THE MINIMUM ENTROPY PRINCIPLE FOR FLUID FLOWS IN A NOZZLE WITH VARIABLE CROSS-SECTIONS

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Abstract. We prove that the minimum entropy principle for entropy solution of fluid flows in a nozzle with variable cross-sections still holds true. Furthermore, we show that the numerical solutions given by [16] also satisfy the entropy minimum principle.

Key words. nozzle problem, conservation law, source term, weak solution, entropy, numerical scheme.

AMS subject classifications. Primary: 35L65, 74XX. Secondary: 65Bxx, 76N10, 76L05.

1. Introduction. The dynamics of compressible flows in a nozzle with variable cross-section is described by the following system of conservation laws with source terms [6, 26, 22, 20, 16]:

$$\begin{aligned}\partial_t(a\rho) + \partial_x(a\rho u) &= 0,\\ \partial_t(a\rho u) + \partial_x(a(\rho u^2 + p)) &= p\partial_x a,\\ \partial_t(a\rho e) + \partial_x(au(\rho e + p)) &= 0, \quad x \in \boldsymbol{R},\, t > 0,\end{aligned} \tag{1.1}$$

where a is the cross-section, u is the velocity, the thermodynamical variables $\epsilon, \rho, v = 1/\rho, p, T, S$ are the internal energy, density, specific volume, pressure, absolute temperature, and specific entropy, respectively, and $e = \epsilon + u^2/2$ is the total energy. See Figure 1.1.

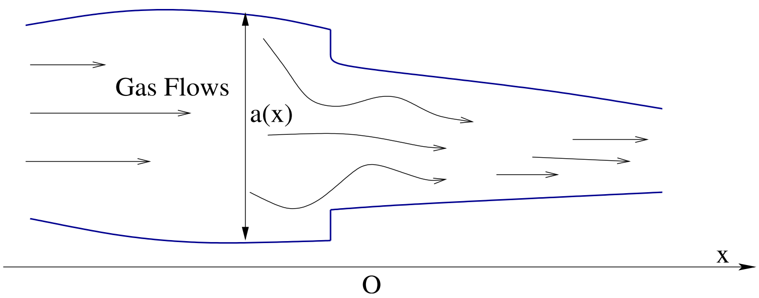

FIG. 1.1. *Gas flow in a nozzle*

In 1986, Tadmor [25] published a paper revealing the *Minimum entropy principle* for the gas dynamics equations that the entropy should be increasing in time (see also [24] for some initiatives).

In this paper, we deal with the same problem but for the system depending on a *source term* (1.1). Because of the source term on the right-hand side of the system

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(1.1), where the nozzle function a may have singularity, the usual notion of weak solutions of system of conservation laws cannot be applicable.

One way to study the system (1.1) is to supplement it with the trivial equation

$$a_t = 0. \tag{1.2}$$

Then, if one considers a as if it is an unknown (see [21, 22]) the system can be written in systems of balance laws in *nonconservative form* (see [7])

$$\partial_t U + A(U) \cdot \partial_x U = 0. \tag{1.3}$$

Then, we can use the theory of generalized weak solutions of balance laws by Dal Maso-LeFloch-Murat [7].

In fact, we will provide in this paper (Section 2) detailed calculations to show that the system (1.1) supplemented with the equation (1.2) can be written in the form (1.3) by choosing the unknown function U to be either of single components, i.e., $U = (\rho, u, S, a)$ for example, or of conservative quantities, i.e., $U = (a\rho, a\rho u, a\rho e, a)$. This reveals that the theory of shock waves of systems of balance laws in nonconservative form of Dal Maso-LeFloch-Murat [7] can be thus applied to the system (1.1)-(1.2). We note that solutions in the sense of [7] depend on a prescribed family of Lipschitz paths by which the obstacle of δ-Dirac with respect to the Lebesgue integration can be overcome. By starting from a viscous model of (1.1), we will establish the entropy inequality for (1.1) in the divergence form (Section 3). We also prove in Section 4 that the entropy inequality for system of balance laws in nonconservative form defined by Dal Maso-LeFloch-Murat in [7] for the system (1.1) can be reduced in the divergence form. This means in particular that the entropy inequality in the sense of [7] is independent of the Lipschitz paths in this case. The fact that the entropy inequality is written in divergence form enables us, in Section 4, to derive the minimum entropy principle that *entropy is increasing in time*. This result is similar to the one by Tadmor [25] for the usual gas dynamics equations (see also [24] for some initiatives). This minimum entropy principle is in particular useful when it comes to the question of the convergence of entropy stable schemes in certain cases, see [8]. In Section 5, we will investigate properties of numerical solutions of (1.1). Note that because of the source term on the right-hand side of (1.1), the usual treatments for the system (1.1) do not give satisfactory results (see [16]). Numerical methods for conservation laws with source terms have been therefore searched and tested (see [12, 11, 4, 5, 16, 10, 3], and also the references therein). A fast and stable numerical scheme for (1.1) was recently constructed by Kröner-Thanh [16]. We will show in Section 5 that this scheme not only conserve equilibrium states as shown in [16], but also conserve the nonnegativity of the density. Moreover, the approximate solutions by this scheme also satisfy the entropy minimum principle that the entropy is increasing in time.

We note that the Riemann problem for isentropic compressible flows in a nozzle was solved in [19]. The Riemann problem for several typical systems of balance laws in nonconservative form was considered by [15, 14, 9, 2], etc.

Some important results of this paper (without proofs) were appeared in [17].

