Optimality Conditions for Nonsmooth Multiobjective Optimization Using Hadamard Directional Derivatives[1]

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Abstract. We establish both necessary and sufficient optimality conditions for weak efficiency and firm efficiency by using Hadamard directional derivatives and scalarizing the multiobjective problem under consideration via signed distances. For first-order conditions the data of the problem need not even be continuous and for the second-order ones we assume only that the first-order derivatives of the data are calm. We include examples showing the advantages of our results over some recent papers in the literature.

Key Words. Multiobjective optimization, weak efficiency, firm efficiency, signed distances, Hadamard directional derivatives, calmness.

1. Introduction

The topic of optimality conditions for nonsmooth and nonconvex problems is a lively subject. As observed in Ref. 1, the notions of generalized convexity and of generalized derivatives are "so abundant and sometimes so exotic that these two fields evoke the richness of a luxuriant nature rather than the purity of classical architecture". This abundance proves also the importance and the attractiveness of the subject. Refs. 2-4 are recent excellent books that contain systematic expositions and references on generalized differentiation and their applications to optimization-related problems, including optimality conditions. Besides, Refs 1 and 5 are also detailed treatments on the issues. Let us mention in more details some recent works on this topic. In Refs. 6-7, the Clarke generalized Jacobian and second-order subdifferential are employed for problems whose data have locally Lipschitz derivatives. First and second-order approximations are the tools in the treatment of optimality conditions in Refs. 8-11. The approximate Jacobian and approximate Hessian are also effective generalized derivatives to investigate optimality conditions, see Refs. 12-14. In Refs. 15-27, various generalized first and second-order differentiability are considered with applications in optimality conditions and sensitivity.

Although a whole spectrum of definitions of differentiability can be given in analytical and/or geometrical ways we can observe that the use of kinds of directional derivatives is often the first step for a differential construction, see e.g. Refs. 1, 5 and 21 for often-met notions of directional derivatives. Therefore, applying directional derivatives would be a simple way to deal with optimization-related problems in general and with optimality conditions in particular. This idea motivates the aim of this paper. We observe several recent papers in this direction: to establish optimality conditions the Dini directional derivative is employed in Refs. 28-30 and the Hadamard directional derivative in Refs. 31-33 (for scalar optimization).

In this paper we consider the following vector optimization problem. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^k$ be given. Let $C \subseteq \mathbb{R}^m$ and $K \subseteq \mathbb{R}^k$ be closed convex cones. The problem under our consideration is

$$\text{(P)} \qquad \min f(x), \text{ s.t. } g(x) \in -K.$$

We use the Hadamard directional derivative for a vector function and a scalarization technique via signed distances to establish both necessary and sufficient optimality conditions for weak efficiency as well as firm efficiency. We do not impose even continuity assumptions for studying first-order conditions. For the second-order ones we assume only the calmness property of derivatives. Our optimality conditions are also proved to be more advantageous in many circumstances than results of Refs. 6, 7, 11, 19, 28, 29 and 30 by several examples.

The organization of the paper is as follows. In Section 2 we present notions needed in the sequel and recall or develop preliminary facts. First-order optimality conditions are established in Section 3. Section 4 is devoted to second-order conditions.

2. Preliminaries

Recall first optimality notions of vector optimization, applied to (P). A feasible point x_0 (i.e. $g(x_0) \in -K$) is said to be a local efficient point of problem (P) if there exists a neighborhood U of x_0 such that

$$(f(U \cap g^{-1}(-K)) - f(x_0)) \cap (-C) \subseteq C \cap (-C). \tag{1}$$

If (1) is replaced by

$$(f(U \cap g^{-1}(-K)) - f(x_0)) \cap (-\text{int } C) = \emptyset,$$

then x_0 is called a local weakly efficient point of (P). Note that if int $C = \emptyset$ then every feasible point is a local weakly efficient point.

Scalarization is a commonly used method for studying vector optimization. One of the ways to scalarize problem (P) is using the signed distance following an idea of C. Malivert and used first in Refs. 34-35. Let $y \in \mathbb{R}^m$ and $A \subseteq \mathbb{R}^m$. The signed distance from y to A, denoted by $D(y, A)$, is defined as

$$D(y, A) = d(y, A) - d(y, \mathbb{R}^m \setminus A),$$

where $d(y, A) = \inf_{a \in A} \|y - a\|$ is the usual distance from y to A. (A signed distance is also called an oriented distance or directed distance in the literature. This concept is widely used in Ref. 36 to study second-order conditions.)

Lemma 2.1 See Ref. 37. If $y \in \mathbb{R}^m$ and $A \subseteq \mathbb{R}^m$ is a convex cone, then

$$D(y, -A) = \sup_{v^* \in A^*, \|v^*\|=1} \langle v^*, y \rangle,$$

where $A^* = \{v^* \in (\mathbb{R}^m)^* : \langle v^*, a \rangle \geq 0, \forall a \in A\}$ is the (positive) polar cone of A.

Lemma 2.2 See Ref. 28. x_0 is a local weakly efficient point of (P) if and only if x_0 is a local minimizer of the scalar optimization problem

$$\min \varphi(x), \text{ s.t. } g(x) \in -K,$$

where

$$\varphi(x) = D(f(x) - f(x_0), -C), \forall x \in \mathbb{R}^n. \tag{2}$$

By this lemma the following definition is natural.

Definition 2.1 See Refs. 28, 21, 38. A feasible point x_0 of (P) is termed a local firm efficient point of order $\alpha > 0$ of (P) if there are $h > 0$ and a neighborhood U of x_0 such that, with φ given in (2),

$$\varphi(x) \geq \varphi(x_0) + h\|x - x_0\|^\alpha, \forall x \in U \cap g^{-1}(-K).$$

(There are other names for a firm efficiency (of order α) in the literature: strict efficiency, isolated efficiency, we used the word "firm" suggested by an anonymous referee.) For scalar functions, firm minimizer concepts were introduced in Ref. 39 for $\alpha = 1$ and in Ref. 40 for $\alpha \geq 1$. In Refs. 41 and 36, a stronger notion of essential local minimizer of order α, for $\alpha = 2$, is discussed, where the above requirement for f (in terms of φ) is fulfilled also for the constrained function g.

