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Characterizations of optimal solution sets of convex infinite programs

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Abstract In this paper, several Lagrange multiplier characterizations of the solution set of a convex infinite programming problem are given. Characterizations of solution sets of cone-constrained convex programs are derived as well. The procedure is then adopted to a semi-convex problem with convex constraints. For this problem, we present firstly a necessary and sufficient condition for optimality and secondly a characterization of its optimal solution set, based on a Lagrange multiplier associated with a given solution and on directional derivatives of the objective function.

Keywords Convex infinite programs · Lagrange multipliers · Solution sets · Semi-convex programs

Mathematics Subject Classification (2000) 90C25 · 90C26 · 90C32 · 90C46

1 Introduction

Consider the following convex infinite optimization problem

$$\begin{array}{lll} \text{(P)} & \text{Minimize} & f(x) \\ & \text{subject to} & f_t(x) \le 0, \quad \forall t \in T, \\ & & x \in C, \end{array}$$

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where X is a locally convex Hausdorff topological vector space, $f, f_t : X \to \mathbb{R} \cup \{+\infty\}$, $t \in T$, are proper, lower semi-continuous and convex functions, C is a nonempty, closed convex subset of X, and T is an arbitrary index set (not necessarily finite). The convex infinite problem of model (P) was considered in several papers published recently (see Dinh et al. 2006a, 2007; Jeyakumar 1992, and references therein), where optimality conditions, duality, and stability results were established under various assumptions on the data. In this paper, we derive characterizations of the solution set of (P) for the two cases: when a solution of (P) is known and when a minimizing sequence of (P) is known instead. Characterizations of solution set for a class of nonconvex problems are given as well.

Characterizations and properties of solution sets are important for understanding the behavior of solution methods for nonlinear optimization problems that have multiple solutions. These have attracted attention of many mathematicians since the last decades of the last century (see Burke and Ferris 1991; Dinh et al. 2006b; Jeyakumar et al. 2004, 2006; Jeyakumar and Yang 1995; Mangasarian 1988; Penot 2003). When a solution is known *a fortiori*, simple and elegant characterizations for an optimization problem of minimizing a (finite-valued) convex function over a convex set were initially introduced in Mangasarian (1988). These results have since been extended to various classes of optimization problems: the problem of model (P) where $T = \emptyset$ and f is an extended real-valued function (Burke and Ferris 1991); the problem where f is pseudo-linear and $T = \emptyset$ (or T is finite and f_t is linear) in Jeyakumar and Yang (1995) (in Dinh et al. 2006b, respectively); vector minimization problems in Jeyakumar et al. (2006); quasi-convex problems in Penot (2003). In Jeyakumar et al. (2004), various characterizations of the solution set of a general cone-constrained convex problem in Banach spaces (with applications to other classes of problems, such as, semi-definite and fractional programs) were given.

In this work, motivated by the mentioned papers, we show that such kinds of characterizations of solution sets can be extended to convex infinite programs of the model (P). Moreover, other kinds of characterizations of solution sets are proposed. Firstly, we establish Lagrange multiplier characterizations of the solution set of (P) for the case where a solution of the problem is known. It is shown that the results cover the ones for cone-constrained convex programs given in Jeyakumar et al. (2004) and hence, the others in Burke and Ferris (1991) and Mangasarian (1988). Secondly, we consider the case where an exact solution of the problem (P) is not known. We present a new result on characterization of the solution set of (P) when a minimizing sequence is known instead (this is often the case when numerical methods are applied). As a by-product, an optimality condition based on a known minimizing sequence is given. Thirdly, in the last part of the paper we give an extension of the method to nonconvex problems. Concretely, in this part, we consider a problem of minimizing a semi-convex function under convex constraints and a set constraint. We establish necessary and sufficient optimality conditions for this problem and then, based on these, derive several characterizations of the solution set for the problem under consideration.

The layout of the paper is as follows: In Sect. 2, we recall some notations and preliminary results which will be used in the sequel. In Sect. 3, basing on the fact that the Lagrangian function with a fixed Lagrange multiplier associated with a known solution of (P) is constant on the solution set of (P), we establish several characterizations of the solution set of (P). Next, we give a characterization of the solution set of (P) when a minimizing sequence (instead of a solution) of (P) is known. Consequences for cone-constrained convex problems are obtained as well. In the last section, Sect. 4, we consider a semi-convex program with convex constraints. An optimality condition for the problem is established and characterizations of its solution set are given, based on a Lagrange multiplier associated with a given solution and on directional derivatives of the objective function.

2 Preliminaries

Throughout this paper (except for Sect. 4) X denotes a locally convex Hausdorff topological vector space and X^* is its topological dual endowed with weak*-topology. For a subset $D \subset X$, the *convex cone generated by* D will be denoted by cone$D$. If A is a convex subset of X and $\bar{x} \in A$, the *normal cone* to A at $\bar{x}$ is defined by

$$N_A(\bar{x}) := \big\{x^* \in X^* \mid \langle x^*, x - \bar{x}\rangle \leq 0,\ \forall x \in A\big\}.$$

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and convex function. The *conjugate function* of f, $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(v) = \sup\big\{\langle v, x\rangle - f(x) \mid x \in \operatorname{dom} f\big\}.$$

The *subdifferential* of f at $a \in \operatorname{dom} f$, denoted by $\partial f(a)$, is

$$\partial f(a) = \big\{v \in X^* \mid f(x) - f(a) \geq \langle v, x - a\rangle, \forall x \in X\big\}.$$

The subdifferential of f at $a \in \operatorname{dom} f$ is a weak*-closed (possibly empty) subset of X^*. For $\epsilon \geq 0$, ϵ*-subdifferential* of f at $a \in \operatorname{dom} f$ is

$$\partial_\epsilon f(a) = \big\{v \in X^* \mid f(x) - f(a) \geq \langle v, x - a\rangle - \epsilon, \forall x \in \operatorname{dom} f\big\}.$$

When $\epsilon > 0$, $\partial_\epsilon f(a)$ is a non-empty, weak*-closed convex subset of X^*. For more details, see Zalinescu (2002). It is well-known that the epigraph of f^*, epif^*, can be represented in the form (see Jeyakumar 1997)

$$\operatorname{epi} f^* = \bigcup_{\epsilon \geq 0} \big\{\big(v, \langle v, a\rangle + \epsilon - f(a)\big) \mid v \in \partial_\epsilon f(a)\big\}, \tag{1}$$

for any $a \in \operatorname{dom} f$.

Now let Y be another locally convex Hausdorff topological vector space and let K be a closed convex cone in Y. A mapping $g : X \to Y$ is said to be K*-convex* if for every $u, v \in X$ and for every $t \in [0, 1]$,

$$g\big(tu + (1-t)v\big) - tg(u) - (1-t)g(v) \in -K.$$

From now on, such a mapping is always assumed to be continuous. Note that if g is K-convex and continuous then the set $g^{-1}(-K) := \{x \in X \mid g(x) \in -K\}$ is a closed and convex subset of X (Craven 1978).

For a closed convex cone $K \subset Y$, the *positive polar cone* of K, denoted by K^+, is defined by

$$K^+ := \{y^* \in Y^* \mid \langle y^*, k\rangle \geq 0,\ \forall k \in K\}.$$

It is easy to see that

$$y \in K \iff \langle \mu, y\rangle \geq 0, \quad \forall \mu \in K^+.$$

In particular, $g(x) \in -K$ if and only if $\mu g(x) := \langle \mu, g(x)\rangle \leq 0$ for all $\mu \in K^+$.

For a non-empty index set T, denote

$$\mathbb{R}_+^{(T)} := \{\lambda = (\lambda_t) \in \mathbb{R}^{(T)} \mid \lambda_t = 0 \text{ for all } t \in T,\ \text{except a finite number } t \in T\}.$$

The *support* of an element $\lambda = (\lambda_t)_{t\in T} \in \mathbb{R}_+^{(T)}$ will be denoted by $T(\lambda)$, i.e., $T(\lambda) = \{t \in T \mid \lambda_t > 0\}$. Moreover, let $\widetilde{T}(\lambda) = T \setminus T(\lambda)$. Lastly, by convention, we understand that $0 \cdot (+\infty) = 0$.

3 Characterizations of the solution set of the problem (P)

In this section we will establish characterizations of the solution set of the problem (P), based on the Lagrange multipliers associated to a known solution of (P). In the case where a minimizing sequence (instead of a solution) is known, we also give a characterization of the solution set based on this minimizing sequence.

Consider the convex infinite programming problem (P) in Sect. 1. Let A be the feasible set of (P), i.e., $A := \{x \in X \mid x \in C, f_t(x) \leq 0, \forall t \in T\}$. From now on, assume that the solution set S of (P), $S := \{x \in A \mid f(x) \leq f(y), \forall y \in A\}$, is nonempty.

It is known that, under some constraint qualification condition (see, e.g., Dinh et al. 2006a, 2007), a feasible point z is a solution of (P) if and only if there exists $\lambda = (\lambda_t) \in \mathbb{R}_+^{(T)}$ such that

$$0 \in \partial f(z) + \sum_{t\in T} \lambda_t \partial f_t(z) + N_C(z), \qquad \lambda_t f_t(z) = 0, \quad \forall t \in T. \tag{2}$$

The element $\lambda \in \mathbb{R}_+^{(T)}$ (may not be unique) satisfying (2) is called a Lagrange multiplier corresponding to the solution z. The set of all Lagrange multipliers corresponding to a solution z of (P) will be denoted by $M(z)$. As usual, the Lagrangian function associated to (P) is defined as

$$L(x, \lambda) := \begin{cases} f(x) + \sum_{t\in T} \lambda_t f_t(x), & \text{if } x \in C, \lambda \in \mathbb{R}_+^{(T)}, \\ +\infty, & \text{otherwise.} \end{cases}$$

For the cone-constrained convex problems (see Jeyakumar et al. 2004) or pseudolinear problems (see Dinh et al. 2006b), the Lagrangian function with a fixed Lagrange multiplier associated with a known solution is constant on the solution set of the problem in consideration. This conclusion still holds for the class of problems of model (P) and will be given in the next lemma. Its proof is quite similar to those in the mentioned papers and will be omitted.

Lemma 3.1 *Suppose that $z \in S$ and (2) holds with $\lambda \in M(z)$. Then $L(\cdot, \lambda)$ is a constant function on S. Moreover, $f_t(x) = 0$ for all $t \in T(\lambda)$ and for all $x \in S$.*

We are now in a position to give characterizations of the optimal solution set of (P).

Theorem 3.1 *For the problem* (P), *suppose that $z \in S$ and* (2) *holds with $\lambda \in \mathbb{R}_+^{(T)}$. Then $S = S_1 = \bar{S}_1$, where*

$$S_1 := \big\{x \in C \mid \exists u \in \partial f(z) \cap \partial f(x), \langle u, x - z\rangle = 0, f_t(x) = 0, \forall t \in T(\lambda), \\ f_t(x) \leq 0, \forall t \in \widetilde{T}(\lambda)\big\},$$

$$\bar{S}_1 := \big\{x \in C \mid \exists u \in \partial f(x), \langle u, x - z\rangle = 0, f_t(x) = 0, \forall t \in T(\lambda), \\ f_t(x) \leq 0, \forall t \in \widetilde{T}(\lambda)\big\}.$$

Proof We will prove that $S \subset S_1 \subset \bar{S}_1 \subset S$. It is obvious that $S_1 \subset \bar{S}_1$. Firstly, we show that $\bar{S}_1 \subset S$. Let $x \in \bar{S}_1$. Then $x \in A$, and there exists $u \in \partial f(x)$ such that $\langle u, x - z\rangle = 0$. This yields $f(z) - f(x) \geq \langle u, z - x\rangle = 0$, which shows $x \in S$. So, $\bar{S}_1 \subset S$.

We now prove that $S \subset S_1$. It follows from (2) that there exist $u \in \partial f(z)$, $v \in \sum_{t \in T(\lambda)} \lambda_t \partial f_t(z)$, $w \in N_C(z)$ such that $u + v + w = 0$, and $f_t(z) = 0$ for all $t \in T(\lambda)$. As $w \in N_C(z)$, $\langle w, y - z\rangle \leq 0$ for all $y \in C$. Moreover, since $v \in \sum_{t \in T(\lambda)} \lambda_t \partial f_t(z) \subset \partial(\sum_{t \in T(\lambda)} \lambda_t f_t)(z)$, we get

$$\sum_{t \in T(\lambda)} (\lambda_t f_t)(y) - \sum_{t \in T(\lambda)} (\lambda_t f_t)(z) \geq \langle v, y - z\rangle, \quad \forall y \in X. \tag{3}$$

Observe that if $y \in A$ then the left-hand side of (3) is less than or equal to zero as $\lambda = (\lambda_t) \in \mathbb{R}_+^{(T)}$ and $\lambda_t f_t(z) = 0$ for all $t \in T$. Hence, $\langle v, y - z\rangle \leq 0$ for all $y \in A$. This, together with the facts that $u + v + w = 0$ and $\langle w, y - z\rangle \leq 0$ for all $y \in C$, implies $\langle u, y - z\rangle = -\langle v + w, y - z\rangle \geq 0$ for all $y \in A$.

If $x \in S$ then $x \in A$ and $f(x) = f(z)$. Hence,

$$0 = f(x) - f(z) \geq \langle u, x - z\rangle \geq 0,$$

which ensures $\langle u, x - z\rangle = 0$. Moreover, for all $y \in X$,

$$\begin{aligned} f(y) - f(x) &= f(y) - f(z) \\ &\geq \langle u, y - z\rangle \\ &= \langle u, y - x\rangle + \langle u, x - z\rangle \\ &= \langle u, y - x\rangle. \end{aligned}$$

Therefore, $u \in \partial f(x)$, and hence, $u \in \partial f(z) \cap \partial f(x)$. Thus, $x \in S_1$ and the inclusion $S \subset S_1$ holds. The proof is complete. □

It is worth observing from the definitions of the sets S_1 and $\bar{S}_1$ that the active constraints corresponding to $\lambda_t > 0$ (the number of such constraints is finite) remain active for all other solutions $x \in S$.

Example 3.1 Let $X = l^2$, the Banach space of all real sequences $x = (\xi_n)_n$ with $\|x\| := (\sum_{n=1}^{\infty} {\xi_n}^2)^{\frac{1}{2}} < +\infty$. Let further $C := \{x = (\xi_n)_n \in l^2 \mid 0 \le \xi_n \le n,\ \forall n \in \mathbb{N}\}$. It is clear that C is a closed convex subset of X. Consider the problem

$$\begin{array}{ll} \text{Minimize} & f(x) := \sum_{n=2}^{\infty} \frac{\xi_n}{n^3} \\ \text{subject to} & 1 - t\xi_1 \le 0, \quad t \in (2, +\infty), \\ & x = (\xi_n)_n \in C. \end{array}$$

Let $T := (2, +\infty)$ and $f_t(x) := 1 - t\xi_1$ for all $t \in T$. It is obvious that f is continuous and convex on C. The feasible set of the problem is:

$$A := \big\{(\xi_n) \in l^2 \mid 1/2 \le \xi_1 \le 1, 0 \le \xi_n \le n, n = 2, 3, \ldots\big\}.$$

A solution of the problem is $z = (1, 0, \ldots)$. On the other hand, for $x \in C$,

$$\partial f(x) = \big\{\big(0, 1/2^3, 1/3^3, \ldots\big)\big\},$$

$$\partial f_t(z) = \big\{(-t, 0, 0, \ldots)\big\}, \quad \forall t \in T,$$

$$N_C(z) = \left\{(u_n)_n \in l^2 \mid u_1(\xi_1 - 1) + \sum_{n=2}^{\infty} u_n \xi_n \le 0, \forall x = (\xi_n)_n \in C\right\}.$$

Let $\lambda = (\lambda_t)$ with $\lambda_t = 0$, for all $t \in T$. Then $\lambda \in M(z)$. By Theorem 3.1, the solution set of the problem is

$$\begin{aligned} S &= \big\{x = (\xi_n)_n \in C \mid \langle u, x - z\rangle = 0,\ u \in \partial f(x),\ f_t(x) \le 0,\ \forall t \in T\big\} \\ &= \left\{(\xi_n)_n \in A \mid u_1(\xi_1 - 1) + \sum_{n=2}^{\infty} u_n \xi_n = 0, (u_n)_n \in \partial f(x)\right\} \\ &= \big\{(\xi_n)_n \in l^2 \mid 1/2 \le \xi_1 \le 1, \xi_i = 0, \forall i \ge 2\big\}. \end{aligned}$$

We now give a characterization of S using subdifferentials of the Lagrangian function. Let $z \in S$ and let $\lambda \in M(z)$. It follows from (2) that

$$0 \in \partial\Bigg(f + \sum_{t \in T(\lambda)} \lambda_t f_t\Bigg)(z) + N_C(z), \qquad \lambda_t f_t(z) = 0, \quad \forall t \in T,$$

which means that

$$\partial_x L(z, \lambda) \cap \big(-N_C(z)\big) \ne \emptyset, \qquad \lambda_t f_t(z) = 0, \quad \forall t \in T. \tag{4}$$

This suggests a way of characterizing the solution set S of (P) in terms of subdifferentials of the Lagrangian function $L(\cdot, \lambda)$. To this aim, we need the following lemma.