2. Systems of Conservation Laws in Nonconservative Form.

2.1. Definition. Let us consider a general system of conservation laws in nonconservative form

$$\partial_t U + A(U)\cdot \partial_x U = 0. \tag{2.1}$$

(see Dal Maso-LeFloch-Murat [7]). The notion of weak solutions of the system (2.1) [7] is relied on a prescribed family of Lipschitz paths in $\boldsymbol{R}^N$. Precisely, let us be given a *family of Lipschitz paths* $\phi : [0,1]\times \boldsymbol{R}^N \times \boldsymbol{R}^N \to \boldsymbol{R}^N$ which satisfies

$$\begin{aligned} &\phi(0;U,V) = U \quad \phi(1;U,V) = V,\\ &|\partial_s\phi(s;U,V)| \le K|V-U|,\\ &|\partial_s\phi(s;U_1,V_1) - \partial_s\phi(s;U_2,V_2)| \le K(|V_1-V_2| + |U_1-U_2|), \end{aligned} \tag{2.2}$$

for some $K>0$, for all $s\in[0,1], U,V,U_1,U_2,V_1,V_2 \in \boldsymbol{R}^N$.

Due to this family of Lipschitz paths, one can determine a Borel measure, as follows. Let U be a function with bounded variation in $[a,b]$ (in the following we will call it a BV-function for simplicity), then dU is a Borel measure which coincides with the distributional derivative of U, i.e.,

$$\int_a^b U\varphi' dx = -\int_a^b \varphi dU, \quad \forall \varphi \in C_0^\infty[a,b],$$

(see [1]).

Definition 2.1. *[7] Let $U=U(x), x\in[a,b]$, be a function with bounded variation. Then, the nonconservative product $\mu := \Big[g(U)\cdot dU\Big]_\phi$ of a locally Borel bounded function $g:\boldsymbol{R}^N \to \boldsymbol{R}^N$ by the vector-valued Borel measure dU is a real-valued bounded Borel measure μ with the following properties:*

(i) For any Borel set B, s.t. U is continuous on B:

$$\mu(B) = \int_B g(U)dU \tag{2.3}$$

(ii) For any $x_0\in[a,b]$:

$$\mu(x_0) = \int_0^1 g(\phi(s;U(x_0-),U(x_0+)))\partial_s\phi(s;U(x_0-),U(x_0+))ds \tag{2.4}$$

The following definition of weak solutions of the system (2.1) was initially proposed by LeFloch [21].

Definition 2.2. *$U\in L^\infty \cap BV_{loc}(\boldsymbol{R}\times\boldsymbol{R}_+,\boldsymbol{R}^N)$ is a weak solution of (2.1) if the Borel measure*

$$\partial_t U + \Big[A(U(.,t))\partial_x U(.,t)\Big]_\phi \tag{2.5}$$

is equal to zero.

2.2. Entropy Inequality. The general Definition 2.2 has an advantage that it inherits automatically the formulation of the entropies from standard theory of systems of conservation laws to the ones in nonconservative form. In fact, a convex function $\mathcal{U}$ is called a *generalized entropy* of the system (2.1) provided

$$D_U^2\mathcal{U}(U)\cdot A(U) = A(U)^T\cdot D_U^2\mathcal{U}(U). \tag{2.6}$$

The only difference between the standard notion of entropy and the generalized one is that the entropy-flux function may not exist in the nonconservative systems. However, as a family of Lipschitz paths ϕ is given, we have the following *generalized entropy inequality* [20]:

$$\partial_t\mathcal{U}(U) + \Big[D_U\mathcal{U}(U)\cdot A(U(.,t))\partial_x U(.,t)\Big]_\phi \le 0, \tag{2.7}$$

in the space of measures.

2.3. The nozzle model. The model of fluid flows in a duct with variable cross-section (1.1) supplemented with the equation (1.2) has the form (2.1), depending on the choice of variables, as follows. If we choose the variable U to be

$$U = (\rho, u, S, a), \tag{2.8}$$

where S is the specific entropy, then the system (1.1)-(1.2) can be written under the form (2.1) with

$$A(U) = \begin{pmatrix} u & \rho & 0 & \dfrac{u\rho}{a} \\ \dfrac{p_\rho}{\rho} & u & \dfrac{p_S}{\rho} & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.9}$$

However, as we will see in the sequel, it will be interesting to re-write the system (1.1)-(1.2) under the form (2.1) with the variable to be *the conservative quantities.* In fact, let us set

$$U = (a\rho, a\rho u, a\rho e, a) := (w_1, w_2, w_3, w_4). \tag{2.10}$$

Replacing the variable U in (2.4) into the system (1.1)-(1.2), we get

$$\begin{aligned}
&\partial_t w_1 + \partial_x w_2 \;=\; 0,\\
&\partial_t w_2 + \frac{w_2}{w_1}\partial_x w_2 + w_2\partial_x\big(\frac{w_2}{w_1}\big) + w_4\partial_x p \;=\; 0,\\
&\partial_t w_3 + w_2\partial_x\big(\frac{w_3}{w_1}\big) + \frac{w_3}{w_1}\partial_x w_2 + \partial_x(aup) \;=\; 0,\\
&\partial_t w_4 = 0,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\partial_t w_1 + \partial_x w_2 \;=\; 0,\\
&\partial_t w_2 + \frac{w_2}{w_1}\partial_x w_2 + w_2\frac{w_1\partial_x w_2 - w_2\partial_x w_1}{w_1^2} + w_4\partial_x p \;=\; 0,\\
&\partial_t w_3 + w_2\frac{w_1\partial_x w_3 - w_3\partial_x w_1}{w_1^2} + \frac{w_3}{w_1}\partial_x w_2 + \partial_x(aup) \;=\; 0,\\
&\partial_t w_4 = 0, \quad x\in\boldsymbol{R},\, t>0.
\end{aligned} \tag{2.11}$$

Besides, the pressure p is given by certain equations of state. And we can assume that $p = p(\rho, \epsilon)$. Thus,

$$p = p\Big(\frac{w_1}{w_4}, \frac{w_3}{w_1} - \frac{1}{2}\big(\frac{w_2}{w_1}\big)^2\Big). \tag{2.12}$$