For problem (P), given a feasible point x_0 the following cones are important,

$$K(g(x_0)) = \{\gamma(z + g(x_0)) : \gamma \geq 0, z \in K\},$$

$$K^* = \{v^* \in (\mathbb{R}^k)^* : \langle v^*, z\rangle \geq 0, \forall z \in K\}.$$

It is not hard to see that

$$K(g(x_0))^* = \{v^* \in K^* : \langle v^*, g(x_0)\rangle = 0\}.$$

In the sequel, for a cone A, we set

$$A_1 = \{v \in A : \|v\| = 1\}.$$

Lemma 2.3. If $g(x_0) \in -K$ and $t_n^{-1}(z_n - g(x_0)) \to z_0 \in -\text{int } K(g(x_0))$ as $t_n \to 0^+$, then $z_n \in -K$ for all n large enough.

Proof. Fix $\bar{v}^* \in K_1^*$. We claim that there exists a neighborhood $V(\bar{v}^*)$ of $\bar{v}^*$ such that, for all $v^* \in V(\bar{v}^*) \cap K_1^*$ and n large enough,

$$\langle v^*, z_n \rangle < 0. \tag{3}$$

Indeed, if $\bar{v}^* \in K(g(x_0))_1^*$ then $\langle \bar{v}^*, z_0 \rangle < 0$. Hence a neighborhood $V(\bar{v}^*)$ of $\bar{v}^*$ exists such that, for all $v^* \in V(\bar{v}^*) \cap K_1^*$ and large n, $\langle v^*, t_n^{-1}(z_n - g(x_0)\rangle < 0$, i.e.,

$$\langle v^*, z_n \rangle < \langle v^*, g(x_0) \rangle \leq 0.$$

If $\bar{v}^* \in K_1^* \setminus K(g(x_0))_1^*$, then $\langle \bar{v}^*, g(x_0) \rangle < 0$. Since $z_n \to g(x_0)$, there also exists a neighborhood $V(\bar{v}^*)$ such that one has (3) for all $v^* \in V(\bar{v}^*)$ and large n. By the compactness of K_1^*, there are finite number of $\bar{v}_1^*, ..., \bar{v}_s^* \in K_1^*$ such that $K_1^* \subseteq V(\bar{v}_1^*) \cup ... \cup V(\bar{v}_s^*)$. Hence one has (3) for all $v^* \in K_1^*$ and large n. Consequently $z_n \in -K$ for all n large enough. □

Definition 2.2 See Ref. 42. The Hadamard (upper) directional derivative of $\psi : \mathbb{R}^n \to \mathbb{R}$ at $x_0 \in \mathbb{R}^n$ in the direction $u \in \mathbb{R}^n$, denoted by $\psi_H^{'}(x_0, u)$, is defined as

$$\psi'_H(x_0, u) = \limsup_{t\to 0^+, v\to u} t^{-1}(\psi(x_0 + tv) - \psi(x_0)).$$

The Dini (upper) directional derivative of ψ at x_0 in the direction u is defined as

$$\psi'_D(x_0, u) = \limsup_{t\to 0^+} t^{-1}(\psi(x_0 + tu) - \psi(x_0)).$$

In Refs. 42-43 (and others) the name "Hadamard" is replaced by Dini (or Dini-Hadamard). Here we use the name "Dini" for the other kind of directional derivative, like in Refs. 31-33 (and others) and the lower directional derivatives with "liminf" replacing "limsup" in the definitions are not in use for our study in the sequel. Note that in Ref. 43 calculus is developed for the Hadamard subdifferential and ϵ-subdifferential which are defined via the Hadamard directional derivatives.

A natural extension to the vector function case is obtained by using upper limits of multifunctions in the sense of Painlevé-Kuratowski instead of the upper limits of single-valued scalar functions in Definition 2.2 as follows. (This definition is introduced in Ref. 15 for general multifunctions.)

Definition 2.3. The Hadamard (upper) directional derivative of $h : \mathbb{R}^n \to \mathbb{R}^k$ at $x_0 \in \mathbb{R}^n$ in the direction $u \in \mathbb{R}^n$ is defined by

$$Dh(x_0, u) = \limsup_{t\to 0^+, v\to u} t^{-1}[h(x_0 + tv) - h(x_0)]$$

$$= \{y \in \mathbb{R}^k : \exists (t_n, u_n) \to (0^+, u), y = \lim_{n\to\infty} t_n^{-1}(h(x_0 + t_n u_n) - h(x_0))\}.$$

The Dini (upper) directional derivative of h at x_0 in the direction u is defined by

$$dh(x_0, u) = \limsup_{t \to 0^+} t^{-1}[h(x_0 + tu) - h(x_0)]$$

$$= \{y \in \mathbb{R}^k : \exists t_n \to 0^+, y = \lim_{n \to \infty} t_n^{-1}(h(x_0 + t_n u) - h(x_0))\}.$$

Considering h as a multifunction the Hadamard directional derivative is more often met under the name "the contingent derivative".

Proposition 2.1. If $h : \mathbb{R}^n \to \mathbb{R}^k$ is locally Lipschitz at x_0, then $Dh(x_0, u) = dh(x_0, u)$ for all $u \in \mathbb{R}^n$.

Proof. Clearly $dh(x_0, u) \subseteq Dh(x_0, u)$. Let $y \in Dh(x_0, u)$, i.e. there are t_n and u_n as in the definition of the Hadamard directional derivative $Dh(x_0, u)$. By the local Lipschitz property, there is L such that

$$\|t_n^{-1}(h(x_0 + t_n u_n) - h(x_0)) - t_n^{-1}(h(x_0 + t_n u) - h(x_0))\|$$

$$= t_n^{-1}\|h(x_0 + t_n u_n) - h(x_0 + t_n u)\| \leq L\|u_n - u\| \to 0.$$

Hence, $y \in dh(x_0, u)$ since

$$\lim_{n \to \infty} t_n^{-1}(h(x_0 + t_n u) - h(x_0)) = y. \qquad \square$$

The following example shows a case where $dh(x_0, u) \subsetneqq Dh(x_0, u)$ and a case where h is not locally Lipschitz but $dh(x_0, u) = Dh(x_0, u)$.