Lemma 3.2 *Suppose that $z \in S$ and (2) holds with $\lambda \in M(z)$. Then for each $x \in S$,*

$$\partial_x L(x, \lambda) \cap \big(-N_C(x)\big) = \partial_x L(z, \lambda) \cap \big(-N_C(z)\big).$$

Proof If $u \in \partial_x L(x,\lambda) \cap (-N_C(x))$ then

$$\begin{cases} L(y,\lambda) - L(x,\lambda) \geq \langle u, y-x \rangle, & \forall y \in X, \\ \langle u, y-x \rangle \geq 0, & \forall y \in C. \end{cases} \tag{5}$$

Since $L(\cdot,\lambda)$ is constant on S (Lemma 3.1) and $z \in S$, it follows from (5) that $\langle u, z-x \rangle = 0$. Thus, $\langle u, y-x \rangle = \langle u, y-z \rangle + \langle u, z-x \rangle = \langle u, y-z \rangle$ for all $y \in X$. This together with (5) entails

$$\begin{cases} L(y,\lambda) - L(z,\lambda) \geq \langle u, y-z \rangle, & \forall y \in X, \\ \langle u, y-z \rangle \geq 0, & \forall y \in C, \end{cases}$$

which shows that $u \in \partial_x L(z,\lambda) \cap (-N_C(z))$. Thus,

$$\partial_x L(x,\lambda) \cap \big(-N_C(x)\big) \subset \partial_x L(z,\lambda) \cap \big(-N_C(z)\big).$$

The converse inclusion holds by a similar argument. □

Theorem 3.2 *Suppose that $z \in S$ and (2) holds with $\lambda \in M(z)$. Then $S = \hat{S}$, where the later set is defined by*

$$\hat{S} := \big\{x \in C \mid \partial_x L(x,\lambda) \cap \big(-N_C(x)\big) = \partial_x L(z,\lambda) \cap \big(-N_C(z)\big),$$
$$f_t(x) = 0, \forall t \in T(\lambda),\ f_t(x) \leq 0, \forall t \in \widetilde{T}(\lambda)\big\}.$$

Proof The inclusion $S \subset \hat{S}$ follows from Lemma 3.1 and Lemma 3.2. For the converse inclusion, let $x \in \hat{S}$. Then by (4) and Lemma 3.2,

$$\partial_x L(x,\lambda) \cap \big(-N_C(x)\big) = \partial_x L(z,\lambda) \cap \big(-N_C(z)\big) \neq \emptyset.$$

Then there exists $u \in \partial_x L(x,\lambda)$ with $-u \in N_C(x)$, and we get

$$\begin{cases} L(z,\lambda) - L(x,\lambda) \geq \langle u, z-x \rangle, \\ \langle u, z-x \rangle \geq 0. \end{cases} \tag{6}$$

Note that for all $t \in T(\lambda)$, $f_t(x) = f_t(z) = 0$. It now follows from (6) that $f(x) \leq f(z)$, which ensures $x \in S$. □

The characterizations of the solution set S given in Theorems 3.1, 3.2 (also in Burke and Ferris 1991; Dinh et al. 2006b; Jeyakumar et al. 2004; Jeyakumar and Yang 1995; Mangasarian 1988) base upon an a priori assumption that one solution of the considered problem is known.

We now turn to the case where the mentioned assumption fails to hold, i.e., an exact solution of (P) is not known. Suppose that $\mathrm{Inf}(P) = \alpha$ is finite and a minimizing sequence $(a_n)_n$ of (P) is known, i.e., $(a_n)_n \subset A$ such that $\lim_{n\to\infty} f(a_n) = \alpha$. This is often the case when some numerical method applies. In such a situation, we propose a way to characterize the solution set of (P), based on minimizing sequences as shown in the next theorem.

Theorem 3.3 *Suppose that* $\operatorname{Inf}(P) = \alpha$ *is finite. If* $(a_n)_n$ *is a minimizing sequence of* (P) *and* $\operatorname{epi} f^* + \operatorname{epi} \delta_A^*$ *is weak***-closed, then*

$$S = \left\{x \in C \mid \exists u \in \partial f(x), \langle u, a_n - x\rangle \geq 0, \forall n \in \mathbb{N}, f_t(x) \leq 0, \forall t \in T\right\}.$$

Proof Let $B := \{x \in C \mid \exists u \in \partial f(x), \langle u, a_n - x\rangle \geq 0, f_t(x) \leq 0, \forall t \in T\}$. If $x \in B$ then $x \in A$ and there exists $u \in \partial f(x)$ such that $\langle u, a_n - x\rangle \geq 0$ for all $n \in \mathbb{N}$. This gives

$$f(a_n) - f(x) \geq \langle u, a_n - x\rangle \geq 0, \quad \forall n \in \mathbb{N},$$

which entails $f(x) \leq f(a_n)$ for all $n \in \mathbb{N}$. Letting $n \to +\infty$, we get $f(x) \leq \alpha$. Thus $x \in S$.

Now, if $x \in S$ then $x \in C$ and $f_t(x) \leq 0$ for all $t \in T$. Moreover,

$$0 \in \partial(f + \delta_A)(x). \tag{7}$$

Since $\operatorname{epi} f^* + \operatorname{epi} \delta_A^*$ is weak*-closed, by Corollary 1 in Burachik and Jeyakumar (2005) (see also Lemma 2 in Dinh et al. 2007), (7) implies

$$0 \in \partial f(x) + N_A(x). \tag{8}$$

Therefore, there exists $u \in \partial f(x)$ such that $-u \in N_A(x)$, which implies that $\langle u, a_n - x\rangle \geq 0$ for all $n \in \mathbb{N}$, since $(a_n)_n \subset A$. Thus $x \in B$. The proof is complete. □

It is worth noting that the proofs of Theorems 3.1, 3.2 base on the optimality condition (2), which is often established under some constraint qualifications (see Dinh et al. 2006a, 2007), while that of Theorem 3.3 bases on (8) instead. The assumption that the set $\operatorname{epi} f^* + \operatorname{epi} \delta_A^*$ is weak*-closed guarantees the validity of (8). So, the presence of this condition in the statement of Theorem 3.3 is not a restriction.

The Theorem 3.3 also gives us an optimality condition for problem (P) based on a known minimizing sequence which is given in the corollary below.

Corollary 3.1 (Optimality condition) *Suppose that* $\operatorname{Inf}(P) = \alpha$ *is finite,* $(a_n)_n$ *is a minimizing sequence of* (P), *and* $z \in A$. *Suppose further that* $\operatorname{epi} f^* + \operatorname{epi} \delta_A^*$ *is weak***-closed and the solution set* S *of* (P) *is none-empty. Then* $z \in S$ *if and only if there exists* $u \in \partial f(z)$ *such that* $\langle u, a_n - z\rangle \geq 0$ *for all* $n \in \mathbb{N}$.

Proof It is obvious from the proof of Theorem 3.3. □

As an application of the previous results, let us consider the cone-constrained convex minimization problem (Jeyakumar et al. 2004)

$$\begin{array}{lll} \text{(P1)} & \text{Minimize} & f(x) \\ & \text{subject to} & g(x) \in -K, \\ & & x \in C, \end{array}$$

where f, X, C are as in the beginning of this section, Y is a locally convex Hausdorff topological vector space, K is a closed convex cone in Y, and $g : X \to Y$ is a continuous and K-convex mapping. Let A and S be the feasible set and the solution set of (P1), respectively. Assume that $S \neq \emptyset$.

Since $g(x) \in -K$ if and only if $\mu g(x) \leq 0$ for all $\mu \in K^+$, the problem (P1) can be rewritten as

$$(\widetilde{\mathrm{P}1}) \quad \begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & \mu g(x) \leq 0, \quad \mu \in K^+, \\ & x \in C. \end{array}$$

Thus, the problem (P1) can be reduced to the model of (P). It is worth emphasizing that the converse is not true. Indeed, all the functions f_t in (P) allowed to be extended real-valued functions while the corresponding ones in $(\widetilde{\mathrm{P}1})$ have the form μg, $\mu \in K^+$, which are always real-valued functions.

We now show that the results established in Theorems 3.1 and 3.2 are extensions of the ones introduced in Jeyakumar et al. (2004).

Note that for the Problem $(\widetilde{\mathrm{P}1})$ the optimality condition (2) reads as

$$0 \in \partial f(z) + \sum_{\mu \in T(\lambda)} \lambda_\mu \partial(\mu g)(z) + N_C(z), \qquad \lambda_\mu \mu g(z) = 0, \quad \forall \mu \in K^+ \tag{9}$$

for some $\lambda = (\lambda_\mu)_{\mu \in K^+} \in \mathbb{R}_+^{(K^+)}$. Here, $T(\lambda) := \{\mu \in K^+ \mid \lambda_\mu > 0\}$.

Set $\bar{\lambda} = \sum_{\mu \in T(\lambda)} \lambda_\mu \mu$. Then $\bar{\lambda} \in K^+$, and (9) can be rewritten in the form:

$$0 \in \partial f(z) + \partial(\bar{\lambda} g)(z) + N_C(z), \bar{\lambda} g(z) = 0. \tag{10}$$

We now can apply Theorem 3.1 to $(\widetilde{\mathrm{P}1})$ to get a characterization of the solution set S of (P1) in terms of subdifferentials of f and the Lagrange multiplier $\bar{\lambda} \in K^+$.

Corollary 3.2 *Assume that $z \in A$ is a solution of* (P1) *and* (10) *satisfies with $\bar{\lambda} \in K^+$. Then $S = S_2 = \bar{S}_2$, where*

$$S_2 := \left\{x \in C \mid \exists u \in \partial f(z) \cap \partial f(x), \langle u, x - z\rangle = 0, \bar{\lambda} g(x) = 0\right\},$$
$$\bar{S}_2 := \left\{x \in C \mid \exists u \in \partial f(x), \langle u, x - z\rangle = 0, \bar{\lambda} g(x) = 0\right\}.$$

Proof The conclusion follows from Theorem 3.1 (applies to $(\widetilde{\mathrm{P}1})$) and from the fact that

$$\lambda_\mu (\mu g)(x) = 0,\ \forall \mu \in K^+ \Longleftrightarrow \sum_{\mu \in T(\lambda)} (\lambda_\mu \mu) g(x) = \left\langle \sum_{\mu \in T(\lambda)} \lambda_\mu \mu, g(x) \right\rangle$$
$$= \bar{\lambda} g(x) = 0.$$
□

Similarly, we can get a characterization of the solution set of (P1) in terms of subdifferentials of the Lagrange function. The same conclusion as in Corollary 3.2 was established recently in Jeyakumar et al. (2004) for problem (P1) where X is a

Banach space and $f : X \to \mathbb{R}$ is a continuous, convex function. So, Corollary 3.2 improves the results in Jeyakumar et al. (2004), and Theorems 3.1, 3.2 extend the results in Jeyakumar et al. (2004) to convex infinite problems.

To conclude this section, note that the same argument as in the proof of Theorem 3.3 leads to a characterization of the solution set of (P1) where a minimizing sequence (instead of an exact solution) of this problem is known.

Corollary 3.3 *Suppose that* Inf(P1) $= \alpha$ *is finite. If* $(a_n)_n$ *is a minimizing sequence of* (P1) *and* $\operatorname{epi} f^* + \operatorname{epi} \delta_A^*$ *is weak***-closed, then*

$$S = \left\{x \in A \mid \exists u \in \partial f(x), \langle u, a_n - x\rangle \geq 0, \forall n \in \mathbb{N}\right\}.$$

4 Semi-convex programs

In this section we will consider a semi-convex problem with convex constraints. We first establish an optimality condition and then give a characterization for the solution set of this problem in terms of the directional derivatives of the objective functional.

Throughout this section, X is a Banach space. We first recall some definitions and notations on locally Lipschitz functions and semi-convex functions.

Let $f : X \to \mathbb{R}$ be a locally Lipschitz function at $x \in X$. The *generalized directional derivative* of f at x in the direction $d \in X$ is defined by (see Clarke 1983, page 25)

$$f^\circ(x; d) := \limsup_{\substack{h \to 0 \\ t \downarrow 0}} \frac{f(x+h+td) - f(x+h)}{t},$$

and the *Clarke's subdifferential* of f at x, denoted by $\partial^c f(x)$, is

$$\partial^c f(x) := \left\{u \in X^* \mid \langle u, d\rangle \leq f^\circ(x; d), \forall d \in X\right\}.$$

For $d \in X$, if the limit

$$\lim_{t \downarrow 0} \frac{f(x+td) - f(x)}{t}$$

exists then it is called the *directional derivative* of f at x in the direction d and is denoted by $f'(x; d)$. The function f is said to be *regular* at x (in the sense of F.H. Clarke) if $f'(x; d)$ exists and equals to $f^\circ(x; d)$ for each $d \in X$ (see Clarke 1983; Clarke et al. 1998).

Definition 4.1 (Mifflin 1977) Let C be a closed convex subset of X. The function $f : X \to \mathbb{R}$ is said to be *semi-convex* at $x \in C$ if f is locally Lipschitz and regular at x, and satisfies the following condition

$$x + d \in C,\ f'(x; d) \geq 0 \implies f(x+d) \geq f(x).$$

The function f is said to be semi-convex on C if f is semi-convex at every point $x \in C$.

Remark 4.1 Suppose that f is semi-convex at $x \in C$. Then it is easy to see that if there exists $u \in \partial^c f(x)$ such that $\langle u, z - x\rangle \geq 0$ then $f(z) \geq f(x)$.

The Lemma 4.1 below was proved in Mifflin (1977) (Theorem 8) for the case where $X = \mathbb{R}^n$ and C is a closed convex subset of X. The conclusion holds for an arbitrary closed convex subset C of a Banach space X without any change in the proof.

Lemma 4.1 *Suppose that f is semi-convex on a closed convex set $C \subset X$. Then for $x \in C$, $d \in X$ with $x + d \in C$,*

$$f(x+d) \leq f(x) \Longrightarrow f'(x; d) \leq 0.$$

We now consider the following semi-convex minimization problem under convex constraints

$$\begin{array}{lll} \text{(SP)} & \text{Minimize} & f(x) \\ & \text{subject to} & f_t(x) \leq 0, \quad t \in T, \\ & & x \in C, \end{array}$$

where C, T, f_t are as in Sect. 3 and f is a semi-convex function on an open subset containing C. Let A and S be the feasible set and the solution set of (SP), respectively. Assume that $S \neq \emptyset$.

The following lemma can be derived from Corollary 2 in Dinh et al. (2007) where the corollary was proved by using some optimality condition for a convex infinite problem established therein. Here, we give a direct proof.

Lemma 4.2 *Let $z \in A$. If* $\text{cone}\{\bigcup_{t\in T} \text{epi} f_t^* \cup \text{epi}\,\delta_C^*\}$ *is weak*-closed then $v \in N_A(z)$ if and only if there exists $\lambda = (\lambda_t) \in R_+^{(T)}$ such that $f_t(z) = 0$ for all $t \in T(\lambda)$ and*

$$v \in \sum_{t\in T(\lambda)} \lambda_t \partial f_t(z) + N_C(z).$$

Proof Let $v \in N_A(z)$. Then

$$x \in A \Longrightarrow \langle v, x\rangle \leq \langle v, z\rangle. \tag{11}$$

Since the set $\text{cone}\{\bigcup_{t\in T} \text{epi} f_t^* \cup \text{epi}\,\delta_C^*\}$ is weak*-closed, it follows from Farkas lemma (Theorem 4.1 in Dinh et al. 2006a) that (11) is equivalent to

$$\big(v, \langle v, z\rangle\big) \in \text{cone}\left\{\bigcup_{t\in T} \text{epi} f_t^* \cup \text{epi}\,\delta_C^*\right\}.$$

Hence, there exists $\lambda = (\lambda_t) \in \mathbb{R}_+^{(T)}$ such that

$$\big(v, \langle v, z\rangle\big) \in \sum_{t\in T(\lambda)} \lambda_t \text{epi} f_t^* + \text{epi}\,\delta_C^*.$$

Using representation (1) for $\operatorname{epi} f_t^*$ and $\operatorname{epi}\delta_C^*$, there exist $u_t, w \in X^*$, $\epsilon_t, \rho \in \mathbb{R}_+$, $t \in T(\lambda)$ such that

$$\big(v, \langle v, z\rangle\big) = \sum_{t\in T(\lambda)} \lambda_t\big(u_t, \langle u_t, z\rangle + \epsilon_t - f_t(z)\big) + \big(w, \langle w, z\rangle + \rho - \delta_C(z)\big),$$

where $u_t \in \partial_{\epsilon_t} f_t(z)$ and $w \in \partial_\rho \delta_C(z)$. Hence,

$$\begin{cases} v = \sum_{t\in T(\lambda)} \lambda_t u_t + w, \\ \langle v, z\rangle = \sum_{t\in T(\lambda)} \lambda_t\big[\langle u_t, z\rangle + \epsilon_t - f_t(z)\big] + \langle w, z\rangle + \rho - \delta_C(z). \end{cases}$$

This implies that

$$\sum_{t\in T(\lambda)} \lambda_t \epsilon_t - \sum_{t\in T(\lambda)} \lambda_t f_t(z) + \rho = 0.$$

Since $z \in A$, $f_t(z) \le 0$ for all $t \in T$. Moreover, $\lambda_t \ge 0$ for all $t \in T$. The last equality yields $\rho = 0$, $\lambda_t\epsilon_t = 0$, and $\lambda_t f_t(z) = 0$ for all $t \in T(\lambda)$. As $\lambda_t > 0$ for all $t \in T(\lambda)$, we get $\rho = 0$, $\epsilon_t = f_t(z) = 0$ for all $t \in T(\lambda)$, which ensures $w \in \partial\delta_C(z) = N_C(z)$, $u_t \in \partial f_t(z)$. Consequently,

$$v \in \sum_{t\in T(\lambda)} \lambda_t \partial f_t(z) + N_C(z).$$

Conversely, let $v \in \sum_{t\in T(\lambda)} \lambda_t \partial f_t(z) + N_C(z)$ with $\lambda = (\lambda_t) \in \mathbb{R}_+^{(T)}$ satisfying $f_t(z) = 0$ for all $t \in T(\lambda)$. Then

$$v \in \partial\Big(\sum_{t\in T(\lambda)} \lambda_t f_t + \delta_C\Big)(z),$$

and hence,

$$\Big(\sum_{t\in T(\lambda)} \lambda_t f_t + \delta_C\Big)(x) - \Big(\sum_{t\in T(\lambda)} \lambda_t f_t + \delta_C\Big)(z) \ge \langle v, x - z\rangle, \quad \forall x \in X.$$

If $x \in A$ then $\delta_C(x) = 0$ and $f_t(x) \le 0$ for all $t \in T$. Taking the fact that $f_t(z) = 0$ for all $t \in T(\lambda)$ into account, the last inequality gives $\langle v, x - z\rangle \le 0$ for all $x \in A$, which shows that $v \in N_A(z)$. □

We are now in a position to establish an optimality condition for the problem (SP), which paves the way to characterize its solution set (see Theorem 4.2).

