Spectral Representation of Multiply Self-decomposable Stochastic Processes*

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Abstract. In the present paper we prove an integral representation for multiply self-decomposable processes which is similar to a known representation of Rajput, B.S. and Rosinski J. [3] for Gaussian , stable and infinitely divisible processes .

I.Introduction, notation and prelimilaries.

The classical spectral representation theory for Gaussian processes has been widely studied and applied in

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various branches of sciences such as prediction and filtration problems, signal transmissions, biological models, quantum mechanics, ... It is then a natural idea to develop the theory for wider classes of processes containing stable processes, semi-stable processes and infinitely divisible (i.d.) processes.In the sequel, we study the above problem for multiply self-decomposable (shortly, m.s.d.) real-valued processes only although our method as well as results are true for general separable Banach spacesses.

We say that a stochastic process $X_t, t \geq 0$, is a Lévy process (cf. Sato [4]), if

(i) $X(0) = 0, (P.1)$;

(ii) It is an independent increment process;

(iii) It is temporally homogeneous;

(iv) With P.1 its realizations are CADLAG, i.e. continuous from the right having the left limits.

Let $\mathbb{P}$ denote the class of all p.m.'s on the σ-field $\mathcal{B}$ of Borel subsets of the real line $\mathbb{R}$ equipped with the weak convergence. Given a positive number c and a p.m. $\mu \in \mathbb{P}$ let $T_c\mu$ denote a p.m. given by

$$(T_c\mu)(E) = \mu(\{c^{-1}x : x \in E\}), \tag{1}$$

where $E \in \mathcal{B}$.

Suppose that $X_1, X_2, ...$ is a sequence of real-valued independent r.v.'s and $\{a_n\}, \{b_n\}$ are sequences of real numbers such that $a_n > 0, n = 1, 2, ...$ and the triangular array $X_{n,k} := a_n X_k$ obey the uniformly asymptotically negligible (UAN), that is $X_{n,k} \to 0$ uniformly in k in probability. The class $L = L_1$ of possible shift limit laws μ (distributions) of the sum $S_n = \Sigma_{k=1}^{n} X_{n,k}$ is non-empty and consists of all *self-decomposable p.m.'s on* $\mathbb{R}$ (cf. Loe've [2], p. 323).

Recall that a p.m. $\mu \in \mathbb{P}$ is called *self-decomposable (shortly, s.d.)*, if for every $c \in (0, 1)$ there exists a p.m. μ_c such that

$$\mu = (T_c\mu) * \mu_c, \tag{2}$$

where $*$ denotes the ordinary convolution of p.m.'s.

It is known ([2],p. 323) that if μ is s.d. then μ and μ_c are both i.d. Let L_0 denote the class of all i.d.p.m.'s on $\mathbb{R}$. The classes $L_n, n = 1, 2, ...$ of n-times s.d.p.m.'s were first introduced and studied by Urbanik [7] and then studied further by many other authors (cf., for example, [1,5,6]).They are defined recurssively as follows: A p.m. $\mu \in L_n, n = 2, 3, ...$ if and only if $\mu \in L_1$ and for each $c \in (0, 1)$ the component μ_c in (1.2) belongs to L_{n-1}.

It has been proved by Thu ([6], Proposition 1.1) that a p.m. $\mu \in L_n, n = 1, 2, ...,$ if and only if, for every

$c \in (0,1)$ there exists a p.m. $\nu := \mu_{c,n} \in L_0$ such that the following equality holds:

$$\mu = *_{k=1}^{\infty} (T_{c^k} \nu)^{*r_{k,n}}, \tag{3}$$

where the power is taken in the convolution sense and, for n=1,2, ... ; k=0,1,2,... we put

$$r_{k,n} = \binom{n+k-1}{k}. \tag{4}$$

The formulas (3) and (4) lead to the following interpolation of classes L_n (cf.Thu [6]):

For each $\alpha > 0$ we put

$$\binom{\alpha}{k} = \begin{cases} 1 & k = 0, \\ \alpha(\alpha-1)...(\alpha-k+1)/k! & k = 1, 2, ... \end{cases} \tag{5}$$

and introduce the class α-times s.d. p.m.'s as the following:

1. Definition (cf.Thu [6]). *A p.m.* $\mu \in L_\alpha, \alpha > 0$, *if and only if, for every* $c \in (0,1)$ *there exists a p.m.* $\nu := \mu_{c,\alpha} \in L_0$ *such that the following equality holds:*

$$\mu = *_{k=1}^{\infty} (T_{c^k} \nu)^{*r_{k,\alpha}}, \tag{6}$$

where, the power is taken in the convolution sense and, for $\alpha > 0; k = 0, 1, 2, ...$ *we put*

$$r_{k,\alpha} = \binom{\alpha+k-1}{k}. \tag{7}$$

It should be noted that the infinite convolution (6) is weakly convergent if and only if

$$(8) \qquad \int_{-\infty}^{\infty} log^{\alpha}(1+|x|)\nu(dx) < \infty.$$

The above definition is equivalent to the following two definitions:

1.Definition *(cf.Thu [6]). A p.m.* $\mu \in L_\alpha, \alpha > 0$ *if and only if, for every* $c \in (0,1)$ *there exists a p.m.* $\nu := \mu_{c,\alpha} \in L_0$ *such that the condition (5) is satisfied and* μ *is the distribution of the following random series*

$$(9) \qquad \Sigma_{k=0}^{\infty} Z_k c^k,$$

where $Z_k, k = 0, 2, 3, ...$ *are independent r.v.'s with distributions* $\nu^{*r_{k,\alpha}}$*, respectively.*

1". Definition *(cf.Hong [1]). A p.m.* $\mu \in L_\alpha, \alpha > 0$ *if and only if there exists a Lévery process* $(X(t))$ *such that*

$$(10) \qquad \mu \stackrel{\mathrm{d}}{=} \int_0^{\infty} exp(t^{1/\alpha})X(dt),$$

where the Lévy process (X(t)) must satisfy the condition

$$Elog^{\alpha}(1+|X(1)| < \infty.$$

It should be noted that for any $0 < \alpha < \beta$ *we have* $L_\beta \subset L_\alpha$ *and the intersection* L_∞ *of all classes* $L_\alpha, \alpha >$

0 *is non-void, since it contains Gaussian and stable p.m.'s. The p.s.'s in the class* L_∞ *are called completely s.d., or, mixed stable (cf. Thu [5,6], Urbanik [7]).*

In the sequel we shall need the following representation of i.d. and m.s.d.p.m.'s:

2. Theorem *(cf.Loéve [2])A p.m.* μ *is i.d. if and only if its characteristic function* $\hat{\mu}(t)$ *is of the unique form:*

$$(11) \qquad \begin{cases} -log\hat{\mu}(t) = iat + \sigma^2 t^2 \\ -\int_{-\infty}^{\infty}(e^{itx} - 1 - i\tau(x))M(dx), \end{cases}$$

where a, σ^2 *are real constants; M is a Lévy measure on* $\mathbb{R}$ *characterized by the property that* $M(0) = 0$ *, M is finite ouside of very neighberhood of the origin and*

$$\int_1^1 \frac{x^2}{1+x^2} M(dx) < \infty;$$

the function $\tau(x)$ *is defined by*

$$(12) \qquad \tau(x) = \begin{cases} x & x \in I; \\ 1 & x > 1; \\ -1 & x < -1, \end{cases}$$

I being the closed unit interval [-1,1].

3. Theorem *(cf.Thu [6], Theorem 3.2) A p.m.* $\mu \in L_\alpha, \alpha > 0$ *if and only if its characteristic function* $\hat{\mu}(t)$ *is of the unique form*

$$(13)\qquad \begin{cases} -log\hat{\mu}(t) = iat + \sigma^2 t^2 - \\ \int_{-\infty}^{\infty} v_\alpha(x)(\int_0^\infty k(e^{-u}x, t)u^{\alpha-1}du)m(dx), \end{cases}$$

where m *is a finite measure on* $\mathbb{R}$ *vanishing at the origin;* a, σ^2 *are real constants; the weight function* $v_\alpha(x)$ *and the kernel* $k(y,t)$ *are defined by*