Example 2.1. (a) Let $x_0 = (0,0), u = (1,0)$ (in $\mathbb{R}^2$) and $h : \mathbb{R}^2 \to \mathbb{R}$ be

$$h(x) = \begin{cases} 0 & \text{if } r = 0 \text{ or } \theta = 0, \\ \theta^{-1} r \sin\theta & \text{otherwise,} \end{cases}$$

where $x = r(\cos\theta, \sin\theta) \in \mathbb{R}^2, r \geq 0$ and $\theta \in [0, 2\pi)$. Then

$$dh(x_0, u) = \{0\}, Dh(x_0, u) = \{0, 1\}.$$

(b) Let $x_0 = 0$ and $u = 1$ (in $\mathbb{R}$) and $h : \mathbb{R} \to \mathbb{R}$ be

$$h(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then h is not locally Lipschitz at x_0 and

$$dh(x_0, u) = Dh(x_0, u) = [-1, 1].$$

Definition 2.4. Let $h : \mathbb{R}^n \to \mathbb{R}^k$ be Fréchet differentiable at $x_0 \in \mathbb{R}^n$. The second-order Hadamard (upper) directional derivative of h at x_0 in the direction $u \in \mathbb{R}^n$ is defined by

$$D_2 h(x_0, u) = \limsup_{t \to 0^+, v \to u} 2t^{-2}(h(x_0 + tv) - h(x_0) - th^{'}(x_0)u)$$

$$= \{y \in \mathbb{R}^k : \exists (t_n, u_n) \to (0^+, u), y = \lim_{n\to\infty} 2t_n^{-2}(h(x_0 + t_n u_n) - h(x_0) - t_n h^{'}(x_0)u)\}. \quad (4)$$

The second-order Dini (upper) directional derivative of h at x_0 in the direction u is defined by

$$d_2 h(x_0, u) = \limsup_{t \to 0^+} 2t^{-2}(h(x_0 + tu) - h(x_0) - th^{'}(x_0)u)$$

$$= \{y \in \mathbb{R}^k : \exists t_n \to 0^+, y = \lim_{n\to\infty} 2t_n^{-2}(h(x_0 + t_n u) - h(x_0) - t_n h^{'}(x_0)u)\}.$$

Remark 2.1. For scalar functions, there are various definitions for $D_2h(x_0, u)$ or $d_2h(x_0, u)$. In Refs. 31, 33 and 40, $th^{'}(x_0)u$ in (4) is replaced by $tDh(x_0, u)$. In Ref. 32, another kind of second-order derivative based on finite differences

$$D_2h(x_0, u) = \limsup_{t\to 0^+, v\to u} t^{-2}(h(x_0 + 2tv) - 2h(x_0 + tv) + h(x_0))$$

was also discussed. In Ref. 44, a different type of second-order directional derivative in a double direction (u, v) was introduced

$$f^{00}(x_0, u, v) = \limsup_{t\to 0^+, x\to x_0} t^{-1}(\langle f^{'}(x + tu), v\rangle - \langle f^{'}(x), v\rangle)$$

and used also for vector optimization.

To relax the locally Lipschitz property commonly used in the literature we apply the following weak property.

Definition 2.5 See Ref. 2. $h : \mathbb{R}^n \to \mathbb{R}^k$ is called calm at $x_0 \in \mathbb{R}$ if there are $L > 0$ and a neighborhood U of x_0 such that, $\forall x \in U$,

$$\|h(x) - h(x_0)\| \leq L\|x - x_0\|.$$

Note that in Refs. 45-47, this property was extended to multifunctions under the name "weak Lipschitz" property. They considered mainly the weak upper Lipschitz property. (As for the usual continuity of multifunctions, we have the upper and lower properties.)

Proposition 2.2. If $h : \mathbb{R}^n \to \mathbb{R}^k$ is Fréchet differentiable around $x_0 \in \mathbb{R}^n$ with h' being calm at x_0 and with $h'(x_0) = 0$, then $d_2h(x_0, u) = D_2h(x_0, u), \forall u \in \mathbb{R}^n$.

Proof. It suffices to prove $D_2h(x_0, u) \subseteq d_2h(x_0, u)$, $\forall u \in \mathbb{R}^n$. If $y \in D_2h(x_0, u)$, i.e. there are $t_n \to 0^+$ and $u_n \to u$ such that

$$y = \lim_{n\to\infty} 2t_n^{-2}[h(x_0 + t_n u_n) - h(x_0) - t_n h'(x_0)u].$$

Then, by the mean value theorem, $\alpha_n \in (0, 1)$ exists such that

$$A := 2t_n^{-2}[h(x_0+t_n u_n)-h(x_0)-t_n h'(x_0)u]-2t_n^{-2}[h(x_0+t_n u)-h(x_0)-t_n h'(x_0)u]$$

$$= 2t_n^{-2}[h(x_0 + t_n u_n) - h(x_0 + t_n u)]$$

$$= 2t_n^{-2}[h'(x_0 + t_n((1 - \alpha_n)u + \alpha_n u_n)) - h'(x_0)]t_n(u_n - u).$$

By the calmness of f', one has $y \in d_2h(x_0, u)$ as

$$\|A\| \le 2L\|(1-\alpha_n)u+\alpha_n u_n\|.\|u_n-u\| \to 0. \qquad \square$$

The example below gives a case where $d_2h(x_0, u) \subsetneq D_2h(x_0, u)$ and shows also that the condition given in Proposition 2.2 is not necessary.

Example 2.2. (a) Let $x_0, u \in \mathbb{R}^2$ be arbitrary with $u \neq 0$. Let $h : \mathbb{R}^2 \to \mathbb{R}$

be defined by $h(x_1, x_2) = x_2$. Then, $d_2h(x_0, u) = \{0\}$ and $D_2h(x_0, u) = \mathbb{R}$.

(b) Let $x_0 = 0$ and $u = 1$ (in $\mathbb{R}$) and $h : \mathbb{R} \to \mathbb{R}$ be

$$h(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then h is Fréchet differentiable with $h^{'}(x_0) = 0$, but not calm at x_0. However,

$$d_2h(0, 1) = D_2h(0, 1) = [-2, 2].$$

3. First-Order Optimality Conditions

Let us consider problem (P). The contingent cone of a subset $A \subseteq \mathbb{R}^n$ at $x \in \mathrm{cl}A$ (the closure of A) is defined by

$$T(A, x) = \{u \in \mathbb{R}^n : \exists t_n \to 0^+, \exists x_n \in A, y = \lim_{n\to\infty} t_n^{-1}(x_n - x)\}.$$

The following general necessary optimality condition seems to be known, but it is more difficult to find the reference than to prove it directly.

Proposition 3.1. If x_0 is a local weakly efficient point of (P) then there is a neighborhood U of x_0 such that

$$T((f, g)(U) + C \times K, (f, g)(x_0)) \cap -\text{ int } (C \times K(g(x_0))) = \emptyset. \tag{5}$$

Proof. By the weak efficiency of x_0 there is a neighborhood U of x_0 such that $(f(U \cap g^{-1}(-K)) - f(x_0)) \cap -\text{int } C = \emptyset$. Suppose (y_0, z_0) in the left-hand side of

(5) exists (if int $(C \times K(g(x_0))) = \emptyset$ then (5) is automatically satisfied). Then there are $t_n \to 0^+$ and $(y_n, z_n) \in (f,g)(U) + C \times K$ (i.e. $x_n \in U, c_n \in C$ and $k_n \in K$ exist with $y_n = f(x_n) + c_n, z_n = g(x_n) + k_n$) such that $t_n^{-1}(y_n - f(x_0)) \to y_0$ and $t_n^{-1}(z_n - g(x_0)) \to z_0$. Taking into account Lemma 2.3 one sees that $z_n \in -K$, for all large n. Hence $g(x_n) = z_n - k_n \in -K$, and then x_n is feasible for all large n. On the other hand, since

$$t_n^{-1}(f(x_n) + c_n - f(x_0)) = t_n^{-1}(y_n - f(x_0)) \to y_0 \in -\text{int } C,$$

one has the following contradiction to the weak efficiency of x_0:

$$f(x_n) - f(x_0) \in -\text{int } C - C \subseteq -\text{int } C. \qquad \square$$

Theorem 3.1 (first-order necessary condition for weak efficiency). If x_0 is a local weakly efficient point of (P) then, $\forall u \in \mathbb{R}^n, \forall (y_0, z_0) \in D(f,g)(x_0,u), \exists (c^*, k^*) \in C^* \times K(g(x_0))^* \setminus \{(0,0)\}$ such that

$$\langle c^*, y_0 \rangle + \langle k^*, z_0 \rangle \geq 0. \tag{6}$$

Proof. For all $u \in \mathbb{R}^n$ and $(y_0, z_0) \in D(f,g)(x_0,u)$, there are $t_n \to 0^+, u_n \to u$ such that

$$(y_0, z_0) = \lim_{n\to\infty} t_n^{-1}((f,g)(x_0 + t_n u_n) - (f,g)(x_0)).$$

Then, by the proof of Proposition 3.1, we see that $(y_0, z_0) \in T((f,g)(U) + C \times$

$K, (f, g)(x_0))$. Hence $(y_0, z_0) \notin -\text{int}\ (C \times K(g(x_0)))$. By the separation theorem, one finds $(c^*, k^*) \in C^* \times K(g(x_0))^* \setminus \{(0, 0)\}$ satisfying (6). $\square$

Remark 3.1. (i) The Lagrange multipliers c^* and k^* in Theorem 3.1 and all the coming results of this paper depend on the given direction u. Theorem 3.1 of Ref. 48 has the same conclusion as this theorem replacing $D(f, g)(x_0, u)$ by the classical directional derivative $(f, g)'(x_0, u)$, but under the strict assumption that f and g are quasidifferentiable in the sense of Ref. 49. Our result clearly includes Theorem 3.1 of Ref. 48 (and with a weaker assumption and a simpler proof) since $(f, g)'(x_0, u) \in D(f, g)(x_0, u)$.

(ii) In Ref. 29, the same conclusion (6) is proved under the additional assumption that f and g are locally Lipschitz at x_0, since in this case the Hadamard directional derivative and Dini directional derivative coincide. The following example provides a case where Theorem 3.1 can be used to reject a suspected point but many recent necessary conditions using other generalized derivatives cannot be applied.

Example 3.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } r = 0 \text{ or } \theta = 0, \\ -r(\sin\theta + 1) & \text{otherwise,} \end{cases}$$

where $x = r(\cos\theta, \sin\theta) \in \mathbb{R}^2$ for $r \geq 0, \theta \in [0, 2\pi)$. Let $C = \mathbb{R}_+, x_0 = (0,0)$ and $g(x) \equiv 0, \forall x \in \mathbb{R}^2$. Then f is not locally Lipschitz and we cannot apply necessary conditions using the Clarke generalized gradients e.g. in Refs. 6-7, the Fréchet subdifferential in Ref. 19 or the Dini directional derivatives but under local Lipschitz assumptions in Refs. 28-30. Choose $u = (1,0)$, then $Df(x_0, u) = \{0, -1\}$. Taking $y_0 = -1$ we see that for all $c^* \in C^* \setminus \{0\}, \langle c^*, y_0 \rangle = -c^* < 0$. Hence x_0 is not a local (weakly) efficient point by Theorem 3.1. Now we try to apply Theorem 3.3 of Ref. 11. We calculate the first-order approximation