The last expression leads to the following calculation

$$\begin{aligned}
\partial_x p &= p_\rho \cdot \rho_x + p_\epsilon \cdot \epsilon_x \\
&= p_\rho \cdot \partial_x\big(\frac{w_1}{w_4}\big) + p_\epsilon \cdot \partial_x\Big(\frac{w_3}{w_1} - \frac{1}{2}\big(\frac{w_2}{w_1}\big)^2\Big) \\
&= p_\rho \cdot \frac{w_4\partial_x w_1 - w_1\partial_x w_4}{w_4^2} + p_\epsilon\Big(\frac{w_1\partial_x w_3 - w_3\partial_x w_1}{w_1^2} - \frac{w_2}{w_1}\frac{w_1\partial_x w_2 - w_2\partial_x w_1}{w_1^2}\Big) \\
&= \Big(\frac{p_\rho}{w_4} - \frac{p_\epsilon w_3}{w_1^2} + \frac{p_\epsilon w_2^2}{w_1^3}\Big) \cdot \partial_x w_1 - \frac{p_\epsilon w_2}{w_1^2}\partial_x w_2 + \frac{p_\epsilon}{w_1}\partial_x w_3 - \frac{p_\rho w_1}{w_4^2}\partial_x w_4.
\end{aligned} \tag{2.13}$$

It is thus derived from (2.13) that

$$\begin{aligned}
\partial_x(aup) &= p\partial_x(au) + au\partial_x p = p\partial_x\big(\frac{w_2w_4}{w_1}\big) + \frac{w_2w_4}{w_1}\partial_x p \\
&= p\frac{(w_4\partial_x w_2 + w_2\partial_x w_4)w_1 - w_2w_4\partial_x w_1}{w_1^2} + \frac{w_2w_4}{w_1}\partial_x p \\
&= \frac{w_2w_4}{w_1}\Big(\frac{p_\rho}{w_4} - \frac{p_\epsilon w_3}{w_1^2} + \frac{p_\epsilon w_2^2}{w_1^3} - \frac{p}{w_1}\Big) \cdot \partial_x w_1 + \frac{w_4}{w_1}\Big(p - \frac{p_\epsilon w_2^2}{w_1^2}\Big)\partial_x w_2 \\
&\qquad + \frac{p_\epsilon w_2 w_4}{w_1^2}\partial_x w_3 + \frac{w_2}{w_1}\Big(p - \frac{p_\rho w_1}{w_4}\Big)\partial_x w_4.
\end{aligned} \tag{2.14}$$

Substituting the expressions (2.13), (2.14) into the system (2.5), after arranging terms, we obtain

$$\begin{aligned}
&\partial_t w_1 + \partial_x w_2 \;=\; 0, \\
&\partial_t w_2 + \Big(p_\rho - \frac{p_\epsilon w_3 w_4}{w_1^2} + \frac{p_\epsilon w_2^2 w_4}{w_1^3} - \frac{w_2^2}{w_1^2}\Big) \cdot \partial_x w_1 + \Big(\frac{2w_2}{w_1} - \frac{p_\epsilon w_2 w_4}{w_1^2}\Big)\partial_x w_2 \\
&\qquad + \frac{p_\epsilon w_4}{w_1}\partial_x w_3 - \frac{p_\rho w_1}{w_4}\partial_x w_4 \;=\; 0, \\
&\partial_t w_3 + \Big[\frac{w_2w_4}{w_1}\Big(\frac{p_\rho}{w_4} - \frac{p_\epsilon w_3}{w_1^2} + \frac{p_\epsilon w_2^2}{w_1^3} - \frac{p}{w_1}\Big) - \frac{w_2w_3}{w_1^2}\Big] \cdot \partial_x w_1 \\
&\quad + \Big(\frac{w_3 + pw_4}{w_1} - \frac{p_\epsilon w_2^2 w_4}{w_1^3}\Big)\partial_x w_2 + \Big(\frac{p_\epsilon w_2 w_4}{w_1^2} + \frac{p + w_2}{w_1}\Big)\partial_x w_3 \\
&\qquad + \frac{w_2}{w_1}\Big(p - \frac{p_\rho w_1}{w_4}\Big)\partial_x w_4 \;=\; 0, \\
&\partial_t w_4 = 0, \quad x \in \boldsymbol{R},\, t > 0.
\end{aligned}$$

The last system has the canonical form (2.1), where the variable is chosen to be conservative:

$$U = (a\rho, a\rho u, a\rho e, a) = (w_1, w_2, w_3, w_4),$$

and $A(U) = \big(a_{ij}(U)\big)_{i,j=\overline{1,4}}$:

$$
\begin{aligned}
& a_{11} = 0, \quad a_{12} = 1, \quad a_{13} = 0, \quad a_{14} = 0, \\
& a_{21} = p_\rho - \frac{p_\epsilon w_3 w_4}{w_1^2} + \frac{p_\epsilon w_2^2 w_4}{w_1^3} - \frac{w_2^2}{w_1^2} = p_\rho - u^2 + \frac{p_\epsilon}{\rho}(u^2 - e), \\
& a_{22} = \frac{2w_2}{w_1} - \frac{p_\epsilon w_2 w_4}{w_1^2} = 2u - \frac{p_\epsilon u}{\rho}, \\
& a_{23} = \frac{p_\epsilon w_4}{w_1} = \frac{p_\epsilon}{\rho}, \\
& a_{24} = -\frac{p_\rho w_1}{w_4} = -p_\rho \rho, \\
& a_{31} = \frac{w_2 w_4}{w_1}\Big(\frac{p_\rho}{w_4} - \frac{p_\epsilon w_3}{w_1^2} + \frac{p_\epsilon w_2^2}{w_1^3} - \frac{p}{w_1}\Big) - \frac{w_2 w_3}{w_1^2} = u(p_\rho - e + \frac{p_\epsilon(u^2 - e) - p}{\rho}), \\
& a_{32} = \frac{w_3 + p w_4}{w_1} - \frac{p_\epsilon w_2^2 w_4}{w_1^3} = e + \frac{p}{\rho} - \frac{p_\epsilon u^2}{\rho}, \\
& a_{33} = \frac{p_\epsilon w_2 w_4}{w_1^2} + \frac{p + w_2}{w_1} = \frac{p_\epsilon u}{\rho} + u, \\
& a_{34} = \frac{w_2}{w_1}\Big(p - \frac{p_\rho w_1}{w_4}\Big) = u(p - p_\rho \rho), \\
& a_{41} = a_{42} = a_{43} = a_{44} = 0.
\end{aligned}
\tag{2.15}
$$