$$(14)\qquad v_\alpha^{-1}(x) = \int_0^\infty \frac{e^{-2t}x^2}{1 + e^{-2t}x^2} t^{\alpha-1} dt$$

and

$$k(y,t) = exp(ity - 1 - i\tau(y)),$$

where the function $\tau(y)$ *is given by the formula (12). Consequently, the Lévy measure* M *of* μ *is of the form* (15)

$$M(A) = \int_{-\infty}^{\infty} v_\alpha(x)(\int_0^\infty \chi_A(e^{-u}x)u^{\alpha-1}du)m(dx),$$

where A *is a Borel subset of the real line separated from 0.*

4. Theorem *(cf.Thu [5]) A p.m.* μ *is mixed-stable (i.e.* $\mu \in L_\infty$*) if and only if its characteristic function* $\hat{\mu}(t)$ *is of the unique form*

$$(16)\qquad \begin{cases} -log\hat{\mu}(t) = iat + \sigma^2 t^2 \\ -\int_{-1}^{1}\int_0^\infty q(x,t)h(x)\nu(du), \end{cases}$$

where the constants a, σ^2 *are the same as in the above theorems and* ν *is a finite measure on the open interval (-1,1),*

$$(17) \quad \begin{cases} q(x,t) = \\ -|tx|^2|x| - itan\pi|x|signxt^{2|x|+1} \qquad 2|x| \neq 1 \\ -|x|t - \frac{2i}{\pi}txlog|tx| \qquad 2|x| = 1. \end{cases}$$

Consequently, the Lévy measure of μ *is given by*

$$(18) \qquad M(A) = \int_{-1}^{1} \int_{0}^{\infty} \chi_A(tx) \frac{dt}{t^2|x|+1} h(x)\nu(dx)$$

2. Definition *Let* T *be a parameter set* Z *of all integers or* R *of all real numbers. A stochastic process* $X_t, t \in T$ *is said to be i.d., stable, mixed-stable,* $\alpha - s.d.$*, if for any* $t_1, t_2, ..., t_n \in T$ *and* $\lambda_1, \lambda_2, ..., \lambda_n, n = 1, 2, ...$ *the r.v.* $\Sigma_1^n \lambda_j X_{t_j}$ *is i.d.,stable,mixed-stable,* $\alpha - s.d$*, respectively.*

3. Definition *Let* $\Lambda = \{\Lambda(A) : A \in \mathcal{S}\}$*be a real stochastic process defined on a probability space* $(\Omega, \mathcal{F}, \mathcal{P})$*, where* $\mathcal{S}$ *stands for a* $\sigma-$*ring of subsets of an arbitrary non-empty set* S *satisfying the following condition :*

There exists an increasing sequence $S_n, n = 1, 2, ...$ *of sets in* S *with* $\bigcap_n S_n = S$.

We call Λ *to be an independently scattered random measure (or r.m. for short), if, for every sequence* $\{A_n\}$ *of disjoint sets in* $\mathcal{S}$*, the random variables* $\Lambda(A_n), n = 1, 2, ...$ *are independent, and , if* $\cup_n A_n$ *belongs to* $\mathcal{S}$*, then we also have*

$$\Lambda(\cup_n A_n) = \Sigma_n \Lambda(A_n) \quad a.s.,$$

where the series is assumed to be convergent a.s. In addition, if for every $A \in \mathcal{S}$ *the distribution of* $\Lambda(A)$ *is i.d., stable, mixed-stable, m.s.d. respectively, then we say that* Λ *is an i.d., stable,mixed stable,* $\alpha - s.d.r.m.$

By virtue of Theorem 2 each random variable $\Lambda(A), A \in S$ *has the characteristic function*

$$(19) \qquad \begin{cases} -logEexp(it\Lambda(A) = it\nu_0(A) + \frac{1}{2}t^2\nu_1(A) \\ -\int_{-\infty}^{\infty}(e^{itx} - 1 - it\tau(x))F_A(dx), \end{cases}$$

where $t \in \mathbb{R}, A \in S$ *and* $-\infty < v_0(A) < \infty, 0 \leqslant v_1(A) < \infty$ *and* F_A *is a Lévy measure on* $\mathbb{R}$*. