds of considered problems. For subsets A and B under consideration we adopt the notations

$$(u,v)\,\mathrm{w}\,A \times B \quad \text{means} \quad \forall u \in A, \exists v \in B,$$

$$(u,v)\,\mathrm{m}\,A \times B \quad \text{means} \quad \exists v \in B, \forall u \in A,$$

$$(u,v)\,\mathrm{s}\,A \times B \quad \text{means} \quad \forall u \in A, \forall v \in B,$$

$$\alpha_1(A,B) \quad \text{means} \quad A \cap B \neq \emptyset,$$

$$\alpha_2(A,B) \quad \text{means} \quad A \subseteq B,$$

$$(u,v)\,\bar{\mathrm{w}}\,A \times B \;\text{ means}\; \exists u \in A, \forall v \in B \;\text{ and similarly for}\; \bar{\mathrm{m}} \;\text{and}\; \bar{\mathrm{s}}.$$

Let $\mathrm{r} \in \{\mathrm{w},\mathrm{m},\mathrm{s}\}$, $\bar{\mathrm{r}} \in \{\bar{\mathrm{w}},\bar{\mathrm{m}},\bar{\mathrm{s}}\}$ and $\alpha \in \{\alpha_1,\alpha_2\}$. Our general parametric multivalued vector quasiequilibrium problem is the following, for $(\lambda,\mu,\eta) \in \Lambda \times M \times N$,

$$(\mathrm{P}_{\mathrm{r}\alpha}) \text{ find } \bar{\mathrm{x}} \in \mathrm{clK}(\bar{\mathrm{x}},\lambda) \text{ such that } (\mathrm{y},\bar{\mathrm{x}}^*)\,\mathrm{r}\,\mathrm{K}(\bar{\mathrm{x}},\lambda) \times \mathrm{G}(\bar{\mathrm{x}},\eta),$$

$$\alpha\big(F(\bar{x}^*,y,\mu), Y \setminus -\mathrm{int}C\big).$$

Let $S_{r\alpha}(\lambda, \mu, \eta)$ be the solution set of ($\mathrm{P}_{r\alpha}$) corresponding to λ, μ and η. If λ, μ and η are fixed and clearly recognized from the context we write simply $S_{r\alpha}$. Moreover, $S_{r\alpha}(.,.,.)$ stands for the corresponding solution multifunction, where λ, μ and η change the values as variables.

By the definition the following relations are clear:

$$\begin{array}{ccccc} S_{w\alpha_1} & \supseteq & S_{m\alpha_1} & \supseteq & S_{s\alpha_1} \\ \cup\!| & & \cup\!| & & \cup\!| \\ S_{w\alpha_2} & \supseteq & S_{m\alpha_2} & \supseteq & S_{s\alpha_2} \end{array}$$

The following examples show that there are not inclusions in the remaining relations between: $S_{m\alpha_1}$ and $S_{w\alpha_2}$, $S_{s\alpha_1}$ and $S_{w\alpha_2}$, $S_{s\alpha_1}$ and $S_{m\alpha_2}$.

Example 1.1 ($S_{m\alpha_1} \not\subseteq S_{w\alpha_2}$). Let $X = Y = R$, $\Lambda \equiv M \equiv N = [0, 1]$, $C = R_+$, $K(x, \lambda) = [\lambda, \lambda + 1]$, $\lambda_0 = 0$, $G(x, \lambda) = [x, x + \lambda + 1]$ and $F(x, y, \lambda) = (-\infty, x - y + \lambda]$. Then, it is not hard to see that $S_{m\alpha_1}(0) = [0, 1]$ and $S_{w\alpha_2}(0) = \emptyset$.

Example 1.2 ($S_{w\alpha_2} \not\subseteq S_{m\alpha_1}$ and $S_{w\alpha_2} \not\subseteq S_{s\alpha_1}$). Let X, Y, Λ, M, N and C be as above. Let $K(x, \lambda) = [0, \frac{3\pi}{2} + \lambda]$, $\lambda_0 = 0$, $G(x, \lambda) = [0, \frac{3\pi}{2} + 2\lambda]$ and $F(x, y, \lambda) = \{\sin(x - y + 3\lambda)\}$. Then, for any $x^* \in [0, \frac{3\pi}{2}]$, there is $y \in [0, \frac{3\pi}{2}]$ such that $\sin(x^* - y) < 0$. Indeed, if $x^* < \frac{3\pi}{2}$ then take $y = x^* + \varepsilon \in [0, \frac{3\pi}{2}]$, $0 < \varepsilon < \pi$; if $x^* = \frac{3\pi}{2}$ then take $y = 0$. Thus, $S_{m\alpha_1}(0) = \emptyset$, and hence $S_{s\alpha_1}(0) = \emptyset$. While

$S_{\mathrm{w}\alpha_2}(0) = [0, \frac{3\pi}{2}]$ (put $x^* = y$ for each $y \in [0, \frac{3\pi}{2}]$).

Example 1.3 ($S_{\mathrm{s}\alpha_1} \not\subseteq S_{\mathrm{w}\alpha_2}$ and $S_{\mathrm{s}\alpha_1} \not\subseteq S_{\mathrm{m}\alpha_2}$). Let $X, Y, \Lambda, M, N, C, K(x, \lambda)$ and λ_0 be as in Example 1.1. Let $G(x, \lambda) = [0, \lambda + 1]$ and $F(x, y, \lambda) = (-\infty, x + y + \lambda]$. Then, $S_{\mathrm{s}\alpha_1}(0) = [0, 1]$ and $S_{\mathrm{w}\alpha_2}(0) = S_{\mathrm{m}\alpha_2}(0) = \emptyset$.

Example 1.4 ($S_{\mathrm{m}\alpha_2} \not\subseteq S_{\mathrm{s}\alpha_1}$). Let $X, Y, \Lambda, M, N, C, K(x, \lambda)$ and λ_0 be as in Example 1.1. Let $G(x, \lambda) = [-x + \lambda, 2 - x + \lambda]$ and $F(x, y, \lambda) = \{x(y - x) + \lambda\}$. Then, $S_{\mathrm{m}\alpha_2}(0) = [0, 1]$ (take $x^* = 0 \in [-x, 2 - x]$), and $S_{\mathrm{s}\alpha_1}(0) = \emptyset$ (for each $x \in [0, 1]$ take $x^* = 1 \in [-x, 2 - x]$ and $y = 0 \in [0, 1]$).

Recall now some notions. Let X and Y be as above and $Q : X \to 2^Y$ be a multifunction. Q is said to be lower semicontinuous (lsc) at x_0 if: $Q(x_0) \cap U \neq \emptyset$ for some open set $U \subseteq Y$ implies the existence of a neighborhood V of x_0 such that, for all $x \in V, Q(x) \cap U \neq \emptyset$. An equivalent formulation is that: Q is lsc at x_0 if $\forall x_\alpha \to x_0$, $\forall y \in Q(x_0)$, $\exists y_\alpha \in Q(x_\alpha)$, $y_\alpha \to y$. Q is called upper semicontinuous (usc) at x_0 if for each open set $U \supseteq Q(x_0)$, there is a neighborhood V of x_0 such that $U \supseteq Q(V)$. Q is termed Hausdorff upper semicontinuious (H-usc) at x_0 if for each neighborhood B of the origin in Y, there is a neighborhood V of x_0 such that $Q(V) \subseteq Q(x_0) + B$. Q is said to be continuous at x_0 if it is both lsc and usc at x_0 and to be H-continuous at x_0 if it is both lsc and H-usc at x_0.

Q is called closed at x_0 if for each net $(x_\alpha, y_\alpha) \in \text{graph}Q := \{(x, y) \mid y \in Q(x)\}$, $(x_\alpha, y_\alpha) \to (x_0, y_0)$, then $y_0 \in Q(x_0)$. The closedness is closely related to the upper (and Hausdorff upper) semicontinuity (see Section 3). We say that Q satisfies a certain property in a subset $A \subseteq X$ if Q satisfies it at every point of A. If $A = \text{dom}Q := \{x \mid Q(x) \neq \emptyset\}$ we omit "in dom$Q$" in the statement.

A topological space Z is called arcwisely connected if for each pair of points x and y in Z, there is a continuous mapping $\varphi : [0, 1] \to Z$ such that $\varphi(0) = x$ and $\varphi(1) = y$.

Note finally that for equilibrium problems considered in the literature usually $G(x, \eta) = \{x\}$. However, the appearance of general multifunction G make the problem setting include more practical situations.

2. Lower Semicontinuity

For $\lambda \in \Lambda$, let $E(\lambda) = \{x \in X \mid x \in \text{cl}K(x, \lambda)\}$. Throughout the paper assume that all the solution sets under consideration are nonempty for all (λ, μ, η) in a neighborhood of $(\lambda_0, \mu_0, \eta_0) \in \Lambda \times M \times N$.

Theorem 2.1. Assume that $E(.)$ is lsc at λ_0 and the following set is open in $\text{cl}K(X, \Lambda) \times \{(\lambda_0, \mu_0, \eta_0)\}$:

$$U_{\text{r}\alpha} := \{(x, \lambda, \mu, \eta) \in X \times \Lambda \times M \times N | \, (y, x^*) \, \bar{\text{r}} \, K(x, \lambda) \times G(x, \eta),$$

$\alpha(F(x^*, y, \mu), Y \setminus -\text{int}C\}$.

Then $S_{\text{r}\alpha}$ is lsc at $(\lambda_0, \mu_0, \eta_0)$.

Proof. Since $\text{r} \in \{\text{w}, \text{m}, \text{s}\}$ and $\alpha \in \{\alpha_1, \alpha_2\}$, we have in fact six cases corresponding to six different combinations of values of r and α. However, the proof techniques are similar. We consider only the case where $\text{r} = \text{w}$ and $\alpha = \alpha_1$. Suppose to the contrary that $S_{\text{w}\alpha_1}(.,.,.)$ is not lsc at $(\lambda_0, \mu_0, \eta_0)$, i.e., $\exists x_0 \in S_{\text{w}\alpha_1}(\lambda_0, \mu_0, \eta_0)$, $\exists(\lambda_\gamma, \mu_\gamma, \eta_\gamma) \to (\lambda_0, \mu_0, \eta_0)$, $\forall x_\gamma \in S_{\text{w}\alpha_1}(\lambda_\gamma, \mu_\gamma, \eta_\gamma)$, $x_\gamma \not\to x_0$.

Since $E(.)$ is lsc at λ_0, there is a net $\bar{x}_\gamma \in E(\lambda_\gamma)$, $\bar{x}_\gamma \to x_0$. By the contradiction assumption, there must be a subnet $\bar{x}_\beta$ such that, $\forall \beta$, $\bar{x}_\beta \notin S_{\text{w}\alpha_1}(\lambda_\beta, \mu_\beta, \eta_\beta)$, i.e., for some $y_\beta \in K(\bar{x}_\beta, \lambda_\beta)$, $\forall \bar{x}^*_\beta \in G(\bar{x}_\beta, \eta_\beta)$,

$$F(\bar{x}^*_\beta, y_\beta, \mu_\beta) \subseteq -\text{int}C. \tag{1}$$

Hence, $(\bar{x}_\beta, \lambda_\beta, \mu_\beta, \eta_\beta) \notin U_{\text{w}\alpha_1}$. By the assumed openness, $(x_0, \lambda_0, \mu_0, \eta_0) \notin U_{\text{w}\alpha_1}$, contradicting the fact that $x_0 \in S_{\text{w}\alpha_1}(\lambda_0, \mu_0, \eta_0)$. □

To compare this theorem with the corresponding ones of Ref. 3 recall a notion.

Definition 2.1 (Ref. 3). Let X and Y be as above and $C \subseteq Y$ be such that $\text{int}C \neq \emptyset$.

(a) A multifunction $Q : X \longrightarrow 2^Y$ is said to have the $C-$inclusion property at x_0 if, for any $x_\gamma \to x_0$, $Q(x_0) \cap (Y \setminus -\text{int}C) \neq \emptyset \Rightarrow \exists \bar{\gamma}, Q(x_{\bar{\gamma}}) \cap$

$(Y\backslash -\text{int}C) \neq \emptyset$.

(b) Q is called to have the strict $C-$ inclusion property at x_0 if, for all $x_\gamma \to x_0$, $Q(x_0) \subseteq Y\backslash -\text{int}C \Rightarrow \exists \bar{\gamma}, Q(x_{\bar{\gamma}}) \subseteq Y\backslash -\text{int}C$.

Remark 2.1. Assume that $K(.,.)$ is usc and has compact values in $\text{cl}K(X, \Lambda) \times \{\lambda_0\}$ and $F(.,.,.)$ has the $C-$inclusion property in $\text{cl}K(X,\Lambda) \times \{\mu_0\}$. Then

(i) if $G(.,.)$ is lsc in $\text{cl}K(X,\Lambda) \times \{\eta_0\}$, then $U_{\text{w}\alpha_1}$ and $U_{\text{m}\alpha_1}$ are open in $\text{cl}K(X,\Lambda) \times \{(\lambda_0, \mu_0, \eta_0)\}$;

(ii) if $G(.,.)$ is usc and compact-valued in $\text{cl}K(X,\Lambda) \times \{\eta_0\}$, then $U_{\text{s}\alpha_1}$ is open in $\text{cl}K(X,\Lambda) \times \{(\lambda_0, \mu_0, \eta_0)\}$.

By the similarity we consider only $U_{\text{w}\alpha_1}$ in assertion (i). To show that the complement $U^c_{\text{w}\alpha_1}$ is closed, let $(x_\gamma, y_\gamma, \mu_\gamma, \eta_\gamma) \to (x_0, y_0, \mu_0, \eta_0)$ such that $\exists y_\gamma \in K(x_\gamma, \lambda_\gamma), \forall x^*_\gamma \in G(x_\gamma, \eta_\gamma), F(x^*_\gamma, y_\gamma, \mu_\gamma) \subseteq -\text{int}C$. As $K(.,.)$ is usc and compact-valued at (x_0, λ_0), we we can assume that $y_\gamma \to y_0$ for some $y_0 \in K(x_0, \lambda_0)$. By the assumed lower semicontinuity of $G(.,.)$ at (x_0, η_0) , $\forall x^*_0 \in G(x_0, \eta_0)$, $\exists x^*_\gamma \in G(x_\gamma, \eta_\gamma)$, $x^*_\gamma \to x^*_0$. Suppose that

$$F(x^*_0, y_0, \mu_0) \cap (Y \setminus -\text{int}C) \neq \emptyset.$$

By the $C-$inclusion property of $F(.,.,.)$, $\exists \bar{\gamma}$ such that $F(x^*_{\bar{\gamma}}, y_{\bar{\gamma}}, \mu_{\bar{\gamma}}) \cap (Y \backslash -\text{int}C) \neq \emptyset$, which is impossible. Hence $(x_0, \lambda_0, \mu_0, \eta_0) \in U^c_{\text{w}\alpha_1}$ □

If $G(x,\eta) = \{x\}$ then the problems $(P_{\mathrm{w}\alpha_1}), (P_{\mathrm{m}\alpha_1})$ and $(P_{\mathrm{s}\alpha_1})$ collapse to problem (QEP) studied in Ref. 3. Remark 2.1 indicates that in this special case Theorem 2.1 implies Theorem 2.2 of Ref. 3. The following three examples point out that none of the three assertions of Remark 2.1 has the converse which is true and hence Theorem 2.1 is strictly stronger than Theorem 2.2 of Ref. 3. They show also that the assumption of Theorem 2.1 (and also that of the coming results of the paper) is not difficult to be checked. (See also examples in Ref. 3.)

Example 2.1. Let $X = Y = R$, $\Lambda \equiv M \equiv N = R$, $C = R_+$, $K(x,\lambda) = [0,1]$, $\lambda_0 = 0$ and

$$G(x,\lambda) = \begin{cases} [0,1] & \text{if} \quad \lambda \in Q, \\ [2,3] & \text{otherwise,} \end{cases}$$

$$F(x,y,\lambda) = (x, +\infty),$$

where Q is the set of all rational numbers. Then, $U_{\mathrm{w}\alpha_1}$ is open and in fact $S_{\mathrm{w}\alpha_1}(\lambda) = [0,1]$, $\forall \lambda \in R$, is lsc but $G(.,.)$ is not lsc at any (x, λ_0).

Example 2.2. Let $X, Y, \Lambda, M, N, C, K$ and λ_0 be as in Example 2.1 and let

$$G(x,\lambda) = \begin{cases} [0,1] & \text{if} \quad \lambda \in Q, \\ [1,2] & \text{otherwise,} \end{cases}$$

$$F(x,y,\lambda) = (-\infty, x-y].$$

Then $U_{\mathrm{m}\alpha_1}$ is open and $S_{\mathrm{m}\alpha_1}(\lambda) = [0,1]$, $\forall \lambda \in R$, is lsc but $G(.,.)$ is not lsc at any point (x, λ_0).

Example 2.3. Let X, Y, C and λ_0 be as in Example 2.1. Let $\Lambda \equiv M \equiv N = [0,1]$, $K(x,\lambda) = [\lambda, \lambda+1]$ and

$$G(x,\lambda) = \begin{cases} [1,+\infty) & \text{if} \quad \lambda \in Q, \\ (-\infty,-1] & \text{otherwise,} \end{cases}$$

$$F(x,y,\lambda) = \begin{cases} \{1\} & \text{if} \quad \lambda \in Q, \\ \{0\} & \text{otherwise.} \end{cases}$$

Then $U_{\mathrm{s}\alpha_1}$ is open and $S_{\mathrm{s}\alpha_1}(\lambda) = [\lambda, \lambda+1]$, $\forall \lambda \in [0,1]$, is lsc but $G(.,.)$ is not usc at any (x, λ_0) and does not have compact values.

Remark 2.2. Assume that $K(.,.)$ is usc and has compact values in $\mathrm{cl}K(X, \Lambda) \times \{\lambda_0\}$ and $F(.,.,.)$ has the strict $C-$inclusion property in $\mathrm{cl}K(X,\Lambda) \times \{\mu_0\}$. Then the following assertions hold.

(i) If $G(.,.)$ is lsc in $\mathrm{cl}K(X,\Lambda) \times \{\eta_0\}$, then $U_{\mathrm{w}\alpha_2}$ and $U_{\mathrm{m}\alpha_2}$ are open in $\mathrm{cl}K(X,\Lambda) \times \{(\lambda_0, \mu_0, \eta_0)\}$.

(ii) If $G(.,.)$ is usc and compact values in $\mathrm{cl}K(X,\Lambda) \times \{\eta_0\}$, then $U_{\mathrm{s}\alpha_2}$ is open in $\mathrm{cl}K(X,\Lambda) \times \{(\lambda_0, \mu_0, \eta_0)\}$.

We can check the assertions similarly as for Remark 2.1.

This remark shows that, for the special case where $G(x,\eta) = \{x\}$, Theorem 2.1 derives Theorem 2.4 of Ref. 3. The following three examples demonstrate that none of the three assertions in Remark 2.2 has the converse which is valid and hence Theorem 2.1 is strictly stronger than this Theorem 2.4.

Example 2.4. Let $X, Y, \Lambda, M, N, C, K$ and λ_0 be as in Example 2.1 and

$$G(x,\lambda) = \begin{cases} (1,+\infty) & \text{if} \quad \lambda \in Q, \\ (-\infty, 1) & \text{otherwise,} \end{cases}$$

$$F(x,y,\lambda) = [x,+\infty).$$

Then $U_{\mathrm{w}\alpha_2}$ is open and $S_{\mathrm{w}\alpha_2}(\lambda) = [0,1]$, $\forall\lambda \in R$, is lsc but $G(.,.)$ is not lsc at any (x,λ_0).

Example 2.5. Let $X, Y, \Lambda, M, N, C, K$ and λ_0 be as in Example 2.3 and let

$$G(x,\lambda) = \begin{cases} [-1,+\infty) & \text{if} \quad \lambda \in Q, \\ (-\infty, 1] & \text{otherwise,} \end{cases}$$

$$F(x,y,\lambda) = [x(x-y),+\infty).$$

The $U_{\mathrm{m}\alpha_2}$ is open and $S_{\mathrm{m}\alpha_2}(\lambda) = [\lambda, \lambda+1]$, $\forall\lambda \in [0,1]$, but $G(.,.)$ is not lsc at any (x,λ_0).

Example 2.6. Example 2.3 can be used here since, $F(.,.,.)$ is single-valued.

The openness assumptions imposed in Theorem 2.1 can be replaced by more usual semicontinuity assumptions as follows (but we have to impose additional assumptions).

Theorem 2.2. Assume that $K(.,.)$ is usc and has compact values in $\mathrm{cl}K(X,\Lambda)\times\{\lambda_0\}$ and $E(.)$ is lsc at λ_0. Assume further that $\forall x\in S_{\mathrm{r}\alpha}(\lambda_0,\mu_0,\eta_0)$, $(y,x^*)\,\mathrm{r}\,K(x,\lambda_0)\times G(x,\eta_0), \alpha(F(x^*,y,\mu_0),Y\setminus C)$. Then the following assertions hold.

(i) If $\alpha=\alpha_1$ and $\mathrm{r}=\mathrm{w}$ (or m), $G(.,.)$ is lsc in $\mathrm{cl}K(X,\Lambda)\times\{\eta_0\}$ and $F(.,.,.)$ is lsc in $\mathrm{cl}K(X,\Lambda)\times\mathrm{cl}K(X,\Lambda)\times\{\mu_0\}$, then $S_{\mathrm{w}\alpha_1}$ (or $S_{\mathrm{m}\alpha_1}$, respectively) is lsc at (λ_0,μ_0,η_0).

(ii) If $\alpha=\alpha_1$ and $\mathrm{r}=\mathrm{s}$, $G(.,.)$ is usc and compact-valued in $\mathrm{cl}K(X,\Lambda)\times\{\eta_0\}$ and $F(.,.,.)$ is lsc in $\mathrm{cl}K(X,\Lambda)\times\mathrm{cl}K(X,\Lambda)\times\{\mu_0\}$, then $S_{\mathrm{s}\alpha_1}$ is lsc at (λ_0,μ_0,η_0).

(iii) If $\alpha=\alpha_2$ and $\mathrm{r}=\mathrm{w}$ (or m), $G(.,.)$ is lsc in $\mathrm{cl}K(X,\Lambda)\times\{\eta_0\}$ and $F(.,.,.)$ is usc in $\mathrm{cl}K(X,\Lambda)\times\mathrm{cl}K(X,\Lambda)\times\{\mu_0\}$, then $S_{\mathrm{w}\alpha_2}$ (or $S_{\mathrm{m}\alpha_2}$) is lsc at (λ_0,μ_0,η_0).

(iv) If $\alpha=\alpha_2$ and $\mathrm{r}=\mathrm{s}$, $G(.,.)$ is usc and compact-valued in $\mathrm{cl}K(X,\Lambda)\times\{\eta_0\}$ and $F(.,.,.)$ is usc in $\mathrm{cl}K(X,\Lambda)\times\mathrm{cl}K(X,\Lambda)\times\{\mu_0\}$, then $S_{\mathrm{s}\alpha_2}$ is lsc at (λ_0,μ_0,η_0).

Proof. As an example we demonstrate only (ii). Suppose $\exists x_0 \in S_{s\alpha_1}(\lambda_0, \mu_0, \eta_0)$, $\exists(\lambda_\gamma, \mu_\gamma, \eta_\gamma) \to (\lambda_0, \mu_0, \eta_0)$, $\forall x_\gamma \in S_{s\alpha_1}(\lambda_\gamma, \mu_\gamma, \eta_\gamma), x_\gamma \not\to x_0$. By the lower semicontinuity of $E(.)$, there is $\bar{x}_\gamma \in E(\lambda_\gamma)$, $\bar{x}_\gamma \to x_0$. The contradiction assumption yields a subnet $\bar{x}_\beta$ such that $\bar{x}_\beta \notin S_{s\alpha_1}(\lambda_\beta, \mu_\beta, \eta_\beta)$, $\forall\beta$, i.e. for some $y_\beta \in K(\bar{x}_\beta, \lambda_\beta)$ and $\bar{x}^*_\beta \in G(\bar{x}_\beta, \eta_\beta)$ one has

$$F(\bar{x}^*_\beta, y_\beta, \mu_\beta) \subseteq -\text{int}C.$$

Since $K(.,.)$ and $G(.,.)$ are usc and have compact values in $\text{cl}K(X, \Lambda) \times \{\lambda_0\}$ and $\text{cl}K(X, \Lambda) \times \{\eta_0\}$, respectively, one can assume that $y_\beta \to y_0 \in K(x_0, \lambda_0)$ and $x^*_\beta \to x^*_0 \in G(x_0, \eta_0)$. By the common assumption of the theorem there exists $f_0 \in F(x^*_0, y_0, \mu_0) \setminus -C$. From the lower semicontinuity of $F(.,.,.)$ there is $f_\beta \in F(\bar{x}^*_\beta, y_\beta, \mu_\beta)$ such that $f_\beta \to f_0 \notin -C$, which is a contradiction, since $f_\beta \in -\text{int}C$, $\forall\beta$. □

Remark 2.3.

(a) If $G(x, \eta) = \{x\}$, then Theorem 2.2 is reduced to Theorems 2.1 and 2.3 together of Ref. 3.

(b) If $G(x, \eta) = \{x\}$ and $F(x, y, \mu) = (T(x, \mu), y - g(x, \mu))$, where $T : X \times M \to 2^{L(X,Y)}$ and $g : X \times M \to X$ is continuous ($L(X, Y)$ is the space of all continuous linear mappings of X into Y), then our problem becomes vector quasivariational inequalities. If, furthermore, $Y = R$, then Theorem 2.2 collapses to Theorems 3.1, 3.2 and 3.3 together of Ref. 20.

(c) Even for the case, where G and F are as in (b), Theorem 2.1 is new for vector quasivariational inequalities.

3. Upper Semicontinuity

In this section we investigate sufficient conditions for the solution multifunctions to be usc in each of the three senses mentioned in Section 1.

Mention first some relations between the three notions of upper semicontinuity. Let X and Y be as before and $G : X \to 2^Y$ be a multifunction.

Proposition 3.1 (Ref. 3).

(i) If G is usc at x_0 then G is H-usc at x_0. Conversely if G is H-usc at x_0 and if $G(x_0)$ is compact, then G is usc at x_0.

(ii) If G is H-usc at x_0 and $G(x_0)$ is closed, then G is closed at x_0.

(iii) If $G(A)$ is compact for any compact subset A of dom$G$ and G is closed at x_0, then G is usc at x_0.

(iv) If Y is compact and G is closed at x_0, then G is usc at x_0.

Theorem 3.1. Assume that E(.) is usc at λ_0 and $E(\lambda_0)$ is compact. Assume further that the set $U_{r\alpha}$ (defined in Theorem 2.1) is closed in cl$K(X,\mathit{\Lambda}) \times \{(\lambda_0, \mu_0, \eta_0)\}$. Then $S_{\mathrm{r}\alpha}$ is both usc and closed at $(\lambda_0, \mu_0, \eta_0)$.

Proof. Similar arguments can be applied to prove the six cases. We present only the proof for the case where r = m and $\alpha = \alpha_1$. Suppose to the contrary that there is an open superset U of $S_{\mathrm{m}\alpha_1}(\lambda_0, \mu_0, \eta_0)$ such that, for any $(\lambda_\gamma, \mu_\gamma, \eta_\gamma) \to (\lambda_0, \mu_0, \eta_0)$, there exists $x_\gamma \in S_{\mathrm{m}\alpha_1}(\lambda_\gamma, \mu_\gamma, \eta_\gamma) \setminus U$, $\forall\gamma$. Since $E(.)$ is usc and $E(\lambda_0)$ is compact, one can assume that $x_\gamma \to x_0$ for some $x_0 \in E(\lambda_0)$. As $x_\gamma \in S_{\mathrm{m}\alpha_1}(\lambda_\gamma, \mu_\gamma, \eta_\gamma)$, $\exists x_\gamma^* \in G(x_\gamma, \eta_\gamma)$, $\forall y_\gamma \in K(x_\gamma, \lambda_\gamma)$, $F(x_\gamma^*, y_\gamma, \mu_\gamma) \nsubseteq -\mathrm{int}C$. By the closedness assumption one has $(x_0, \alpha_0, \mu_0, \eta_0) \in U_{\mathrm{m}\alpha_1}$, i.e., $\exists x_0^* \in G(x_0, \eta_0)$, $\forall y_0 \in K(x_0, \lambda_0)$, $F(x_0^*, y_0, \mu_0) \nsubseteq -\mathrm{int}C$. This means that $x_0 \in S_{\mathrm{m}\alpha_1}(\lambda_0, \mu_0, \eta_0) \subseteq U$, which contradicts the fact that $x_\gamma \notin U$, $\forall\gamma$.

The proof of the closedness of $S_{\mathrm{m}\alpha_1}(.,.,.)$ is similar. □

Remark 3.1. Assume that $K(.,.)$ is lsc in cl$K(X,\mathit{\Lambda}) \times \{\lambda_0\}$, $G(.,.)$ is usc and compact-valued in cl$K(X,\mathit{\Lambda}) \times \{\eta_0\}$ and $F(.,.,.)$ is usc in cl$K(X,\mathit{\Lambda}) \times$ cl$K(X,\mathit{\Lambda}) \times \{\mu_0\}$. Then $U_{\mathrm{w}\alpha_1}$ and $U_{\mathrm{m}\alpha_1}$ are closed in cl$K(X,\mathit{\Lambda}) \times \{(\lambda_0, \mu_0, \eta_0)\}$. Indeed, consider $U_{\mathrm{w}\alpha_1}$ for instance. Assume that $(x_\gamma, \lambda_\gamma, \mu_\gamma, \eta_\gamma) \to (x_0, \lambda_0, \mu_0, \eta_0)$, such that $\forall y_\gamma \in K(x_\gamma,\lambda_\gamma), \exists x^*_\gamma \in G(x_\gamma,\eta_\gamma), F(x^*_\gamma,y_\gamma,\mu_\gamma) \not\subseteq -\text{int}C$. By the assumption about $G(.,.)$, there is a subnet x^*_β and $x^*_0 \in G(x_0,\eta_0)$ such that $x^*_\beta \to x^*_0$. As $K(.,.)$ is lsc, $\forall y_0 \in K(x_0,\lambda_0), \exists y_\beta \in K(x_\beta,\lambda_\beta)$, $y_\beta \to y_0$. Since $F(.,.,.)$ is usc and $F(x^*_\beta,y_\beta,\mu_\beta) \not\subseteq -\text{int}C$, one has $F(x^*_0,y_0,\mu_0) \not\subseteq -\text{int}C$, i.e. $U_{\text{w}\alpha_1}$ is closed.

The following examples tell us that the converse is not true.

Example 3.1. Let $X = Y = R$, $\Lambda \equiv M \equiv N = R$, $C = R_+$, $K(x,\lambda) = [\lambda,\lambda+1]$, $\lambda_0 = 0$ and

$$G(x,\lambda) = \begin{cases} \{1\} & \text{if} \quad \lambda \in Q, \\ \{-1\} & \text{otherwise}, \end{cases}$$

$$F(x,y,\lambda) = \begin{cases} \{1\} & \text{if} \quad \lambda \in Q, \\ \{0\} & \text{otherwise}. \end{cases}$$

Then, it is not hard to see that $U_{\text{w}\alpha_1}$ is closed and in fact $S_{\text{w}\alpha_1}(\lambda) = [\lambda,\lambda+1]$, for all $\lambda \in R$, is usc and closed. But $G(.,.)$ and $F(.,.,.)$ are not even H-usc.

Example 3.2. Let $X = Y = R$, $\Lambda \equiv M \equiv N = R$, $C = R_+$, $K(x,\lambda) = [0,1]$, $\lambda_0 = 0$ and

$$G(x,\lambda) = \begin{cases} (-\infty, 2^x) & \text{if} \quad \lambda \in Q, \\ (-2^x, +\infty) & \text{otherwise}, \end{cases}$$

$$F(x,y,\lambda) = \begin{cases} \{x(x^2-y)\} & \text{if} \quad \lambda \in Q, \\ (-\infty, -x^2+1) & \text{otherwise}. \end{cases}$$

Then, we have the closedness of $U_{\text{m}\alpha_1}$ and in fact $S_{\text{m}\alpha_1}(\lambda) = [0,1]$, for all $\lambda \in R$, is usc and closed. But $G(.,.)$ and $F(.,.,.)$ are not usc.

Remark 3.2. If $K(.,.)$ is lsc in $\mathrm{cl}K(X,\Lambda)\times\{\lambda_0\}$, $G(.,.)$ is lsc in $\mathrm{cl}K(X,\Lambda)\times\{\eta_0\}$ and $F(.,.,.)$ is usc in $\mathrm{cl}K(X,\Lambda)\times\mathrm{cl}K(X,\Lambda)\times\{\mu_0\}$, then $U_{\mathrm{s}\alpha_1}$ is closed. The converse is not valid as indicated by

Example 3.3. Let $X, Y, \Lambda, M, N, C, K$ and λ_0 be as in Example 3.2. Let

$$G(x,\lambda)=\begin{cases}[0,+\infty) & \text{if}\quad \lambda\in Q,\\ (-\infty,0] & \text{otherwise,}\end{cases}$$

$$F(x,y,\lambda)=\begin{cases}(1,+\infty) & \text{if}\quad \lambda\in Q,\\ (-\infty,0] & \text{otherwise.}\end{cases}$$

Then $U_{\mathrm{s}\alpha_1}$ is closed and $S_{\mathrm{s}\alpha_1}(\lambda)=[0,1]$, for all $\lambda\in R$. But $G(.,.)$ and $F(.,.,.)$ are neither usc nor lsc in $R\times\{0\}$ and $R\times R\times\{0\}$, respectively.

If $G(x,\eta)=\{x\}$ then the first three cases of our problem coincide with problem (QEP) in Ref. 3. Three remarks and examples above show that even for this special case Theorem 3.1 is strictly stronger than Theorem 3.2 of Ref. 3. It should be noted here that the assumption $\mathrm{cl}K(.,.)$ is usc and has compact values in $X\times\{\lambda_0\}$ in Ref. 3 is not enough (the proof there has a mistake). In fact this assmption should be replaced by "$E(.)$ is usc and has compact values at λ_0" as made in Theorem 3.1.

Remark 3.3. If $G(.,.)$ is usc and has compact values in $\mathrm{cl}K(X,\Lambda)\times\{\eta_0\}$ and $F(.,.,.)$ is lsc in $\mathrm{cl}K(X,\Lambda)\times\mathrm{cl}K(X,\Lambda)\times\{\mu_0\}$, then $U_{\mathrm{w}\alpha_2}$ is closed. However, Example 3.1 proves that the converse is false.

Remark 3.4. If $K(.,.)$ is lsc in $\mathrm{cl}K(X,\Lambda)\times\{\lambda_0\}$, and $G(.,.)$ and $F(.,.,.)$ are as in Remark 3.3, then $U_{\mathrm{m}\alpha_2}$ is closed. The converse does not hold as shown by

Example 3.4. Let $X, Y, \Lambda, M, N, C, K$ and λ_0 be as in Example 3.2. Let

$$G(x,\lambda)=\begin{cases}(1,+\infty) & \text{if} \quad \lambda\in Q,\\ (-\infty,-1) & \text{otherwise,}\end{cases}$$

$$F(x,y,\lambda)=\begin{cases}\{1\} & \text{if} \quad \lambda\in Q,\\ \{0\} & \text{otherwise.}\end{cases}$$

Then $U_{\mathrm{m}\alpha_2}$ is closed and in fact $S_{\mathrm{m}\alpha_2}(\lambda)=[0,1]$, for all $\lambda\in\Lambda$, is both usc and closed. But $G(.,.)$ and $F(.,.,.)$ are neither usc nor lsc in $\mathrm{cl}K(X,\Lambda)\times\{0\}$ and $\mathrm{cl}K(X,\Lambda)\times\mathrm{cl}K(X,\Lambda)\times\{0\}$, respectively.

Remark 3.5. If $K(.,.)$, $G(.,.)$ and $F(.,.,.)$ are lsc in $\mathrm{cl}K(X,\Lambda)\times\{\lambda_0\}$, $\mathrm{cl}K(X,\Lambda)\times\{\eta_0\}$ and $\mathrm{cl}K(X,\Lambda)\times\mathrm{cl}K(X,\Lambda)\times\{\mu_0\}$, respectively, then $U_{\mathrm{s}\alpha_2}$ is closed. The converse is not valid by Example 3.4.

If $G(x,\eta)=\{x\}$ then the last three cases of our problem fall into problem (SQEP) studied in Ref. 3. The previous three remarks indicate that Theorem 3.1 strictly improves Theorem 3.4 of Ref. 3.

Remark 3.6. Theorem 3.1 is strictly stronger than the corresponding result in Refs. 1 and 2, since many assumptions there, like convexity, boundedness, monotonicity are omitted.

Now we pass to considering Hausdorff upper semicontinuity.

Theorem 3.2 Assume that $K(.,.)$ is lsc in $\mathrm{cl}K(X,\varLambda)\times\{\lambda_0\}$, $E(.)$ is H-usc at λ_0, $E(\lambda_0)$ is compact and $F(.,.,.)$ are H-usc in $\mathrm{cl}K(X,\varLambda)\times\mathrm{cl}K(X,\varLambda)\times\{\mu_0\}$. Assume further that $\forall B_X$ (neighborhood of 0 in X), $\forall x\notin S_{\mathrm{r}\alpha_1}(\lambda_0,\mu_0,\eta_0)+B_X,\exists B_Y$ (neighborhood of 0 in Y), $(y,r^*)\,\bar{\mathrm{r}}\,K(x,\lambda_0)\times G(x,\eta_0),\bar{\alpha}_1(F(x^*,y,\mu_0)+B_Y,Y\setminus -\mathrm{int}C)$. Then the following hold.

(i) If r = w (or m) and $G(.,.)$ is H-usc and compact-valued in $\mathrm{cl}K(X,\varLambda)\times\{\eta_0\}$, then $S_{\mathrm{w}\alpha_1}$ (or $S_{\mathrm{m}\alpha_1}$, respectively) is H-usc at (λ_0,μ_0,η_0).

(ii) If r = s and $G(.,.)$ is lsc in $\mathrm{cl}K(X,\varLambda)\times\{\eta_0\}$, then $S_{\mathrm{s}\alpha_1}$ is H-usc at (λ_0,μ_0,η_0).