3. Entropy inequality by vanishing viscosity method. Let us first describe the motivation. Consider the usual one-dimensional gas dynamics equations

$$
\begin{aligned}
\partial_t \rho + \partial_x(\rho u) &= 0, \\
\partial_t(\rho u) + \partial_x(\rho u^2 + p) &= 0, \\
\partial_t(\rho e) + \partial_x(u(\rho e + p)) &= 0, \quad x \in \boldsymbol{R},\ t > 0.
\end{aligned}
\tag{3.1}
$$

On one hand, the notion of entropy is motivated from physics, and the physical entropy is

$$\mathcal{U} = \rho S,$$

where S is the specific entropy. On the other hand, as it was shown by Harten et al [13], necessary and sufficient conditions for a twice differentiable function $\mathcal{U}_c$ of the form:

$$\mathcal{U}_c = \rho g(S), \tag{3.2}$$

to be an entropy of the usual gas dynamics equations is that $g(S)$ satisfies the following properties

(i) $g(S)$ is strictly decreasing as function of ϵ;

(ii) $g(S)$ is strictly convex as function of $(1/\rho, \epsilon)$.

And the authors also showed that this class of entropies is broad enough to justify the point-wise hyperbolicity of the gas dynamics equations (3.1) in the sense that the system is strictly hyperbolic if and only if it admits an entropy of the form (3.2).

Therefore, for the model with source term (1.1), we restrict our consideration to a similar class of entropies. To this end, we first look at the model of viscous flows in a smooth nozzle a_ε:

$$\begin{aligned}
&\partial_t(a_\varepsilon\rho) + \partial_x(a_\varepsilon\rho u) \;=\; 0,\\
&\partial_t(a_\varepsilon\rho u) + \partial_x(a_\varepsilon(\rho u^2+p)) \;=\; p\partial_x a_\varepsilon + \partial_x(b_\varepsilon(\gamma(\varepsilon)+\mu(\varepsilon))\partial_x u),\\
&\partial_t(a_\varepsilon\rho e) + \partial_x(a_\varepsilon u(\rho e+p)) \;=\; \partial_x(b_\varepsilon((\gamma(\varepsilon)+\mu(\varepsilon))u\partial_x u + k\partial_x T)), \quad x\in \boldsymbol{R},\, t>0,
\end{aligned} \tag{3.3}$$

where $b_\varepsilon = b_\varepsilon(x) \geq 0$, $x \in \boldsymbol{R}$, is given, and γ, μ are Lamé coefficients of viscosity and tend to zero as ε tends to zero, and k is the coefficient of thermal conductivity. Note that the coefficients γ, μ and k usually depend only on the absolute temperature T. Let us ignore in the following the subscript $(.)_\varepsilon$ for simplicity. We want to find an equation for the specific entropy. To this end, we assume that the internal energy is given by an equation of state $\epsilon = \epsilon(\rho, S)$. On one hand, thanks to the equation of conservation of mass of (3.3), the equation of balance of momentum in (3.3) can be written as

$$a\rho(u_t+uu_x) + p_x a - (b(\gamma+\mu)uu_x)_x = 0. \tag{3.4}$$

On the other hand, the equation of balance of energy in (3.3) can be written as

$$a\rho e_t + e\big[(a\rho)_t + (a\rho u)_x\big] + a\rho u e_x + (aup)_x \;=\; (b((\gamma+\mu)uu_x + kT_x))_x. \tag{3.5}$$

The second term on the left-hand side of (3.5) is equal to zero due to the conservation of mass. Using the thermodynamical identity

$$d\epsilon = TdS - pdv, \quad v = 1/\rho, \tag{3.6}$$

we can re-write the equation (3.5) as

$$a\rho T(S_t+uS_x) + \frac{ap}{\rho}(\rho_t + u\rho_x) + (aup)_x + a\rho u(u_t+uu_x) \;=\; (b((\gamma+\mu)uu_x + kT_x))_x.$$

Or, after arranging terms, we obtain the following equation of balance of energy

$$\begin{aligned}
a\rho T(S_t+uS_x) + \frac{p}{\rho}\big[(a\rho)_t + (au\rho)_x\big] + u\big[a\rho(u_t+uu_x) + ap_x - (b(\gamma+\mu)u_x)_x\big]&\\
= b(\gamma+\mu)u_x^2 + (kT_x)_x.&
\end{aligned} \tag{3.7}$$

The second and the third term on the left-hand side of (3.7) are equal to zero by the conservation of mass of (3.3) and the balance of momentum (3.4). Thus, we deduce from (3.7) that the specific entropy S should satisfy

$$\partial_t S + u\partial_x S = \frac{b(\gamma+\mu)}{a\rho T}u_x^2 + \frac{(kT_x)_x}{a\rho T}. \tag{3.8}$$

Let $g(S)$ be any smooth function of S. Multiplying the equation (3.8) by $a\rho g'(S)$, we get

$$a\rho\partial_t g(S) + a\rho u\partial_x g(S) = a\rho g'(S)\Big(\frac{\gamma+\mu}{\rho T}u_x^2 + \frac{(kT_x)_x}{a\rho T}\Big). \tag{3.9}$$

Multiplying the conservation of mass of (3.3) by $g(S)$, and then summing up the resulted equation with the equation (3.9), we have

$$\partial_t(a\rho g(S)) + \partial_x(a\rho u g(S)) \;=\; \rho g'(S)\Big(\frac{b(\gamma+\mu)}{\rho T}u_x^2 + \frac{(kT_x)_x}{\rho T}\Big). \qquad (3.10)$$

Assume that the thermal conductivity $k = 0$. Then, the equation (3.10) becomes

$$\partial_t(a\rho g(S)) + \partial_x(a\rho u g(S)) = b\rho g'(S)\frac{\gamma+\mu}{\rho T}u_x^2. \qquad (3.11)$$

The right-hand side is always non-positive whenever g is non-increasing.

