Existence of Solutions to General Quasiequilibrium Problems and Applications[1]

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Abstract. A general quasiequilibrium problem is proposed including, among others, equilibrium problems, implicit variational inequalities, quasivariational inequalities involving multifunctions. Sufficient conditions for the existence of solutions with and without relaxed pseudomonotonicity are established. Even semicontinuity may not be imposed. These conditions improve several resent results in the literature.

Key Words. Quasiequilibrium problems, quasivariational inequalities, 0-level-quasiconcavity, upper semicontinuity, KKM - Fan theorem.

1. Introduction

Equilibrium problems, which include as special cases various problems related to optimization theory such as fixed point problems, coincidence point problems, Nash equilibria problems, variational inequalities, complementarity problems, maximization problems, have been studied by many authors, see e.g., Refs. 1-6. A main attention has been paid to sufficient conditions for the existence of solutions. It is also interested in getting such conditions for more general problem settings and under weaker assumptions about continuity, monotonicity and compactness.

In the present note we propose a general vector quasiequilibrium problem, which includes vector equilibrium problems, vector quasivariational inequalities, quasicomplementarity problems, etc. We establish sufficient conditions for solution existence with and without relaxed pseudomonotonicity.

In the sequel, if not otherwise specified, let X, Y and Z be real topological vector spaces, X be Hausdorff and $A \subseteq X$ be a nonempty closed convex subset. Let $C : A \to 2^Y$, $K : A \to 2^X$ and $T : A \to 2^Z$ be multifunctions such that

$C(x)$ is a closed convex cone with int $C(x) \neq \emptyset$ and $K(x)$ is nonempty convex, for each $x \in A$. Let $f : T(A) \times A \times A \to Y$ be a single-valued mapping. The quasiequilibrium problem under our consideration is of

(QEP) finding $\bar{x} \in A \bigcap \mathrm{cl} K(\bar{x})$ such that for each $y \in K(\bar{x})$, there exists $\bar{t} \in T(\bar{x})$ satisfying

$$f(\bar{t}, y, \bar{x}) \notin \text{int } C(\bar{x}).$$

To motivate the problem setting let us look at several special cases of (QEP).

(a) If $K(x) \equiv A$ and $Z = L(X, Y)$, the space of linear continuous mappings of X into Y, then (QEP) coincides with an implicit vector variational inequality studied in Refs. 7 and 8: find $\bar{x} \in A$ such that for each $y \in A$, there exists $\bar{t} \in T(\bar{x})$ satisfying $f(\bar{t}, y, \bar{x}) \notin \text{int } C(\bar{x})$.

(b) If $K(x) \equiv A$ and T is single-valued, then setting $f(T(x), y, x) := h(y, x)$, (QEP) becomes the vector equilibrium problem of (considered, e.g., in Refs. 1, 2, 3, 5, and 6)

(EP) finding $\bar{x} \in A$ such that, for each $y \in A$,

$$h(y,\bar{x}) \notin \text{int } C(\bar{x}).$$

(c) If $Z = L(X,Y)$, $f(t,y,x) = (t, x-y)$, where (t,x) denotes the value of a linear mapping t at x, then (QEP) reduces to the vector quasivariational inequality problem of (investigated by many authors)

(QVI) finding $\bar{x} \in A \bigcap \text{cl}K(\bar{x})$ such that for each $y \in K(\bar{x})$, there exists $\bar{t} \in T(\bar{x})$ satis fying

$$(\bar{t}, y-\bar{x}) \notin -\text{ int}\, C(\bar{x}).$$

(d) Let X be a Banach space, $Y = R, Z = X^*, C(x) \equiv R_+$, A be a closed convex cone, $T : A \to 2^{X^*}$ and $S : A \to 2^A$. A quasicomplementarity problem is of

(QCP) finding $\bar{x} \in A$ such that $\forall \bar{s} \in K \bigcap S(\bar{x}), \exists\, \bar{t} \in (-A^*) \bigcap T(\bar{x})$ satisfying

$$< \bar{t},\ \bar{s} >= 0,$$

where $< t, s >$ denotes the value of a linear functional t at s.

Then, setting $K(x) := x - A \bigcap S(x) + A$ and $f(t,y,x) :=< t, x-y >$, (QEP) collapses to (QCP), see Ref. 9.

(e) Consider the maximization problem of

(MP) finding a Pareto maximizer of a mapping $J : A \to \mathrm{Y}$,

where Y is ordered by a convex cone C. Then setting $C(x) \equiv C, K(x) \equiv A, T(x) = \{x\}$ and $f(T(x), y, x) := J(y) - J(x)$, (QEP) is equivalent to (MP).

Our aim now is to develop sufficient conditions for existence of solutions to (QEP) under weak assumptions and to derive as consequences several improvements of known results for vector equilibrium problems and vector quasivariational inequalities.

2. Preliminaries

We recall first some definitions needed in the sequel. Let X and Y be topological spaces. A multifunction $F : X \to 2^Y$ is said to be upper semicontinuous (usc) at $x_0 \in \mathrm{dom}F := \{x \in X : F(x) \neq \emptyset\}$ if for each neighborhood U of $F(x_0)$, there is a neighborhood N of x_0 such that $F(N) \subseteq U$. F is called usc if F is usc at each point of dom$F$. In the sequel all properties defined at a point will be extentded to domains in this way. F is called lower semicontinuous (lsc) at $x_0 \in$ dom$F$ if for each open subset U satisfying $U \cap F(x_0) \neq \emptyset$ there exists a neighborhood N of x_0 such that, for all $x \in N$, $U \cap F(x) \neq \emptyset$. F is said to be continuous at $x \in$ dom$F$ if F is both usc and lsc at x. F is termed closed at $x \in$ dom$F$ if $\forall x_\alpha \to x$, $\forall y_\alpha \in F(x_\alpha)$ such that $y_\alpha \to y$, then $y \in F(x)$. It known that if F is usc and has closed values, then F is closed.

A multifunction H of a subset A of a topological vector space X into X is said to be a KKM mapping in A if for each $\{x_1, ..., x_n\} \subseteq A$, one has co$\{x_1, ..., x_n\} \subseteq \bigcup_{i=1}^{n} H(x_i)$, where co$\{.\}$ stands for the convex hull.

The main machinary for proving our results is the following well-known KKM-Fan theorem (Ref. 10).

Theorem 2.1. Assume that X is a topological vector space, $A \subseteq X$ is nonempty and $H : A \to 2^X$ is a KKM mapping with closed values. If there is a subset X_0 contained in a compact convex subset of A such that $\bigcap_{x \in X_0} H(x)$ is compact, then $\bigcap_{x\in A} H(x) \neq \emptyset$.

The following fixed point theorem is a slightly weaker version (suitable for our use) of Tarafdar's theorem (Ref. 11), which is equivalent to Theorem 2.1.

Theorem 2.2. Assume that X is a Hausdorff topological vector space, $A \subseteq X$ is nonempty and convex and $\varphi: A \longrightarrow 2^A$ is a multifunction with nonempty convex values. Assume that

(i) $\varphi^{-1}(y)$ is open in A for each $y \in A$;

(ii) there exists a nonempty subset X_0 contained in a compact convex set of A such that $A \setminus \bigcup_{y\in X_0} \varphi^{-1}(y)$ is compact or empty.

Then, there exists $\hat{x} \in A$ such that $\hat{x} \in \varphi(\hat{x})$.

The next theorem on fixed points is modified (for our use) from a theorem in Ref. 12.

Theorem 2.3. Assume that V is a convex set in a Hausdorff topological vector space and $f : V \longrightarrow 2^V$ is a multifunction with convex values. Assume that

(i) $V = \bigcup_{x\in V} \text{int} f^{-1}(x)$;

(ii) there exists a nonempty compact subset $D \subseteq V$ such that for all finite subsets $M \subseteq V$, there is a compact convex subset L_M of V, containing M, such that $L_M \backslash D \subseteq \bigcup_{x \in L_M} f^{-1}(x)$.

Then, there is a fixed point of f in V.

Using Theorem 2.3 we derive the following modification of Theorem 2.1.

Theorem 2.4. Assume that V is a convex set in a Hausdorff topological vector space and $H : V \to 2^V$ is a KKM mapping in V with closed values. Assume further that there exists a nonempty compact subset $D \subseteq V$ such that for all finite subsets $M \subseteq V$, there is a compact convex subset L_M of V, containing M, such that

$$L_M \backslash D \subseteq \bigcup_{x \in L_M} (V \backslash H(x)). \qquad (2)$$

Then, $\bigcap_{x \in V} H(x) \neq \emptyset$.

Proof. Suppose that $\bigcap_{x \in V} H(x) = \emptyset$. Defined multifunction $g : V \to 2^V$ by $g(y) = \{x \in V : y \notin H(x)\}$. Then $g(y) \neq \emptyset \; \forall y \in V$, and $g^{-1}(x) = V \backslash H(x)$. Hence, $g^{-1}(x)$ is open and $V = \bigcup_{x \in V} g^{-1}(x)$. Define further $f : V \to 2^V$ by $f(x) =$co$g(x)$, where co means the convex hull. One has $V = \bigcup_{x \in V} f^{-1}(x)$.

Moreover, $L_M \backslash D \subseteq \bigcup_{x \in L_M} g^{-1}(x) \subseteq \bigcup_{x \in L_M} f^{-1}(x)$.

By Theorem 2.3 there is $x_0 \in V$ such that $x_0 \in f(x_0)$. Therefore, one can find $x_j \in g(x_0)$ and $\lambda_j \geq 0$, $j = 1, ..., m$, $\sum_{j=1}^m \lambda_j = 1$ such that $x_0 = \sum_{j=1}^m \lambda_j x_j$. By the definition of g, $x_0 \notin H(x_j)$, $j = 1, ..., m$. Thus $x_0 = \sum_{j=1}^m \lambda_j x_j \notin \bigcup_{j=1}^m H(x_j)$, which is impossible, since H is KKM. □

3. Main Results

We propose first a very relaxed quasiconcavity. Let Z, A, C, T and f be as for problem (QEP). For $x \in A$, the mapping f is said to be 0-level-quasiconcave with respect to $T(x)$ if for any finite subsets $\{y_1, ..., y_n\} \subseteq A$, and any $\alpha_i \geq 0$, $i = 1, ..., n$, with $\sum_{i=1}^n \alpha_i = 1$, there exists $t \in T(x)$ such that

$$[f(T(x), y_i, x) \subseteq \text{int } C(x),\ i = 1, ..., n]$$
$$\Rightarrow [f(t, \textstyle\sum_{i=1}^n \alpha_i y_i, x) \in \text{int } C(x)].$$

In the sequel let $E := \{x \in A : x \in \text{cl } K(x)\}$. Our first sufficient condition for the existence of solutions to (QEP) is the following.

Theorem 3.1. Assume for (QEP) the existence of a (single-valued) mapping $g : T(A) \times A \times A \to Y$ such that

(i) for all $x, y \in A$, if $g(T(x), y, x) \nsubseteq \operatorname{int} C(x)$, then $f(T(x), y, x) \nsubseteq \operatorname{int} C(x)$;

(ii) $g(., ., x)$ is 0-level-quasiconcave with respect to $T(x)$ and $g(t, x, x) \notin \operatorname{int} C(x)$ for all $x \in A$ and all $t \in T(x)$;

(iii) for each $y \in A, \{x \in A : f(T(x), y, x) \nsubseteq \operatorname{int} C(x)\}$ is closed;

(iv) $A \cap K(x) \neq \emptyset$ for all $x \in A$, K^{-1}(y) is open in A for all $y \in A$ and cl $K(.)$ is usc;

(v) there exist a nonempty compact subset D of A and a subset X_0 of a compact convex subset of A such that $\forall x \in A \backslash D$, $\exists y_x \in X_0 \cap K(x)$, $f(T(x), y_x, x) \subseteq \operatorname{int} C(x)$.

Then, (QEP) has a solution.

Proof. For $x, y \in$ A and $i = 1, 2$ set

$$P_1(x) := \{\, z \in A \, : \, f(T(x), z, x) \subseteq \operatorname{int} C(x)\},$$

$$P_2(x) := \{\, z \in A \, : \, g(T(x), z, x) \subseteq \operatorname{int} C(x)\},$$

$$\Phi_i(x) \; := \begin{cases} K(x) \cap P_i(x) & \text{if } x \in E, \\ A \cap K(x) & \text{if } x \in A \backslash E, \end{cases}$$

$$Q_i(y) \; := \; A \backslash \Phi_i^{-1}(y).$$

Observe that, by (ii), $x \notin P_2(x)$ and then $y \in Q_2(y)$ for each $y \in A$, by the definition of $Q_2(y)$. Furthermore we claim that $Q_2(.)$ is a KKM mapping in A.