Theorem 4.1 *Let $z \in A$. If the set* $\operatorname{cone}\{\bigcup_{t\in T} \operatorname{epi} f_t^* \cup \operatorname{epi}\delta_C^*\}$ *is weak*-closed then z is the solution of* (SP) *if and only if there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that*

$$0 \in \partial^c f(z) + \sum_{t\in T(\lambda)} \lambda_t \partial f_t(z) + N_C(z), \qquad f_t(z) = 0, \quad \forall t \in T(\lambda). \tag{12}$$

Proof We first prove that z is a solution of (SP) if and only if

$$0 \in \partial^c f(z) + N_A(z). \tag{13}$$

Suppose that z is a minimizer (SP). Then

$$0 \in \partial^c f(z) + N_A(z), \tag{14}$$

(see Clarke 1983, page 52) where $N_A(z)$ is the Clarke normal cone of A at z which coincides with the normal cone of A at z in the sense of convex analysis, since A is convex.

Conversely, suppose that (14) holds. Then there exists $u \in \partial^c f(z)$ satisfying

$$x \in A \Rightarrow \langle u, x - z\rangle \geq 0.$$

Since f is semi-convex, it follows from Remark 4.1 that

$$f(x) \geq f(z).$$

Since this inequality holds for all $x \in A$, z is a solution of (SP). The inclusion (13) holds.

The conclusion of the theorem now follows from (13) and Lemma 4.2. □

It is worth observing that if, in addition, f is convex on C, then the optimality condition (12) collapses to the well-known one (2). For an asymptotic optimality condition (without any constraint qualification condition) for a special case of (SP) where $C = X$ and T is a finite index set, see Jeyakumar (1997).

The following theorem gives a characterization of the solution set of Problem (SP).

Theorem 4.2 *For the problem* (SP), *assume that z is a solution and* (12) *holds with* $\lambda \in \mathbb{R}_+^{(T)}$. *Then $S = S_3$, where*

$$S_3 := \left\{x \in C \mid f'(x, z - x) \geq 0,\ f_t(x) = 0, \forall t \in T(\lambda) \text{ and } f_t(x) \leq 0, \forall t \in \widetilde{T}(\lambda)\right\}.$$

Proof Let $x \in S_3$. Then $x \in A$ and $f'(x; z - x) \geq 0$. Since $z = x + (z - x) \in A$, it follows from the definition of semi-convex function that $f(z) \geq f(x)$. Hence, $x \in S$, since z is a solution of (SP).

Conversely, let $x \in S$. Then $x \in A$. Since $z \in S$ and $\lambda \in \mathbb{R}_+^{(T)}$ satisfies (12), there exists $u \in X^*$ such that

$$\begin{cases} u \in \partial^c f(z), \\ -u \in \displaystyle\sum_{t \in T(\lambda)} \lambda_t \partial f_t(z) + \partial \delta_C(z), \qquad f_t(z) = 0, \quad \forall t \in T(\lambda). \end{cases}$$

Thus,

$$-u \in \partial\left(\sum_{t \in T(\lambda)} \lambda_t f_t + \delta_C\right)(z), \qquad f_t(z) = 0, \quad \forall t \in T(\lambda),$$

which implies that

$$\sum_{t\in T(\lambda)} \lambda_t f_t(y) \geq -\langle u, y-z\rangle, \quad \forall y \in X. \tag{15}$$

In particular, with $y = x \in S$ (note that $\lambda_t f_t(x) \leq 0$ for all $t \in T$), we get

$$\langle u, x-z\rangle \geq 0. \tag{16}$$

Since $u \in \partial^c f(z)$ and f is semi-convex at z, $f^\circ(z,d) = f'(z,d)$, and $\langle u, d\rangle \leq f'(z,d)$ for all $d \in X$. On the other hand, x and z are solutions of (SP), $f(z+(x-z)) = f(x) = f(z)$. By Lemma 4.1,

$$f'(z; x-z) \leq 0.$$

Hence, $\langle u, x-z\rangle \leq 0$. Combining this with (16), we get $\langle u, x-z\rangle = 0$. It now follows from (15) that $\lambda_t f_t(x) = 0$ for all $t \in T(\lambda)$, and hence,

$$f_t(x) = 0, \quad \forall t \in T(\lambda) \quad \text{and} \quad f_t(x) \leq 0, \quad \forall \widetilde{T}(\lambda). \tag{17}$$

On the other hand, since A is convex and $x, z \in A$, $x + t(z-x) = (1-t)x + tz \in A$ for all $t \in (0,1)$. We get

$$f'(x; z-x) = \lim_{t\downarrow 0} \frac{f[x+t(z-x)] - f(x)}{t} \geq 0,$$

because x is a solution of (SP). This inequality, together with (17), implies that $x \in S_3$. The proof is complete. □

Example 4.1 Consider the problem (P2)

$$\begin{array}{ll} \text{Minimize} & \sin(x-y) \\ \text{subject to} & ty - x \leq 0, \quad t \in (0, \tfrac{1}{2}], \\ & (x,y) \in C := \big\{(x, y \in \mathbb{R}^2 \mid x^2+y^2 \leq 1,\ y \leq x\big\}. \end{array}$$

Let $f(x,y) := \sin(x-y)$, $f_t(x,y) := ty - x$ for all $t \in T := (0, \frac{1}{2}]$. It is easy to verify that the feasible set of (P2) is

$$A = \big\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1, y \leq x, x \geq 0\big\},$$

f is semi-convex on A, and $z = (0,0)$ is a minimizer of (P2).