Thus, we arrive at the following statement.

THEOREM 3.1. *Consider viscous flows (3.3) with the following hypotheses: the thermal conductivity coefficient is negligible, and the nozzle is smooth for each choice of viscosity coefficients. Assume that for each ε the system (3.3) has a smooth solution U_ε. Then, the limit of these solutions obtained as the viscosity vanishes when ε tends to zero satisfies the following* entropy inequality

$$\partial_t(a\rho g(S)) + \partial_x(a\rho u g(S)) \le 0, \qquad (3.12)$$

where g is any function satisfying the above items (i)-(ii).

4. Minimum entropy principle.

4.1. The entropy inequality in divergence form. In this subsection, we will show that the entropy inequality in terms of *nonconservative product* [7] reduces to the usual divergence form (3.12) for entropy-pairs of the form

$$(\mathcal{U}, \mathcal{F}) = (a\rho g(S), a\rho u g(S)), \qquad (4.1)$$

where the functions g satisfy the assumptions (i)-(ii) in the previous section.

THEOREM 4.1. *Consider the system (1.1) in the form of (2.1), where the matrix $A(U)$ is given by (2.15). Let g be any function satisfying the hypotheses i) and ii) in the previous section. The function $\mathcal{U} = a\rho g(S)$ of the conservative variables $(a\rho, a\rho u, a\rho e, a)$ is convex. Moreover, it satisfies the following property*

$$D_U\mathcal{U} \cdot A(U) = D_U\mathcal{F}(U), \quad \mathcal{F}(U) = a\rho u g(S). \qquad (4.2)$$

Consequently, the function $\mathcal{U}$ is the entropy of the system (2.1) with the entropy-flux $\mathcal{F}$, i.e., the generalized entropy inequality in the sense of Dal Maso-LeFloch-Murat [7] is reduced to the usual one in the divergence form

$$(a\rho g(S))_t + (a\rho u g(S))_x \le 0. \qquad (4.3)$$

Proof. First, as is shown in [13], the function $\rho g(S)$ is convex in the variable $(\rho, \rho u, \rho e)$. By projection, $\rho g(S)$ is convex in the variable $(\rho, \rho u, \rho e, 1)$. Since the mapping $(\rho, \rho u, \rho e, 1) \mapsto a(\rho, \rho u, \rho e, 1)$ is linear, the Hessian matrix of $\rho g(S)$ as function of $(a\rho, a\rho u, a\rho e, a)$ is the same of that one as a function of $(\rho, \rho u, \rho e, 1)$ at $a(\rho, \rho u, \rho e, 1) = (a\rho, a\rho u, a\rho e, a)$. Then, multiplying the convex function $\rho g(S)$ of variable

$(a\rho, a\rho u, a\rho e, a)$ with the positive a give a convex function in the same variable. This establishes the first conclusion that the function $\mathcal{U} = a\rho g(S)$ of the conservative variables

$$(a\rho, a\rho u, a\rho e, a) = (w_1, w_2, w_3, w_4)$$

is convex.

Second, assume the equation of state for the specific entropy is given by

$$S = S(\rho, \epsilon). \tag{4.4}$$

Then, a straightforward calculation shows that

$$\mathcal{U}(U) = a\rho g(S) = w_1 g\Big(S\Big(\frac{w_1}{w_4}, \frac{w_3}{w_1} - \frac{1}{2}\big(\frac{w_2}{w_1}\big)\Big)\Big), \quad \mathcal{F}(U) = u\mathcal{U}(U). \tag{4.5}$$

Using the thermodynamical identity

$$d\epsilon = TdS - pdv = TdS + \frac{p}{\rho^2}d\rho,$$

we have

$$S_\epsilon = \frac{1}{T}, \quad S_\rho = \frac{-p}{T\rho^2}.$$

Therefore, it is followed from (4.5) that

$$\begin{aligned} D_U\mathcal{U}(U) &= \Big(g(S) + w_1 g'(S)\big(\frac{S_\rho}{w_4} + \frac{S_\epsilon}{w_1^2}\big(\frac{w_2^2}{w_1} - w_3\big)\big), \\ &\qquad - g'(S)S_\epsilon\frac{w_2}{w_1}, g'(S)S_\epsilon, -g'(S)S_\rho\frac{w_1^2}{w_4^2}\Big) \\ &= \big(g(S) + \frac{g'(S)}{T}\big(-\frac{p}{\rho} - e + u^2\big), -\frac{g'(S)u}{T}, \frac{g'(S)}{T}, \frac{g'(S)p}{T}\big). \end{aligned} \tag{4.6}$$

and that

$$\begin{aligned} D_U\mathcal{F}(U) &= uD_U\mathcal{U}(U) + \mathcal{U}(U)D_U\big(\frac{w_2}{w_1}\big) \\ &= \big(u\mathcal{U}_{w_1} - \frac{w_2}{w_1^2}\mathcal{U}(U), u\mathcal{U}_{w_2} + \frac{\mathcal{U}(U)}{w_1}, u\mathcal{U}_{w_3}, u\mathcal{U}_{w_4}\big). \end{aligned} \tag{4.7}$$