$$A_f(x_0) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} \alpha \\ -1 \end{pmatrix} : \alpha \in (-\infty, -1) \cup [1, +\infty) \right\},$$

the pointwise closure

$$\text{p-cl}A_f(x_0) = A_f(x_0) \cup \left\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\},$$

the pointwise recession cone

$$\text{p-}A_f(x_0)_\infty = \left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

and the corresponding sets of the second-order approximation

$$B_f(x_0) = \text{p-cl}B_f(x_0) = \text{p-}B_f(x_0)_\infty = \{0\},$$

and easily see that both assumptions (i) and (ii) of Theorem 3.3 of Ref. 11 are satisfied and this theorem cannot reject x_0. (For the definitions of first and second-order approximations and related sets see Refs. 8, 9 and 11.)

In the following example, Theorem 3.1 can be applied to a constrained problem, but other recent results cannot.

Example 3.2. Let $C = K = \mathbb{R}_+, x_0 = 0, f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -x & \text{if } x \geq 0, \\ \sqrt{-x} & \text{if } x < 0, \end{cases}$$

$$g(x) = \begin{cases} -x & \text{if } x \geq 0, \\ \sqrt[3]{x} & \text{if } x < 0. \end{cases}$$

For $u = 1$ we have $D(f,g)(x_0, u) = \{(-1,-1)\}$. Hence, for all $(c^*, k^*) \in C^* \times K^* \setminus \{(0,0)\}$, $-c^* - k^* < 0$. Thus Theorem 3.1 rejects x_0 from suspected points for local (weak) efficiency. Since (f,g) is not locally Lipschitz at x_0, the results of Refs. 6-7, 28-30 cannot be applied. Theorem 3.1 of Ref. 48 cannot employed since f and g are not directional differentiable. We show now that the following theorem of Ref. 14 does not work either. Let $\partial H(x)$ stands for an approximate Jacobian of a function $H : \mathbb{R}^n \to \mathbb{R}^l$ at x (see Ref. 14), A_∞ be the recession cone of a subset A and co(.) be the convex hull of a set (.).

Theorem 3.1$'$ See Ref. 14. Let $\partial(f,g)(x)$ be an approximate Jacobian of (f,g) at x, which is upper semicontinuous at x_0. Let x_0 be a local weakly efficient point of (P). Then there exists $(c^*, k^*) \in C^* \times K(g(x_0))^*$, $\|(c^*, k^*)\| = 1$, such that

$$0 \in (c^*, k^*)(\mathrm{clco}\partial(f,g)(x_0) \bigcup \mathrm{co}(\partial(f,g)(x_0)_\infty \setminus \{0\})).$$

For Example 3.2, (f, g) admits the following approximate Jacobian

$$\partial(f,g)(x)$$
$$= \begin{cases} \{-1,-1\} & \text{if } x > 0, \\ \{(-1/2\sqrt{-x}, 1/3\sqrt[3]{x^2})\} & \text{if } x < 0, \\ \{(u,v) : u \in (-\infty,\alpha] \cup \{-1\}, v \in \{-1\} \cup [\beta,+\infty), \alpha < 0 < \beta\} & \text{if } x = 0. \end{cases}$$

Then $\partial(f,g)$ is upper semicontinuous at $x_0 = 0$. On the other hand,

$$\mathrm{clco}\partial(f,g)(0) = (-\infty,-1] \times [-1,+\infty),$$

$$\partial(f,g)(0)_\infty = (-\infty,0] \times [0,+\infty).$$

It is now clear that $(c^*, k^*) = (1,0)$ satisfies the two conditions of Theorem 3.1$'$ and then x_0 is not known to be local weakly efficient or not.

Theorem 3.2 (Sufficient condition for firm efficiency of order 1). Assume that f and g are calm at a feasible point x_0 and, $\forall u \in \mathbb{R}^n$: $\|u\| = 1$, $\forall (y_0, z_0) \in D(f,g)(x_0,u), \exists (c^*, k^*) \in C^* \times K(g(x_0))^*$,

$$\langle c^*, y_0 \rangle + \langle k^*, z_0 \rangle > 0.$$

Then x_0 is a local firm efficient point of order 1 of (P).

Proof. Suppose to the contrary that for any $\epsilon_n \to 0^+$, there are $t_n \to 0^+$ and

$u_n \to u$ with $\|u_n\| = \|u\| = 1$ such that, for all n, $g(x_0 + t_n u_n) \in -K$,

$$D(f(x_0 + t_n u_n) - f(x_0), -C) < \epsilon_n t_n.$$

Since f is calm at x_0, there is $L > 0$ such that, for all large n, $\|t_n^{-1}(f(x_0 + t_n u_n) - f(x_0))\| \leq L$ and by extracting a subsequence if necessary we can assume that $t_n^{-1}(f(x_0 + t_n u_n) - f(x_0)) \to y_0$ for some $y_0 \in \mathbb{R}^m$. Similarly, $t_n^{-1}(g(x_0 + t_n u_n) - g(x_0)) \to z_0$ for some $z_0 \in \mathbb{R}^k$. Then, $(y_0, z_0) \in D(f, g)(x_0, u)$.

On the other hand, for each $k^* \in K(g(x_0))^*$ and each n,

$$t_n^{-1}\langle k^*, g(x_0 + t_n u_n) - g(x_0)\rangle = t_n^{-1}\langle k^*, g(x_0 + t_n u_n)\rangle \leq 0.$$

Letting $n \to \infty$ we get $\langle k^*, z_0\rangle \leq 0$.