Indeed, suppose there is a convex combination $\hat{x} := \sum_{j=1}^{n} \alpha_j y_j$ in A such that $\hat{x} \notin \bigcup_{j=1}^{n} Q_2(y_j)$. Then, $\hat{x} \notin Q_2(y_j)$, i.e., $y_j \in \Phi_2(\hat{x})$ for $j = 1, ..., n$. If $\hat{x} \in E$, one has $y_j \in P_2(\hat{x})$, i.e., $g(T(\hat{x}), y_j, \hat{x}) \subseteq \operatorname{int} C(\hat{x})$ for $j = 1, ..., n$. In virtue of the 0-level-quasiconcavity with respect to $T(\hat{x})$ of $g(.,.,\hat{x})$, there is $\hat{t} \in T(\hat{x})$ such that $g(\hat{t}, \hat{x}, \hat{x}) \in \operatorname{int} C(\hat{x})$, contradicting (ii). On the other hand, if $\hat{x} \in A \backslash E$ (i.e., $\hat{x} \notin \operatorname{cl} K(\hat{x})$), then $y_j \in \Phi_2(\hat{x}) = A \cap K(\hat{x})$, $j = 1, ..., n$. So $\hat{x} \in A \cap K(\hat{x})$, another contradiction. Thus, Q_2 must be KKM. By (i), for x $\in A$, one has $P_1(x) \subseteq P_2(x)$ and then $\Phi_1(x) \subseteq \Phi_2(x)$. Hence, $Q_2(y) \subseteq Q_1(y)$ for all y $\in A$, which results in that $Q_1(.)$ is also KKM.

Next we verify the closedness of $Q_1(y)$, $\forall y \in A$. One has

$$\begin{aligned}
\Phi_1^{-1}(y) &= \{x \in E : y \in K(x) \cap P_1(x)\} \cup \{x \in A \backslash E : y \in K(x)\} \\
&= \{x \in E : x \in K^{-1}(y) \cap P_1^{-1}(y)\} \cup \{x \in A \backslash E : x \in K^{-1}(y)\} \\
&= [E \cap K^{-1}(y) \cap P_1^{-1}(y)] \cup [(A \backslash E) \cap K^{-1}(y)] \\
&= [(A \backslash E) \cup P_1^{-1}(y)] \cap K^{-1}(y).
\end{aligned}$$

Therefore,

$$\begin{aligned}
Q_1(y) &= A \backslash \{[(A \backslash E) \cup P_1^{-1}(y)] \cap K^{-1}(y)\} \\
&= \{A \backslash [(A \backslash E) \cup P_1^{-1}(y)]\} \cup (A \backslash K^{-1}(y)] \\
&= [E \cap (A \backslash P_1^{-1}(y))] \cup (A \backslash K^{-1}(y)). \qquad (1)
\end{aligned}$$

Since $A \cap K(x) \neq \emptyset\,, \forall x \in A$, we have $\bigcup_{y \in A} K^{-1}(y) = A$. Theorem 2.2 in turn assures that $K(.)$ has a fixed point in A (hence $E \neq \emptyset$). Indeed, only (ii) of Theorem 2.2 is to be checked. By assumption (v),

$$A \backslash D \subseteq \bigcup_{x \in X_0} K^{-1}(x) \subseteq A$$

and then, $A \backslash \bigcup_{x \in X_0} K^{-1}(x) \subseteq D$ and is compact, i.e. (ii) of Theorem 2.2 is satisfied. Furthermore, since cl $K(.)$ is usc and has closed values, cl $K(.)$ is closed. Hence, E is closed. We have also

$$\begin{aligned} A \backslash P_1^{-1}(y) &= \{\, x \in A \,:\, y \notin P_1(x)\} \\ &= \{\, x \in A \,:\, f(T(x), y, x) \nsubseteq \operatorname{int} C(x)\}, \end{aligned}$$

which is closed by (iii). It follows from (1) that $Q_1(y)$ is closed. By assumption (v), $\forall x \in A \backslash D$, $\exists y_x \in X_0$ such that $y_x \in \Phi_1(x)$. Therefore,

$$A \backslash D \;\subseteq\; \bigcup_{x \in X_0} \Phi_1^{-1}(x) \;\subseteq\; A.$$