From (2.4), (2.15), (4.6), and (4.7), we claim that

$$B := D_U\mathcal{U} \cdot A(U) - D_U\mathcal{F}(U) = 0, \tag{4.8}$$

since set $B = (b_1, b_2, b_3)$, we have

$$\begin{aligned} b_1 &= -w_1 g'(S)S_\epsilon\frac{w_2}{w_1}a_{21} + w_1 g'(S)\frac{S_\epsilon}{w_1}a_{31} - u\mathcal{U}_{w_1} + \frac{w_2}{w_1^2}\mathcal{U} \\ &= g'(S)S_\epsilon\big(a_{31} - \frac{w_2}{w_1}a_{21}\big) - u\mathcal{U}_{w_1} + \frac{w_2}{w_1^2}\mathcal{U} \\ &= g'(S)S_\epsilon\big(\frac{-pw_2w_4}{w_1^2} + \frac{w_2^3}{w_1^3} - \frac{w_2w_3}{w_1^2}\big) - u\mathcal{U}_{w_1} + \frac{w_2}{w_1^2}\mathcal{U} \\ &= g'(S)S_\epsilon\big(u^3 - \frac{pu}{\rho} - ue\big) - u\mathcal{U}_{w_1} + \frac{w_2}{w_1^2}\mathcal{U} \\ &= \frac{g'(S)u}{T}\big(u^2 - \frac{p}{\rho} - e\big) - u\big(g(S) + \frac{g'(S)}{T}\big(\frac{-p}{\rho} + u^2 - e\big)\big) + ug(S) \\ &= 0, \end{aligned} \tag{4.9}$$

and

$$
\begin{aligned}
b_2 &= g(S) + \frac{g'(S)}{T}(\frac{-p}{\rho} + u^2 - e) - \frac{g'(S)u}{T}(2u - p_\epsilon\frac{u}{\rho}) + \frac{g'(S)}{T}(e + \frac{p}{\rho} - p_\epsilon\frac{u^2}{\rho}) \\
&\qquad - (u\mathcal{U}_{w_2} + \frac{\mathcal{U}}{w_1}) \\
&= g(S) - \frac{g'(S)u^2}{T} - u\mathcal{U}_{w_2} - \frac{\mathcal{U}}{w_1}) \\
&= 0,
\end{aligned}
\tag{4.10}
$$

and lastly

$$
\begin{aligned}
b_3 &= -\frac{g'(S)u}{T}\frac{p_\epsilon}{\rho} + \frac{g'(S)}{T}(\frac{p_\epsilon u}{\rho} + u) - u\mathcal{U}_{w_3} \\
&= 0.
\end{aligned}
\tag{4.11}
$$

From (4.9), (4.10), and (4.11), we obtain (4.8). This establishes the second statement (4.2). The remaining conclusions are straightforward. Theorem 4.1 is completely proved. □

4.2. Minimum entropy principle. In this subsection we will establish a minimum entropy principle for entropy solutions in the sense of Definition 4.2 for the model of gas flows in a nozzle (1.1). Note that a similar work for the usual gas dynamics equations was established by Tadmor [25].

THEOREM 4.2. *Assume that the bounded function U is an entropy solution of the system (1.3). The following* Minimum Entropy Principle *holds*

$$
||S(.,t)||_{L^\infty(-R,R)} \geq ||S(.,0)||_{L^\infty(-(R+t||u||_{L^\infty}),R+t||u||_{L^\infty})}. \tag{4.12}
$$

Proof. Let $g = g(S)$, where S is the specific entropy, be any function satisfying the conditions (i) and (ii) in the subsection 3.3, i.e.,

(i) $g(S)$ is strictly decreasing as function of the internal energy ϵ;

(ii) $g(S)$ is strictly convex as function of $(1/\rho, \epsilon)$, where ρ is the density.

Similar arguments as the ones in the proof of Lemma 3.1 of [25] assert that

LEMMA 4.3. *Any bounded entropy solution of the system (1.1) should satisfy*

$$
\int_{|x|\leq R} \rho(x,t)g(S(x,t))dx \leq \int_{|x|\leq R+t||u||_{L^\infty}} \rho(x,0)g(S(x,0))dx. \tag{4.13}
$$

Taking $g(S) = -(S+S_0)^p, p > 1$, where S_0 is some constant such that $S+S_0 > 0$, the inequality (4.13) yields

$$
\begin{aligned}
&||\rho^{1/p}(.,t)(S(.,t) + S_0)||_{L^p(-R,R)} \\
&\qquad \geq ||\rho^{1/p}(.,0)(S(.,0) + S_0)||_{L^p(-(R+t||u||_{L^\infty}),R+t||u||_{L^\infty})}.
\end{aligned}
\tag{4.14}
$$

Letting $p \to +\infty$ in the last inequality and eliminating S_0, we obtain (4.12). Theorem 4.3 is completely proved. □

5. Principles for the numerical solutions. In [16], a new fast and stable (by numerical tests) scheme for system (1.1) was presented. In this section we will investigate properties of approximate solutions generated by this scheme. For definitiveness, we consider stiffened gases which have equations of state of the form

$$p = (\gamma - 1)\rho(\epsilon - \epsilon_\infty) - \gamma p_\infty, \quad 1 < \gamma < 5/3,$$

where $\epsilon_\infty, p_\infty$ are constants, depending on the material under consideration, $p_\infty \geq 0$, (see Menikoff and Plohr [23]). We note that the analysis presented below can be applied to a broader class of gases.