Now, for each $c^* \in C_1^*$ and each n, by Lemma 2.1 we have

$$t_n^{-1}\langle c^*, f(x_0 + t_n u_n) - f(x_0)\rangle \leq t_n^{-1} D(f(x_0 + t_n u_n) - f(x_0), -C) < \epsilon_n.$$

Again letting $n \to \infty$ we obtain $\langle c^*, y_0\rangle \leq 0$. Thus, for all $(c^*, k^*) \in C^* \times K(g(x_0))^*$, we have $\langle c^*, y_0\rangle + \langle k^*, z_0\rangle \leq 0$, contradicting the assumption of the theorem. □

In Ref. 29 the same conclusion as that of Theorem 3.2 is obtained, using the Dini directional derivative, under the additional assumption that f and g are locally Lipschitz at x_0. The following example gives a case where known sufficient conditions are not applied but Theorem 3.2 is.

Example 3.3. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x) = \begin{cases} (|x|, x\sin(1/x^2)) & \text{if } x \neq 0, \\ (0,0) & \text{if } x = 0, \end{cases}$$

$g(x) \equiv 0, x_0 = 0$ and $C = \mathbb{R}^2_+$. Then f is calm at x_0 but not locally Lipschitz at x_0. Then the results in the references mentioned in Example 3.1, using the local Lipschitz property, cannot be applied. Checking the assumptions of Theorem 3.4 of Ref. 11 we can calculate the first-order approximation of f and related sets

$$A_f(0) = \{(\pm 1, \alpha) : \alpha \in [-1, 1]\} = \text{p-cl}A_f(0),$$

$$\text{p-}A_f(0)_\infty = \{(0,0)\}.$$

Taking $h = 1 \in \mathbb{R} \setminus \{0\}$ and

$$M = (-1, -1) \in \text{p-cl}A_f(0) \bigcup (\text{p-}A_f(0)_\infty \setminus \{0\})$$

we get $Mh = (-1, -1) \in -\text{int } C$, i.e. both conditions (i) and (ii) of Theorem 3.4 of Ref. 11 are violated and we still do not know if $x_0 = 0$ is a local efficient point of (P) or not. However, applying Theorem 3.2 we see that, if $u \in \mathbb{R}, |u| = 1$, then $Df(x_0, u) = \{(1, \alpha) : \alpha \in [-1, 1]\}$. Hence, for each $y_0 \in Df(x_0, u)$, with $c^* = (1, 0) \in C^* \setminus \{(0,0)\}$ we have $\langle c^*, y_0 \rangle = 1 > 0$. Consequently x_0 is a local firm efficient point of order 1 and hence also a local efficient point.

4. Second-Order Optimality Conditions

Throughout this section, assume that f and g are continuously Fréchet differentiable at $x_0 \in \mathbb{R}^n$.

Proposition 4.1. If x_0 is a local weakly efficient point of (P) then, $\forall u \in \mathbb{R}^n$,

(i) $(f,g)'(x_0)u \notin -\text{int } (C \times K(g(x_0)))$;

(ii) if $(f,g)'(x_0)u \in -((C \times K(g(x_0))) \setminus \text{ int } (C \times K(g(x_0))))$, then, $\forall (y_0, z_0) \in D_2(f,g)(x_0,u)$,

$$\text{co}\{(y_0,z_0),(f,g)'(x_0)\mathbb{R}^n\} \bigcap -\text{int } (C \times K(g(x_0))) = \emptyset.$$

Proof. If the interior involved in (i) (or (ii)) is empty then the conclusion (i) (or (ii), respectively) is trivial. So we assume that these interiors are nonempty.

(i) It follows directly from Proposition 3.1.

(ii) Arguing by contradiction, for some $u \in \mathbb{R}^n$ and

$$(f,g)'(x_0)u \in -((C \times K(g(x_0))) \setminus \text{ int } (C \times K(g(x_0)))),$$

some $(y_0, z_0) \in D_2(f,g)(x_0,u)$, some $\lambda \in [0,1]$ and $\bar{u} \in \mathbb{R}^n$, suppose

$$(1-\lambda)(y_0,z_0) + \lambda(f,g)'(x_0)\bar{u} \in -\text{int } (C \times K(g(x_0))). \tag{7}$$

By (i), $\lambda < 1$. Since $(y_0, z_0) \in D_2(f,g)(x_0,u)$, there are $t_n \to 0^+, u_n \to u$ such that

$$y_0 = \lim_{n\to\infty} 2t_n^{-2}(f(x_0 + t_n u_n) - f(x_0) - t_n f^{'}(x_0)u),$$

$$z_0 = \lim_{n\to\infty} 2t_n^{-2}(g(x_0 + t_n u_n) - g(x_0) - t_n g^{'}(x_0)u).$$

Let $v_n = u_n + (\lambda/2(1-\lambda))t_n\bar{u}$. Then $v_n \to u$ and by the mean value theorem,

$$f(x_0 + t_n v_n) - f(x_0 + t_n u_n) = f^{'}(\theta_n)(t_n(v_n - u_n)) = (t_n^2\lambda/2(1-\lambda))f^{'}(\theta_n)\bar{u},$$

for some θ_n between $x_0 + t_n u_n$ and $x_0 + t_n v_n$. Therefore,

$$2t_n^{-2}(f(x_0 + t_n v_n) - f(x_0) - t_n f^{'}(x_0)u)$$

$$= 2t_n^{-2}(f(x_0+t_n u_n) - f(x_0) - t_n f^{'}(x_0)u) + 2t_n^{-2}(f(x_0+t_n v_n) - f(x_0+t_n u_n))$$

$$\to y_0 + (\lambda/(1-\lambda))f^{'}(x_0)\bar{u} \in -\operatorname{int} C \tag{8}$$

by (7). Similarly, one has also

$$2t_n^{-2}(g(x_0 + t_n v_n) - g(x_0) - t_n g^{'}(x_0)u)$$

$$\to z_0 + (\lambda/(1-\lambda))g^{'}(x_0)\bar{u} \in -\operatorname{int} K(g(x_0)). \tag{9}$$