Let $(x,y) \in A$, $(d_1, d_2) \in \mathbb{R}^2$, a simple computation gives

$$f'\big((x,y); (d_1,d_2)\big) = (d_1 - d_2)\cos(x-y),$$

$$\partial f^c(0,0) = \big\{(u_1,u_2) \mid u_1 d_1 + u_2 d_2 \leq d_1 - d_2, \forall (d_1,d_2) \in \mathbb{R}^2\big\},$$

and

$$\partial f_t(0,0) = \{(-1,t)\}, \quad \forall t \in (0, \tfrac{1}{2}],$$

$$N_C(0,0) = \{(u,v) \in \mathbb{R}^2 \mid u = -v, v \geq 0\}.$$

Choose $\lambda \in \mathbb{R}_+^{(T)}$ with $\lambda_t = 0$ for all $t \in (0, \frac{1}{2}]$, $(1,-1) \in \partial f^c(0,0)$, and $(-1,1) \in N_C(0,0)$. It is easy to see that (12) holds.

On the hand, if $(x,y) \in A$ then $\cos(x-y) > 0$, and hence,

$$\begin{aligned} f'\big[(x,y);(0,0)-(x,y)\big] \geq 0 &\Leftrightarrow -(x-y)\cos(x-y) \geq 0 \\ &\Leftrightarrow x - y \leq 0 \\ &\Rightarrow x = y \end{aligned}$$

(note that $(x,y) \in A$ then $y \leq x$). By Theorem 4.2,

$$\begin{aligned} S &= \{(x,y) \in C \mid f'\big[(x,y);(0,0)-(x,y)\big] \geq 0, \text{ and } ty - x \leq 0, t \in (0, \tfrac{1}{2}]\} \\ &= \{(x,y) \in C \mid x = y, \text{and } ty - x \leq 0, t \in (0, \frac{1}{2}]\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid x = y, 0 \leq x \leq \tfrac{\sqrt{2}}{2}\}. \end{aligned}$$

It is easy to verify that if the objective function f is convex and locally Lipschitz near $x \in X$ then f is semi-convex at x. In such a case, one may expect that the characterization of the solution set S of (SP) given in Theorem 4.2 ($S = S_3$) can be rewritten in terms of subdifferentials of f as those in Theorem 3.1. This is true, as we will see below. We start with an elementary proposition.

Proposition 4.1 *Suppose that the objective function f is convex locally Lipschitz on an open set containing A. Let z be a solution of* (SP) *and* (12) *holds for some $\lambda \in \mathbb{R}_+^{(T)}$. Then for any feasible point $x \in A$, it holds*

$$\big(\exists u \in \partial f(x) \cap \partial f(z), \langle u, x - z\rangle = 0\big) \Leftrightarrow \big(f'(x; z - x) \geq 0\big).$$

Moreover, in this case, x is also a solution of (SP).

Proof Suppose that there exists $u \in \partial f(x) \cap \partial f(z)$ such that $\langle u, x - z\rangle = 0$. Then $u \in \partial f(x)$ and $\langle u, x - z\rangle = 0$, and hence, $f(z) - f(x) \geq \langle u, z - x\rangle = 0$. So, $f(z) \geq f(x)$, which proves that x is a solution of (SP). The inequality $f'(x; z - x) \geq 0$ holds by an elementary calculation.

Conversely, assume that $f'(x; z - x) \geq 0$. Since f is semi-convex at x, we have $f(x + z - x) = f(z) \geq f(x)$, which shows that x is also a solution of (SP). On the other hand, it follows from (12) that there exists $u \in \partial^c f(z) = \partial f(z)$ such that

$$-u \in \sum_{t \in T(\lambda)} \lambda_t \partial f_t(z) + N_C(z), \qquad f_t(z) = 0, \quad \forall t \in T(\lambda).$$

By the same argument as in the second part of the proof of Theorem 3.1 we get $u \in \partial f(x)$ and $\langle u, x - z\rangle = 0$. The proof is complete. □

Due to Proposition 4.1, when f is locally Lipschitz and convex on an open set containing C, the set S_3 in Theorem 4.2 is the same as the set S_1 defined in Theorem 3.1. So, Theorem 4.2 can be considered as an extension of Theorem 3.1 to semi-convex problems.

To conclude this section, we consider the following semi-convex minimization problem under a cone-constrained and a geometrical set constraint

$$\text{(SP1)} \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \in -K, \\ & x \in C. \end{array}$$

Here X, C, f are the same as in the first part of this section, Y is a locally convex Hausdorff topological vector space, K is a closed convex cone in Y, and $g : X \to Y$ is a continuous and K-convex mapping. Let us again denote the solution set and the feasible set of (SP1) by S and A, respectively. Assume that $S \neq \emptyset$.

Similar to the last part of Sect. 3, the problem (SP1) can be rewritten in the form of (SP), say problem $(\widetilde{\text{SP1}})$, with $T = K^+$ and for each $\mu \in T$, f_μ is defined by $f_\mu(x) := \langle \mu, g(x) \rangle$.

Suppose that the set $\operatorname{cone}\{\bigcup_{\mu \in K^+} \operatorname{epi}(\mu g)^* \cup \operatorname{epi} \delta_C^*\}$ is weak*-closed. Applying Theorem 4.1 and using a similar argument as for Problem (P1), we can conclude that a point $z \in A$ is a minimizer of (SP1) if and only if there exists $\lambda \in K^+$ such that

$$0 \in \partial^c f(z) + \partial(\lambda g)(z) + N_C(z), \quad \lambda g(z) = 0. \tag{18}$$

We now give a characterization of the solution set of (SP1).

Corollary 4.1 *If z is a solution of* (SP1) *and* (18) *holds with $\lambda \in K^+$ then*

$$S = \left\{x \in C \mid f'(x; z - x) \geq 0,\ \lambda g(x) = 0\right\}.$$

Proof The same as the proof of Corollary 3.2, using Theorem 4.2 instead of Theorem 3.1. □

If, in addition, f is convex on C then, by Proposition 4.1, $f'(x; z - x) \geq 0$ is equivalent to the fact that there exists $u \in \partial f(x) \cap \partial f(z)$ such that $\langle u, x - z \rangle = 0$ for any $x \in A$. So, Corollary 4.1 extends Corollary 3.2 to semi-convex problems with convex constraints.

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