5.1. Nonnegativity density principle. In this subsection, we will show that the approximate density by the scheme in [16] always remains non-negative. This numerical scheme is defined as follows: the mesh-size can be chosen to be uniform, i.e., $x_{j+1} - x_j = \Delta x = l, \forall j$, and

$$\begin{aligned}
&\lambda \leq \frac{1}{\max_j\{|u_j^n| + \sqrt{2p_\rho(\rho_j^n, S_j^n)}\}},\\
&\Delta t = \lambda \Delta x,\\
&U := (\rho, \rho u, \rho e), \quad f(U) := (\rho u, (\rho u^2 + p), u(\rho e + p)),\\
U_j^{n+1} &= U_j^n - \lambda(g^{\mathrm{N}}(U_j^n, U_{j+1,-}^n) - g^{\mathrm{N}}(U_{j-1,+}^n, U_j^n)),\\
&= \frac{U_{j-1,+}^n + U_{j+1,-}^n}{2} + \frac{\lambda}{2}(f(U_{j-1,+}^n) - f(U_{j+1,-}^n)),
\end{aligned} \tag{5.1}$$

where $g^{\mathrm{N}}(U, V)$ is the Lax-Friedrichs numerical flux:

$$g^{\mathrm{N}}(U, V) = \frac{1}{2}(f(U) + f(V)) - \frac{1}{2\lambda}(V - U). \tag{5.2}$$

The description of the states

$$U_{j+1,-}^n = (\rho, \rho u, \rho e)_{j+1,-}^n, \quad U_{j-1,+}^n = (\rho, \rho u, \rho e)_{j-1,+}^n \tag{5.3}$$

is given as follows. As was observed in [16], the system (1.1), (2.14) has four characteristic fields corresponding to the following eigenvalues of its Jacobian matrix:

$$\lambda_0 = 0, \quad \lambda_1 = u - \sqrt{p_\rho(\rho, S)}, \quad \lambda_2 = u, \quad \lambda_3 = u + \sqrt{p_\rho(\rho, S)}. \tag{5.4}$$

The phase domain can then be decomposed into the following sub-domains (call each a *phase*):

$$\begin{aligned}
G_1 &= \{U : \lambda_1(U) < \lambda_2(U) < \lambda_3(U) < \lambda_0(U)\},\\
G_2 &= \{U : \lambda_1(U) < \lambda_2(U) < \lambda_0(U) < \lambda_3(U)\},\\
G_3 &= \{U : \lambda_1(U) < \lambda_0(U) < \lambda_2(U) < \lambda_3(U)\},\\
G_4 &= \{U : \lambda_0(U) < \lambda_1(U) < \lambda_2(U) < \lambda_3(U)\},
\end{aligned} \tag{5.5}$$

together with isolated surfaces only on which the system fails to be strictly hyperbolic (call each a *hyperbolic boundary*):

$$\begin{aligned}
\Sigma_+ &= \{U : \lambda_1(U) = \lambda_0(U)\},\\
\Sigma_0 &= \{U : \lambda_2(U) = \lambda_0(U)\},\\
\Sigma_- &= \{U : \lambda_3(U) = \lambda_0(U)\}.
\end{aligned} \tag{5.6}$$

Now, in the scheme (5.1), the states

$$U^n_{j+1,-} = (\rho, \rho u, \rho e)^n_{j+1,-}, \quad U^n_{j-1,+} = (\rho, \rho u, \rho e)^n_{j-1,+}$$

are defined as follows. First, observe that the entropy is constant across each stationary jump. We set $\epsilon^n_{j+1,-} = \epsilon(\rho^n_{j+1,-}, S^n_{j+1}), e^n_{j+1,-} = \epsilon^n_{j+1,-} + (u^n_{j+1,-})^2/2$, and so on. And determine $\rho^n_{j+1,-}, u^n_{j+1,-}$ from the following equations

$$\begin{aligned} &a^n_{j+1}\rho^n_{j+1}u^n_{j+1} = a^n_j \rho^n_{j+1,-}u^n_{j+1,-}, \\ &\frac{(u^n_{j+1})^2}{2} + h(\rho^n_{j+1}, S^n_{j+1}) = \frac{(u^n_{j+1,-})^2}{2} + h(\rho^n_{j+1,-}, S^n_{j+1}), \end{aligned} \tag{5.7}$$

where h is the specific enthalpy: $dh = TdS + vdp$, and h written as a function $h = h(\rho, S)$ of the density and the specific entropy satisfies

$$h_\rho(\rho, S) \;=\; \frac{p_\rho(\rho, S)}{\rho}, \tag{5.8}$$

for any fixed S, and that *the stationary jump always remains in the same phase.* Similarly, we determine $\rho^n_{j-1,+}, u^n_{j-1,+}$ from the equations

$$\begin{aligned} &a^n_{j-1}\rho^n_{j-1}u^n_{j-1} = a^n_j \rho^n_{j-1,+}u^n_{j-1,+}, \\ &\frac{(u^n_{j-1})^2}{2} + h(\rho^n_{j-1}, S^n_{j-1}) = \frac{(u^n_{j-1,+})^2}{2} + h(\rho^n_{j-1,+}, S^n_{j-1}), \end{aligned} \tag{5.9}$$

and also that the stationary jump always remains in the same phase.

We will show that our scheme (5.1) is stable in the sense that the density remains always in the correct phase.

THEOREM 5.1. *Let us consider stiffened gases, where the local sound speed $c := \sqrt{p_\rho(\rho, S)}$ is real. If the initial density ρ_0 is nonnegative, then the approximate density generated by the scheme (5.1) is also nonnegative.*

Proof. The scheme (5.1) computes the density as

$$\begin{aligned} \rho^{n+1}_j &= \frac{\rho^n_{j-1,+} + \rho^n_{j+1,-}}{2} + \frac{\lambda}{2}(\rho^n_{j-1,+}u^n_{j-1,+} - \rho^n_{j+1,-}u^n_{j+1,-}) \\ &\geq \frac{\rho^n_{j-1,+} + \rho^n_{j+1,-}}{2} - \frac{\lambda}{2} 2\max\{u^n_{j-1,+}, u^n_{j+1,-}\}(\rho^n_{j-1,+} + \rho^n_{j+1,-}) \\ &\geq \frac{\rho^n_{j-1,+} + \rho^n_{j+1,-}}{2}(1 - 2\lambda \max\{u^n_{j-1,+}, u^n_{j+1,-}\}). \end{aligned} \tag{5.10}$$

Since stationary waves provided by (5.7) and (5.9) always connect between states with nonnegative densities, it is derived from the inequality (5.10) that

$$\rho^{n+1}_j \geq 0 \quad \text{whenever} \quad 1 - 2\lambda \max\{u^n_{j-1,+}, u^n_{j+1,-}\} \geq 0. \tag{5.11}$$