Hence, $A \backslash \bigcup_{x \in X_0} \Phi_1^{-1}(x) \subseteq D$, i.e., $\bigcap_{x \in X_0} A \backslash \Phi_1^{-1}(x) \subseteq D$ and then $\bigcap_{x \in X_0} Q_1(x)$ is compact. Applying Theorem 2.1 one obtains a point $\bar{x}$ such that

$$\bar{x} \in \bigcap_{y \in A} Q_1(y) \;=\; A \backslash \bigcup_{y \in A} \Phi_1^{-1}(y).$$

So, $\bar{x} \notin \Phi_1^{-1}(y)$, $\forall y \in A$, i.e., $\Phi_1(\bar{x}) = \emptyset$. If $\bar{x} \in A \backslash E$, then, $\Phi_1(\bar{x}) = A \cap K(\bar{x})$, contradicting (iv). In the remaining case, $\bar{x} \in E$, one has $\emptyset = \Phi_1(\bar{x}) = K(\bar{x}) \cap P_1(\bar{x})$. Thus, for all $y \in K(\bar{x})$, $y \notin P_1(\bar{x})$, i.e., $f(T(\bar{x}), y, \bar{x}) \not\subseteq \operatorname{int} C(\bar{x})$, which means that $\bar{x}$ is a solution of (QEP). $\square$

Remark 3.1

(a) Apart from (ii) and (iv), which have clear meanings, we can explain the other assumptions as follows. (i) is a kind of relaxed monotonicity. It may be said to be a pseudomonotonicity of f with respect to g. (iii) defines a kind of lower semicontinuity of $f(T(.), y, .)$ with respect to moving cone $C(.)$. (v) is a coercivity condition.

(b) If $K(x) \equiv A$ and $Z = L(X, Y)$, then (QEP) reduces to the implicit vector variational inequality considered in Refs. 7 and 8. In this case Theorem 3.1 is different from Theorem 3.1 in Refs. 7 and 8. However, we can observe that our theorem avoids strict continuity assumptions for mapping f, needed in Refs. 7 and 8.

(c) Theorem 3.1 is still valid if the coercivity assumption (v) is replaced by

(v') there are a compact subset D of A and $x_0 \in A$ such that, $\forall x \in A \backslash D$, $x_0 \in K(x)$ and $g(T(x), x_0, x) \subseteq \text{int } C(x)$.

So, if $K(x) \equiv A$ and T is single-valued, in nature Theorem 3.1 becomes the main result (Theorem 2.1) of Ref. 14, but with (ii) and (v) being slightly weaker than the corresponding assumptions in Ref. 14.

(d) Theorem 3.1 is also in forte if we replace (i) and (ii) respectively by the following (i') and(ii'):

(i') $\forall x$, $y \in A$, if $g(T(x), y, x) \not\subseteq C(x)$, then $f(T(x), y, x) \not\subseteq \text{int } C(x)$;

(ii') $\forall \{y_1, ..., y_n\} \subseteq A$, $n \geq 2$, $\forall \bar{x} \in co\{y_1, ..., y_n\}$, $\bar{x} \neq y_i$, $i = 1, ..., n$, $\exists j \in \{1, ..., n\}$, $\forall x \in A$, $g(T(\bar{x}), y_j, \bar{x}) \not\subseteq C(\bar{x})$ and $f(T(x), x, x) \not\subseteq C(x)$.

Indeed, in the proof we modify P_2(x) as follows

$$P_2(x) := \{y \in A \,:\, g(T(x), y, x) \subseteq C(x)\} \backslash \{x\}.$$

Then, all what we obtained before from (i) and (ii), namely the fact that $Q_2(.)$ is KKM and that $P_1(x) \subseteq P_2(x)$, $\forall x \in A$, can be derived from (i') and (ii').

If $Y = R$, $C(x) \equiv R_+$ and $K(x) \equiv A$, Theorem 3.1, with (i') and (ii'), is an improvement of Theorem 3.2 of Ref. 3 in the sense that in (v) D needs not be convex and x_0 needs not be fixed, but flexible in a subset X_0.

Assumptions (i) and (i') of Theorem 3.1 about a kind of relaxed pseudomonotonicity are commonly wanted to be avoided. The following result gets rid of this assumption.

Theorem 3.2. Assume for (QEP) that (iv) and (v) of Theorem 3.1 are satisfied. Assume also the following conditions

(ii") this is (ii) with mapping g replaced by f;

(iii') if $x, y \in A$, $x_\alpha \to x$, $x_\alpha \in A$ and $t_\alpha \in T(x_\alpha)$, then there are $t \in T(x)$, $u \in C(x) + f(t, y, x)$ and subnets x_β and t_β such that $f(t_\beta, y, x_\beta) \to u$;

(vi) $Y \setminus \operatorname{int} C(.)$ is closed.

Then, (QEP) has a solution.

Proof. For $x, y \in A$, let $P_1(x)$, $\Phi_1(x)$ and $Q_1(x)$ be as in the proof of Theorem 3.1. As for Theorem 3.1, we have (1). We have also the nonemptiness and closedness of E. To see the closedness of $A \setminus P_1^{-1}(y)$ let $x_\alpha \in A \setminus P_1^{-1}(y)$, $x_\alpha \to \hat{x}$. Then, $y \notin P_1(x_\alpha)$, i.e., there exists $t_\alpha \in T(x_\alpha)$, $f(t_\alpha, y, x_\alpha) \notin \operatorname{int} C(x_\alpha)$. By (iii') there are $t \in T(\hat{x})$, $u \in C(\hat{x}) + f(t, y, \hat{x})$ and subnets x_β and $t_\beta \in T(x_\beta)$ such that $f(t_\beta, y, x_\beta) \to u$. It follows from (vi) that $u \in Y \setminus \operatorname{int} C(\hat{x})$. One has

$$f(t,y,\hat{x}) = u + (f(t,y,\hat{x}) - u) \in Y\setminus \operatorname{int} C(\hat{x}) - C(\hat{x})$$

$$= Y\setminus \operatorname{int} C(\hat{x}),$$

i.e., $y \notin P_1(\hat{x})$. Hence, $\hat{x} \in A\setminus P_1^{-1}(y)$, showing the required closedness. Thus, looking at (1) one sees that $Q_1(y)$ is closed, $\forall y \in A$. Similarly as for Theorem 3.1, we have also that $\bigcap_{x\in X_0} Q_1(x)$ is compact.

Next we verify that $Q_1(.)$ is KKM in A. Suppose the existence of a convex combination $x^* := \sum_{j=1}^{n} \alpha_j y_j$ in A such that $x^* \notin \bigcup_{j=1}^{n} Q_1(y_j)$. Then, $y_j \in \Phi_1(x^*)$, $j = 1,...,n$. If $x^* \in E$, then $y_j \in P_1(x^*)$, i.e., $f(T(x^*), y_j, x^*) \subseteq \operatorname{int} C(x^*)$. Consequently, the quasiconcavity in (ii") gives a $t \in T(x^*)$ such that $f(t, x^*, x^*) \in \operatorname{int} C(x^*)$, a contradiction . Now if $x^* \in A\setminus E$, i.e., $x^* \notin cl\, K(x^*)$, then $y_j \in A \cap K(x^*)$, and hence $x^* \in A \cap K(x^*)$, another contradiction. Thus, Q_1 is KKM. By virtue of Theorem 2.1, there exists $\bar{x} \in \bigcap_{y\in A} Q_1(y)$ and, similarly as in the proof of Theorem 3.1, $\bar{x}$ is a solution of (QEP). $\square$

Remark 3.2. In Ref. 15 a quasiequilibrium problem slightly different from our (QEP) is studied and several existence results different from Theorems 3.1 and 3.2 are obtained. For the special case of (QEP), where $Z = L(X, Y)$ and $K(x) \equiv A$, our Theorem 3.2 is different from Theorem 3.2 in Ref. 8. However, our assumption (iii') is weaker than the corresponding continuity assumption in Ref. 8.

Moreover, if $K(x) \equiv A$ and T is single-valued, (QEP) collapses to the equilibrium problem considered by many authors. Theorem 3.2 contains improvements when compared with several known results. The 0-level-quasiconcavity in (ii") is weaker than concavity used in Ref. 5.

The following example gives a case where our Theorem 3.2 can be applied even when T is neither usc nor lsc and f is discontinuous (so the theorems in Refs. 7 and 8 cannot used).

Example 3.1. Let $X = Y = Z = R$, $A = [0, 1]$, $K(x) \equiv [0, 1]$, $C(x) \equiv R_+$,

$$T(x) = \begin{cases} [-2, -1.5] & \text{if } x = 0.5, \\ [-1, -0.5] & \text{otherwise,} \end{cases}$$

$$f(t, y, x) = \begin{cases} 2t & \text{if } x = 0.5, \\ t & \text{otherwise.} \end{cases}$$

All, but assumption (iii'), are clearly satisfied. We check (iii'). If $x \neq 0.5$, $y \in A$ is arbitrary, $x_n \to x$, $x_n \neq 0.5$ and $t_n \in T(x_n) = [-1, -0.5]$, then there are $t \in [-1, -0.5] = T(x)$ and a subsequence t_{n_k} such that $t_{n_k} \to t$. Taking $u = t \in C(x) + f(t, y, x)$ we see that $f(t_{n_k}, y, x_{n_k}) = t_{n_k} \to u$.

Now assume that $x = 0.5$, $y \in A$ is arbitrary, $x_n \to x$ and $t_n \in T(x_n)$. Since for (iii') we have to find required subsequence x_{n_k}, we have to consider only two possibilities.

If $x_n \equiv 0.5$, then $t_n \in [-2, -1.5]$ and there are $t^* \in [-2, -1.5]$ and t_{n_k} such that $t_{n_k} \to t^*$. Taking $t = -2$ and $u = 2t^*$ we see that (iii') is satisfied.

If $x_n \neq 0.5$, $\forall n$, then $t_n \in [-1, -0.5]$ and there are $t^{**} \in [-1, -0.5]$ and t_{n_k} such that $t_{n_k} \to t^{**}$. Choosing $t = -2$ and $u = t^{**}$ we see also that (iii') is fulfilled. Thus, Theorem 3.2 can be applied.

The next example shows that assumption (ii") of Theorem 3.2 is essential.

Example 3.2. Let X, Y, Z, A, K and $C(x)$ be as in Example 3.1, $T(x) = [0, 1]$ and

$$f(t, y, x) = \begin{cases} -1 & \text{if } y = 0.5, \\ 1 & \text{otherwise.} \end{cases}$$

It is obvious that in this case (QEP) do not have solutions and all assumptions of Theorem 3.2, but (ii"), are fulfilled. To see that (ii") is violated let x be arbitrary, $y_1 = 0$, $y_2 = 1$, $\alpha_1 = \alpha_2 = 0.5$. Then $f(T(x), y_i, x) = \{1\} \subseteq \operatorname{int} C(x)$ but $f(T(x), \alpha_1 y_1 + \alpha_2 y_2, x) = \{-1\}$, which does not meet $\operatorname{int} C(x)$.

We now modify Theorem 3.1 to include some main results in Refs. 7 and 8.

Theorem 3.3. Assume (i)-(iv) of Theorem 3.1 and replace assumption (v) there by

(v") there exists a nonempty compact subset $D \subseteq A$ such that for all finite subsets $M \subseteq A$, there is a compact convex subset L_M of A, containing M, such that $\forall x \in L_M \backslash D$, $\exists y_x \in L_M$, $y_x \in K(x)$ and $f(T(x), y_x, x) \subseteq \operatorname{int} C(x)$.

Then, (QEP) has a solution.

Proof. We define P_i, Φ_i and Q_i, $i = 1, 2$, and argue as for Theorem 3.1 to see that Q_1 is KKM and has closed values. To apply Theorem 2.4 instead of Theorem 2.1 we verify assumption (2) of Theorem 2.4. By (v"), $\forall x \in L_M \backslash D$, $\exists y_x \in \Phi_1(x) \bigcap L_M$. Hence $x \in \Phi_1^{-1}(y_x)$, i.e. $x \in A \backslash Q_1(y_x)$. Thus, $x \in \bigcup_{y \in L_M} A \backslash Q_1(y)$, i.e., (2) is satisfied. Then, by using Theorem 2.4 in the same way as employing Theorem 2.1 for Theorem 3.1 we complete the proof. □

Corollary 3.1. Assume (ii") of Theorem 3.2, (iii) and (iv) of Theorem 3.1 and (v") of Theorem 3.3. Then (QEP) has solutions.