Since $g^{'}(x_0)u \in -K(g(x_0))$, i.e. $g^{'}(x_0)u = -\gamma(k + g(x_0))$ for some $\gamma \geq 0$ and $k \in K$,

$$2t_n^{-2}(g(x_0 + t_n v_n) - g(x_0) - t_n g^{'}(x_0)u)$$

$$= 2t_n^{-2}(1-\gamma t_n)((1-\gamma t_n)^{-1}(g(x_0+t_n v_n)+\gamma t_n k) - g(x_0)). \tag{10}$$

By Lemma 2.3, (9) and (10), for large n, $g(x_0 + t_n v_n) \in -K$ since

$$(1-\gamma t_n)^{-1}(g(x_0+t_nv_n)+\gamma t_nk)\in -K.$$

On the other hand, it follows from (8) that, for large n,

$$f(x_0+t_nv_n)-f(x_0)-t_nf'(x_0)u\in -\mathrm{int}\ C,$$

and hence $f(x_0+t_nv_n)-f(x_0)\in -\mathrm{int}\ C$, a contradiction. □

Theorem 4.1 (Necessary condition for weak efficiency). If int C and int K are nonempty and x_0 is a local weakly efficient point of (P) then,

(i) there exists $(c^*,k^*)\in C^*\times K(g(x_0))^*\setminus\{(0,0)\}$ such that

$$c^*\circ f'(x_0)+k^*\circ g'(x_0)=0. \tag{11}$$

(ii) for $u\in\mathbb{R}^n$ if $(f,g)'(x_0)u\in -((C\times K(g(x_0)))\setminus \mathrm{int}\ (C\times K(g(x_0))))$, then $\forall(y_0,z_0)\in D_2(f,g)(x_0,u)$, $\exists(c^*,k^*)\in C^*\times K(g(x_0))^*\setminus\{(0,0)\}$ such that one has (11) and

$$\langle c^*,y_0\rangle+\langle k^*,z_0\rangle\geq 0. \tag{12}$$

Proof. (i) By Proposition 4.1(i) and the separation theorem, there are $(c^*,k^*)\in(\mathbb{R}^m)^*\times(\mathbb{R}^k)^*\setminus\{(0,0)\}$ and $\alpha\in\mathbb{R}$ such that, $\forall u\in\mathbb{R}^n$, $\forall(c,k)\in -(C\times K(g(x_0))$

$$\langle c^*,f'(x_0)u\rangle+\langle k^*,g'(x_0)u\rangle\geq\alpha, \tag{13}$$

$$\langle c^*,c\rangle+\langle k^*,k\rangle\leq\alpha. \tag{14}$$

Since $(f,g)^{'}(x_0)\mathbb{R}^n$ and $C \times K(g(x_0))$ are cones, $\alpha = 0$. Then (13) implies (11). Letting $k = 0$ in (14) one obtains $c^* \in C^*$. Setting $c = 0$ in (14) gives $k^* \in K(g(x_0))^*$.

(ii) According to Proposition 4.1(ii) and the separation theorem, one has (13), (14) and, in addition,

$$\langle c^*, y_0 \rangle + \langle k^*, z_0 \rangle \geq \alpha.$$

Similarly as in (i), $\alpha = 0, c^* \in C^*, k^* \in K(g(x_0))^*$ and (11) and (12) hold. □

Example 4.1. Let $C = \mathbb{R}_+, x_0 = 0, f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -x^2/2 & \text{if } x \geq 0, \\ x^2/2 - (2/3)x\sqrt{-x} & \text{if } x < 0, \end{cases}$$

and $g(x) \equiv 0$. Then

$$f^{'}(x) = \begin{cases} -x & \text{if } x \geq 0, \\ x - \sqrt{-x} & \text{if } x < 0, \end{cases}$$

is not locally Lipschitz at x_0. Therefore, the second-order necessary conditions provided in Refs. 6, 7, 28, 30 cannot be applied. Applying Theorem 4.1 we choose $u = 1$. Then $D_2 f(x_0, u) = \{-1\}$ and, for all $c^* \in \mathbb{R}_+ \setminus \{0\}, \langle c^*, -1 \rangle < 0$. So Theorem 4.1 asserts that x_0 is not a local (weakly) efficient point. Now we try with Theorem 3.3 of Ref. 11. We have $f^{'}(0) = 0, B_f(0) = \{-1/2\} \cup (\alpha, +\infty)$ for some $\alpha > 0$, p-cl$B_f(0) = \{-1/2\} \cup [\alpha, +\infty)$ and p-$B_f(0)_\infty = [0, +\infty)$. Thus x_0 satisfies the necessary condition stated in Theorem 3.3 of Ref. 11 and cannot be rejected.

Theorem 4.2 (Sufficient condition for firm efficient point of order 2). Assume that the Fréchet derivatives $f^{'}$ and $g^{'}$ are calm at a feasible point x_0. Then, each of the following conditions is sufficient for x_0 to be a local firm efficient point of order 2 of (P).

(i) $\forall u \in \mathbb{R}^n : \|u\| = 1$, $\exists (c^*, k^*) \in C^* \times K(g(x_0))^*$ such that

$$\langle c^*, f^{'}(x_0)u\rangle + \langle k^*, g^{'}(x_0)u\rangle > 0.$$

(ii) $\forall u \in \mathbb{R}^n : \|u\| = 1$, one has

(a) $(f^{'}(x_0)u, g^{'}(x_0)u) \in -((C \times K(g(x_0)))\setminus \text{ int } (C \times K(g(x_0))))$;

(b) $\forall (y_0, z_0) \in D_2(f, g)(x_0, u)$, $\exists (c^*, k^*) \in C^* \times K(g(x_0))^*$ such that (11) hold and

$$\langle c^*, y_0\rangle + \langle k^*, z_0\rangle > 0. \tag{15}$$

Proof. Reasoning by contraposition suppose for any $\epsilon_n \to 0^+$, there are $t_n \to 0^+$ and $u_n \to u$ with $\|u_n\| = \|u\| = 1$ such that, for all n,

$$g(x_0 + t_n u_n) \in -K,$$

$$D(f(x_0+t_n u_n)-f(x_0), -C) < \epsilon_n t_n^2. \tag{16}$$

(i) For all $c^* \in C^*$, $\|c^*\| = 1$, Lemma 2.1 yields, for all n,

$$t_n^{-1}\langle c^*, f(x_0+t_n u_n) - f(x_0)\rangle < \epsilon_n t_n. \tag{17}$$

Letting $n \to \infty$ one gets $\langle c^*, f^{'}(x_0)u\rangle \leq 0$.