Besides, the hypothesis implies that the function h in (5.8) is concave in ρ:

$$h_{\rho\rho}(\rho, S) = (\gamma - 2)\frac{p_\rho(\rho, S)}{\rho^2} \;\leq\; 0. \tag{5.12}$$

It is thus derived from (5.7) and the concavity of the function h that

$$\begin{aligned} \frac{(u^n_{j+1,-})^2}{2} - \frac{(u^n_{j+1})^2}{2} &= -(h(\rho^n_{j+1,-}, S^n_{j+1}) - h(\rho^n_{j+1}, S^n_{j+1})) \\ &\le -h_\rho(\rho^n_{j+1}, S^n_{j+1})(\rho^n_{j+1,-} - \rho^n_{j+1}) \\ &\le -\frac{p_\rho(\rho^n_{j+1}, S^n_{j+1})}{\rho^n_{j+1}}(\rho^n_{j+1,-} - \rho^n_{j+1}) \\ &\le p_\rho(\rho^n_{j+1}, S^n_{j+1}). \end{aligned} \tag{5.13}$$

And therefore,

$$|u^n_{j+1,-}| \le \sqrt{(u^n_{j+1})^2 + 2p_\rho(\rho^n_{j+1}, S^n_{j+1})} \le |u^n_{j+1}| + \sqrt{2p_\rho(\rho^n_{j+1}, S^n_{j+1})}. \tag{5.14}$$

Similarly,

$$|u^n_{j-1,+}| \le |u^n_{j-1}| + \sqrt{2p_\rho(\rho^n_{j-1}, S^n_{j-1})}. \tag{5.15}$$

From (5.1), (5.11), (5.14), and (5.15), we conclude that

$$\rho^{n+1}_j \ge 0, \quad \text{for all} \quad j, n, \tag{5.16}$$

which establishes the statement of the theorem. ▯

5.2. Minimum entropy principle for approximate solutions. In this subsection we also establish the minimum entropy principle for approximate solutions (5.1).

THEOREM 5.2. *Assume that there is a bounded sequence of approximate solutions generated by the scheme (5.1) which converges almost everywhere to a limit U. Then, the following minimum numerical entropy principle holds:*

$$S^{n+1}_j \ \ge\ \min\{S^n_{j-1}, S^n_{j+1}\}. \tag{5.17}$$

Proof. The proof is based on the following result by Lax [18].

LEMMA 5.3. *Assume that $\mathcal{U}$ is a strictly convex function in $\boldsymbol{R}^N$, and the scalar-valued function $\mathcal{F}$, and the vector-valued function f such that*

$$D\mathcal{F} = D\mathcal{U} \cdot Df. \tag{5.18}$$

If U is a vector defined by

$$U = \frac{V + W}{2} + \frac{\lambda}{2}(f(V) - f(W)), \tag{5.19}$$

then

$$\mathcal{U}(U) \le \frac{\mathcal{U}(V) + \mathcal{U}(W)}{2} + \frac{\lambda}{2}(\mathcal{F}(V) - \mathcal{F}(W)). \tag{5.20}$$

Comparing (5.1) and (5.19), we conclude by (5.20) that the scheme (5.1) satisfies the numerical entropy inequality

$$\mathcal{U}(U^{n+1}_j) \le \frac{\mathcal{U}(U^n_{j-1,+}) + \mathcal{U}(U^n_{j+1,-})}{2} + \frac{\lambda}{2}(\mathcal{F}(U^n_{j-1,+}) - \mathcal{F}(U^n_{j+1,-})), \tag{5.21}$$

for any entropy-pair of the form (4.1). Thus, we have

$$a_j^{n+1}\rho_j^{n+1}g(S_j^{n+1}) \leq \frac{a_j^n\rho_{j-1,+}^n g(S_{j-1}^n) + a_j^n\rho_{j+1,-}^n g(S_{j+1}^n)}{2} + \frac{\lambda}{2}(a_j^n\rho_{j-1,+}^n u_{j-1,+}^n g(S_{j-1}^n) - a_j^n\rho_{j+1,-}^n u_{j+1,-}^n g(S_{j+1}^n)),$$

or

$$\frac{a_j^{n+1}}{a_j^n}\rho_j^{n+1}g(S_j^{n+1}) \leq \frac{1}{2}\rho_{j-1,+}^n(1+\lambda u_{j-1,+}^n)g(S_{j-1}^n) + \frac{1}{2}\rho_{j+1,-}^n(1-\lambda u_{j+1,-}^n)g(S_{j+1}^n). \tag{5.22}$$

We know already by Theorem 5.1 that ρ_j^{n+1} is nonnegative. Taking $g(S) = -(S+S_0)^p, p>1$, where S_0 is some constant such that $S+S_0>0$, we obtain from the last inequality that

$$\begin{aligned}
&\Big(\frac{a_j^{n+1}}{a_j^n}\rho_j^{n+1}\Big)^{1/p}(S_j^{n+1}+S_0)\\
&\geq \Big(\frac{1}{2}\rho_{j-1,+}^n(1+\lambda u_{j-1,+}^n)(S_{j-1}^n+S_0)^p + \frac{1}{2}\rho_{j+1,-}^n(1-\lambda u_{j+1,-}^n)(S_{j+1}^n+S_0)^p\Big)^{1/p}\\
&\geq \Big(\frac{1}{2}\rho_{j-1,+}^n(1+\lambda u_{j-1,+}^n) + \frac{1}{2}\rho_{j+1,-}^n(1-\lambda u_{j+1,-}^n)\Big)^{1/p}\cdot\Big(\min\{S_{j-1}^n, S_{j+1}^n\}+S_0\Big).
\end{aligned} \tag{5.23}$$

Sending $p\to+\infty$ from the inequality (5.23), and eliminating S_0, we obtain (5.17). The proof of Theorem 5.2 is complete. □

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