Proof. Apply Theorem 3.3 with $g \equiv f$. □

Corollary 3.1 improves Theorem 3.1 of Ref. 7 and Theorem 3.1 of Ref. 8 by getting rid of many strict assumptions on continuity, compactness, pseudomonotonicity and concavacity. For example, our assumption (iii) can be satisfied even when f is not continuous. To see this take $X = Y = Z = R$, $A = [0,1]$, $C(x) \equiv R_+$, $T(x) \equiv [0,1]$ and

$$f(t,y,x) = \begin{cases} -1 & \text{if } t \neq 0, \\ -0.5 & \text{if } t = 0. \end{cases}$$

Then $\{x \in A : f(T(x), y, x) \not\subseteq R_+\} = [0,1]$ is closed but f is not continuous.

It is not hard to see that for this example all assumptions of Theorem 3.1 are also fulfilled.

Remark 3.3. After submitting the paper we observed Refs. 15 - 20 with recent related results on equilibrium problems. Ref. 15 considers a similar problem setting but requires some assumptions different from ours, e.g. K has compact values, f is continuous and properly quasiconvex (in the second variable) and $C(x) \equiv C$ whose polar cone has a weak* compact base (Theorem 1). Refs. 16 - 20 consider cases where f is multivalued. The problem setting in Refs. 16 and 20 is similar to ours but $K(x) \equiv A$ (i.e. an equilibrium problem, not quasiequilibrium). Refs. 17 and 18 also investigate equilibrium problems, but here f has two variables (not three and not include multifunction T). In Ref. 19 a quasiequilibrium problem with f having two variables is studied. In each of Refs. 16 - 20 there are several assumptions different from that of the present paper.

4. Applications to Quasivariational Inequalities

As aforementioned in the introduction, in the special case, where $Z = L(X, Y)$ and $f(t, y, x) = (t, h(x) - y)$ with $h : A \to A$ being a given mapping, (QEP) collapses to the quasivariational inequality

(QVI), find $\bar{x} \in A \cap \operatorname{cl} K(\bar{x})$ such that for each $y \in K(\bar{x})$, there exists $\bar{t} \in T(\bar{x})$ such that

$$(\bar{t}, y \;-\; h(\bar{x})) \notin -\operatorname{int} C(\bar{x}).$$

In this special case the 0-level-quasiconcavity with respect to $T(x)$ of $f(., ., x)$ is obvious. Rewriting Theorem 3.1 and 3.2 for this case we get the following new results.

Corollary 4.1. Assume that

(ii) $(T(x), h(x) - x) \subseteq Y \backslash {-\text{int}}\ C(x)$, $\forall x \in A$;

(iii) for each $y \in A$, the set $\{x \in A : (T(x),\ h(x) - y) \not\subseteq \operatorname{int} C(x)\}$ is closed;

(iv) $A \cap K(x) \neq \emptyset$ for each $x \in A$, $K^{-1}(y)$ is open in A for each $y \in A$ and cl $K(.)$ is usc;

(v) there exists a nonempty closed compact subset D of A and a subset X_0

of a compact convex subset of A such that $\forall x \in A\backslash D$, $\exists y_x \in X_0 \cap K(x)$, $(T(x), g(x) - y_x) \subseteq \operatorname{int} C(x)$.

Then, (QVI) has a solution.

Corollary 4.2. Assume (ii), (iv) and (v) as in Corollary 4.1. Assume further that

(iii') if $x, y \in A$, $x_\alpha \to x$, $x_\alpha \in A$ and $t_\alpha \in T(x_\alpha)$, $\exists t \in T(x)$, $\exists u \in C(x) + (t, h(x) - y)$, $\exists x_\beta$, $\exists t_\beta$ (subnets), $(t_\beta, h(x_\beta) - y) \to u$;

(vi) $Y \backslash \operatorname{int} C(.)$ is closed.

Then, (QVI) has a solution.

Observe that Corollary 4.2 is in nature an extention of Theorem 2.1 of Ref. 9 to the case where A being noncompact. Assumption (ii) of Corollary 4.2 is slightly more strict but (iii') is weaker than the corresponding assumption in Ref. 9.

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