Proof. We demonstrate only for $S_{\mathrm{w}\alpha_1}$. Arguing by the contradiction suppose $\exists B_X$, $\exists(\lambda_\gamma,\mu_\gamma,\eta_\gamma)\to(\lambda_0,\mu_0,\eta_0)$, $\exists x_\gamma\in S_{\mathrm{w}\alpha_1}(\lambda_\gamma,\mu_\gamma,\eta_\gamma)$, $x_\gamma\notin S_{\mathrm{w}\alpha_1}(\lambda_0,\mu_0,\eta_0)+B_X$. By the Hausdorff upper semicontinuity of $E(.)$ at λ_0 and the compactness of $E(\lambda_0)$ one can assume that $x_\gamma\to x_0$ for some $x_0\in E(\lambda_0)$. If $x_0\notin S_{\mathrm{w}\alpha_1}(\lambda_0,\mu_0,\eta_0)+B_X$, $\exists B_Y$, $\exists y_0\in K(x_0,\lambda_0)$, $\forall x_0^*\in G(x_0,\eta_0)$, $F(x_0^*,y_0,\mu_0)+B_Y\subseteq -\mathrm{int}C$. Since $K(.,.)$ is lsc, $\exists y_\gamma\in K(x_\gamma,\lambda_\gamma)$, $y_\gamma\to y_0$. As $x_\gamma\in$

$S_{w\alpha_1}(\lambda_\gamma, \mu_\gamma, \eta_\gamma)$, $\exists x^*_\gamma \in G(x_\gamma, \eta_\gamma)$,

$$F(x^*_\gamma, y_\gamma, \mu_\gamma) \cap (Y \backslash -\text{int}C) \neq \emptyset. \tag{2}$$

Because of the Hausdorff upper semicontinuity of $G(.,.)$ and the compactness of $G(x_0, \eta_0)$, one has a subnet x^*_β and $x^*_0 \in G(x_0, \eta_0)$, $x^*_\beta \to x^*_0$. Since $F(.,.,.)$ is H-usc, one can assume that

$$F(x^*_\beta, y_\beta, \mu_\beta) \subseteq F(x^*_0, y_0, \mu_0) + B_Y \subseteq -\text{int}C,$$

which contradicts (2). □

Remark 3.7. If $G(x, \eta) = \{x\}$ then Theorem 3.2 collapses to Theorem 3.3 of Ref. 3. However, it should be added that the assumption that $K(.,.)$ is H-usc and has compact values in $X \times \{\lambda_0\}$ in Ref. 3 should be replaced by "$E(.)$ is H- usc and has compact values in $X \times \{\lambda_0\}$" as in Theorem 3.2, since there is an error in the proof in Ref. 3.

4. Comparison of the six solution sets

In this section we investigate the sufficient conditions for possible coincidences of solution sets.

Lemma 4.1. Assume that $\forall x \in S_{w\alpha_1}(\lambda_0, \mu_0, \eta_0), \forall y \in K(x, \lambda_0), F(G(x, \eta_0), y, \mu_0)$ is arcwisely connected and does not meet the boundary of $-C$. Then

$S_{\mathrm{w}\alpha_1}(\lambda_0, \mu_0, \eta_0) = S_{\mathrm{m}\alpha_1}(\lambda_0, \mu_0, \eta_0) = S_{\mathrm{w}\alpha_2}(\lambda_0, \mu_0, \eta_0).$

Proof. We always have $S_{\mathrm{w}\alpha_2}(\lambda_0, \mu_0, \eta_0) \subseteq S_{\mathrm{w}\alpha_1}(\lambda_0, \mu_0, \eta_0)$. To see the reverse inclusion let $x \notin S_{\mathrm{w}\alpha_2}(\lambda_0, \mu_0, \eta_0)$ then $\exists y \in K(x, \lambda_0)$, $\forall x^* \in G(x, \eta_0)$, $\exists z_1 \in F(x^*, y, \mu_0) \cap -\mathrm{int}C$. Suppose that $x \in S_{\mathrm{w}\alpha_1}(\lambda_0, \mu_0, \eta_0)$. Then $\exists x^{**} \in G(x, \eta_0), \exists z_2 \in F(x^{**}, y, \mu_0) \backslash (-C)$. Since $F(G(x, \eta_0), y, \mu_0)$ is arcwisely connected, there exists a continuous mapping $\varphi : [0, 1] \to F(G(x, \eta_0), y, \mu_0)$ such that $\varphi(0) = z_1$ and $\varphi(1) = z_2$. Let $T = \{t \in (0, 1] : \varphi([t, 1]) \subseteq Y \backslash (-C)\}$ and $t_0 = \inf T$. Since $z_1 \in -\mathrm{int}C$ there is an open set A such that $A \cap F(G(x, \eta_0), y, \mu_0)$ is arcwisely connected and $z_1 \in A \subseteq -\mathrm{int}C$. Then $\varphi^{-1}(A \cap F(G(x, \eta_0), y, \mu_0)) \cap T = \emptyset$. Since $\varphi^{-1}(A \cap F(G(x, \eta_0), y, \mu_0))$ is open in $[0, 1]$, it is of the form $[0, t_1)$. So it contains 0 and $0 < t_1 \le t_0$. Similarly, $t_0 < 1$. Then, for all large n, there is $t_n \in \left(t_0 - \frac{1}{n}, t_0\right]$ such that $\varphi(t_n) \in -C$. Then $\varphi(t_0) \in -C$, since $t_n \to t_0$ and $-C$ is closed. On the other hand, for all large n, there is $t_n \in \left(t_0, t_0 + \frac{1}{n}\right)$ such that $\varphi(t_n) \in Y \backslash (-C)$. So $\varphi(t_0) \in \mathrm{cl}(Y \backslash (-C))$. Thus $\varphi(t_0)$ is in the boundary of $-C$, contradicting the fact that $\varphi(t_0) \in F(G(x, \eta_0), y, \mu_0)$.

Similarly, one has $S_{\mathrm{m}\alpha_1}(\lambda_0, \mu_0, \eta_0) \subseteq S_{\mathrm{w}\alpha_1}(\lambda_0, \mu_0, \eta_0)$. To see the reverse inclusion let $x_1 \notin S_{\mathrm{m}\alpha_1}(\lambda_0, \mu_0, \eta_0)$, i.e., $\forall x_1^* \in G(x_1, \eta_0)$, $\exists y_1 \in K(x_1, \lambda_0)$ such that $F(x_1^*, y_1, \mu_0) \subseteq -\mathrm{int}C$. Suppose that $x_1 \in S_{\mathrm{w}\alpha_1}(\lambda_0, \mu_0, \eta_0)$. Then $\exists x_1^{**} \in G(x_1, \eta_0)$ such that $\exists z_3 \in F(x_1^{**}, y_1, \mu_0), z_3 \notin -C$. The further argument is the same as above. $\square$

The following two examples show that the arcwise connectedness condition is essential.

Example 4.1. Let $X = Y = R, \mathit{\Lambda} = M = N = [0, 1], C = R_+, K(x, \lambda) = [0, 1]$, $\lambda_0 = \mu_0 = \eta_0 = 0$ and $G(x, \eta) = \{x\}$, $F(x, y, \mu) = \{-x, x\}$. Then, it is not hard to see that $S_{\mathrm{w}\alpha_1}(0, 0, 0) = [0, 1]$, but $S_{\mathrm{w}\alpha_2}(0, 0, 0) = \{0\}$. The reason is that $F(G(x, \eta_0), y, \mu_0) = \{-x, x\}$ is not arcwisely connected $\forall x \in (0, 1] \subseteq S_{\mathrm{w}\alpha_1}(0, 0, 0)$.

Example 4.2. Let $X, Y, \mathit{\Lambda}, M, N, C, K, \lambda_0, \mu_0$ and η_0 be as in Example 4.1. Let $G(x, \eta) = [0, 1]$ and

$$F(x, y, \mu) = \begin{cases} \{1\} & \text{if} \quad x + y + \mu \in Q, \\ \{-1\} & \text{otherwise.} \end{cases}$$

Since $\forall x \in [0, 1], \forall y \in [0, 1], \exists x^* := 1 - y \in [0, 1] = G(x, \eta_0)$ such that $F(x^*, y, \mu_0) \notin -\mathrm{int}C$, one has $S_{\mathrm{w}\alpha_1}(0, 0, 0) = [0, 1]$. But $S_{\mathrm{m}\alpha_1}(0, 0, 0) = \emptyset$. The reason is that $F(G(x, \eta_0), y, \mu_0) = \{-1, 1\}$ is not arcwisely connected $\forall x \in [0, 1] = S_{\mathrm{w}\alpha_1}(0, 0, 0)$.

Lemma 4.2. Assume that $\forall x \in S_{\mathrm{m}\alpha_1}(\lambda_0, \mu_0, \eta_0), \forall y \in K(x, \lambda_0), F(G(x, \eta_0), y, \mu_0)$ is arcwisely connected and does not meet the boundary of $-C$. Then $S_{\mathrm{m}\alpha_1}(\lambda_0, \mu_0, \eta_0) = S_{\mathrm{s}\alpha_1}(\lambda_0, \mu_0, \eta_0) = S_{\mathrm{m}\alpha_2}(\lambda_0, \mu_0, \eta_0)$.

Proof. The proof is similar to that of Lemma 4.1. The only difference is that the order of y and x^* is disposed vice versa. $\square$

The following two examples explain why we assume that $F(G(x,\eta_0),y,\mu_0)$ does not meet the boundary of $-C$.

Example 4.3. Let $X,Y,\Lambda,M,N,C,K,\lambda_0$ and $G(x,\eta)$ be as in Example 4.1 and let $F(x,y,\mu)=[-x,x]$. Then, $S_{\mathrm{m}\alpha_1}(0,0,0)=[0,1]$, but $S_{\mathrm{m}\alpha_2}(0,0,0)=\{0\}$. The reason is that $F(G(x,\eta_0),\ y,\mu_0)\ =\ [-x,x]$ meet the boundary of $-C,\forall x\in[0,1]=S_{\mathrm{m}\alpha_1}(0,0,0)$.

Example 4.4. Let X,Y,Λ,M,N,C,K and λ_0 be as in Example 4.1 and let $G(x,\eta)=[1-x,1+x]$, $F(x,y,\mu)=(-\infty,x-y+\mu]$. Then, $\forall x\in[0,1]$, $\exists x^*=1\in G(x,\eta_0),\forall y\in[0,1]$, $F(x^*,y,\mu_0)=(-\infty,x^*-y+\mu_0]$, $F(x^*,y,\mu_0)\cap -\mathrm{int}C\neq\emptyset$, so $S_{\mathrm{m}\alpha_1}(0,0,0)=[0,1]$. But $S_{\mathrm{s}\alpha_1}(0,0,0)=\{0\}$, since $\forall x\in(0,1]$, $\exists x^*:=1-x\in G(x,\eta_0)$, $\exists y=1\in[0,1]$ and $F(x^*,y,\mu_0)=(-\infty,x^*-y+\mu_0]$, $F(x^*,y,\mu_0)\cap -\mathrm{int}C=\emptyset$. The reason is that $F(G(x,\eta_0),y,\mu_0)=(-\infty,1+x]$ meet the boundary of $-C,\forall x\in[0,1]=S_{\mathrm{m}\alpha_1}(0,0,0)$.

The proof of the following three lemmas are similar to that of Lemma 4.1.

Lemma 4.3. Assume that $\forall x\in S_{\mathrm{s}\alpha_1}(\lambda_0,\mu_0,\eta_0),\forall y\in K(x,\lambda_0),F(G(x,\eta_0),\ y,\mu_0)$ is arcwisely connected and does not meet the boundary of $-C$. Then $S_{\mathrm{s}\alpha_1}(\lambda_0,\mu_0,\eta_0)=S_{\mathrm{s}\alpha_2}(\lambda_0,\mu_0,\eta_0)$.

Lemma 4.4. Assume that $\forall x\in S_{\mathrm{w}\alpha_2}(\lambda_0,\mu_0,\eta_0),\forall y\in K(x,\lambda_0),F(G(x,$

$\eta_0), y, \mu_0)$ is arcwisely connected and does not meet the boundary of $-C$. Then $S_{\mathrm{w}\alpha_2}(\lambda_0, \mu_0, \eta_0) = S_{\mathrm{m}\alpha_2}(\lambda_0, \mu_0, \eta_0)$.

Lemma 4.5. Assume that $\forall x \in S_{\mathrm{m}\alpha_2}(\lambda_0, \mu_0, \eta_0), \forall y \in K(x, \lambda_0), F(G(x, \eta_0), y, \mu_0)$ is arcwisely connected and does not meet the boundary of $-C$. Then $S_{\mathrm{m}\alpha_2}(\lambda_0, \mu_0, \eta_0) = S_{\mathrm{s}\alpha_2}(\lambda_0, \mu_0, \eta_0)$.

Summarizing the lemmas one obtains

Theorem 4.6. Assume that $\forall x \in S_{\mathrm{w}\alpha_1}(\lambda_0, \mu_0, \eta_0), \forall y \in K(x, \lambda_0), F(G(x, \eta_0), y, \mu_0)$ is arcwisely connected and does not meet the boundary of $-C$. Then all the six solution sets at $(\lambda_0, \mu_0, \eta_0)$ are equal.

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