For all $k^* \in K(g(x_0))^*$, since, for all n,

$$t_n^{-1}\langle k^*, g(x_0 + t_n u_n) - g(x_0)\rangle \leq 0,$$

letting $n \to \infty$ one arrives at $\langle k^*, g^{'}(x_0)u\rangle \leq 0$. This contradiction completes the proof of (i).

(ii) If (a) is not satisfied for u (obtained at the beginning of the proof) the theorem has been established. Now assume (a). By the calmness of $f^{'}$ there are $L > 0$ and $\alpha_n \in (0, 1)$ such that, for all large n,

$$\|f(x_0 + t_n u) - f(x_0) - t_n f^{'}(x_0)u\|$$

$$= \|f^{'}(x_0 + \alpha_n t_n u)(t_n u) - t_n f^{'}(x_0)u\|$$

$$\leq L\alpha_n t_n^2.$$

Hence, using subsequences if necessary, assume that, for some $y_0 \in R^m$, $2t_n^{-2}(f(x_0 + t_n u) - f(x_0) - t_n f^{'}(x_0)u) \to y_0$. Similarly, for some $z_0 \in R^k$, $2t_n^{-2}(g(x_0 + t_n u) - g(x_0) - t_n g^{'}(x_0)u) \to z_0$. Then $(y_0, z_0) \in D_2(f, g)(x_0, u)$. On the other hand, we will show that, if $(c^*, k^*) \in C^* \times K(g(x_0))^*$ with (11), then (15) must be violated.

Indeed, by (11), (16), (17) and the mean value theorem, for large n,

$$2t_n^{-2}(\langle c^*, f(x_0 + t_n u) - f(x_0) - t_n f^{'}(x_0)u\rangle + \langle k^*, g(x_0 + t_n u) - g(x_0) - t_n g^{'}(x_0)u\rangle)$$

$$= 2t_n^{-2}(\langle c^*, f(x_0 + t_n u) - f(x_0)\rangle + \langle k^*, g(x_0 + t_n u) - g(x_0)\rangle)$$

$$= 2t_n^{-2}(\langle c^*, f(x_0 + t_n u_n) - f(x_0)\rangle + \langle k^*, g(x_0 + t_n u_n) - g(x_0)\rangle$$

$$+\langle c^*, f(x_0 + t_n u) - f(x_0 + t_n u_n)\rangle + \langle k^*, g(x_0 + t_n u) - g(x_0 + t_n u_n)\rangle)$$

$$\leq \frac{2}{t_n^2}(\langle c^*, f(x_0+t_n u_n)-f(x_0)\rangle+\langle c^*, (f^{'}(x_0+t_n((1-\alpha_n)u+\alpha_n u_n))-f^{'}(x_0))t_n(u-u_n)\rangle$$

$$+\langle k^*, (g^{'}(x_0 + t_n((1 - \beta_n)u + \beta_n u_n) - g^{'}(x_0)))t_n(u - u_n)\rangle)$$

$$\leq 2\|c^*\|\epsilon_n + 2t_n^{-2}\|c^*\|L\|t_n((1 - \alpha_n)u + \alpha_n u_n)\|\|t_n(u - u_n)\|$$

$$+ 2t_n^{-2}\|k^*\|M\|t_n((1 - \beta_n)u + \beta_n u_n)\|\|t_n(u - u_n)\|$$

$$\leq 2\|c^*\|\epsilon_n + 4L\|c^*\|\|u - u_n\| + 4M\|k^*\|\|u - u_n\|,$$

where L and M are the calmness constants of $f^{'}$ and $g^{'}$ and $\alpha_n, \beta_n \in (0, 1)$. Letting $n \to \infty$ we obtain a contradiction to (15):

$$\langle c^*, y_0\rangle + \langle k^*, z_0\rangle \leq 0.$$

□

Remark 4.1. Theorems 4.1 and 4.2 include the counterparts in Ref. 30 since from the proofs of our Proposition 4.1 and Theorems 4.1 and 4.2 it is seen that these results are still valid if $D_2(f, g)(x_0, u)$ is replaced by $d_2(f, g)(x_0, u)$. With this replacement the conclusions of Theorems 4.1 and 4.2 coincide with the corresponding results of Ref. 30, but we do not need the local Lipschitz assumption.

Example 4.2. Let $C = \mathbb{R}^2_+, x_0 = 0$ in $\mathbb{R}, f : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x) = \begin{cases} (x^2/2, \int_0^x t^2 \sin(1/t^2)dt) & \text{if } x \neq 0, \\ (0,0) & \text{if } x = 0, \end{cases}$$

and $g(x) \equiv 0$. Then f' is calm at x_0 but not locally Lipschitz at x_0. Applying Theorem 4.2 we see that, for $u \in \mathbb{R}, |u| = 1, D_2 f(x_0, u) = \{(1,0)\}$. Then, taking $c^* = (1,0) \in C^* \setminus \{(0,0)\}$, we get $\langle c^*, (1,0)\rangle = 1 > 0$. Thus Theorem 4.2 ensures that x_0 is a local firm efficient point of order 2 of problem (P).

Remark 4.2. One of the desired features of optimality conditions is that the "smaller gaps" between necessary conditions and sufficient conditions are the better. Although the data of the problem under our consideration may be even nonconvex and discontinuous, the gaps between the necessary conditions and sufficient ones both of the first-order in Theorems 3.1 and 3.2 and of the second-order in Theorems 4.1 and 4.2 are rather minimal: the difference is inequalities for "necessary" and strict inequalities for "sufficient".

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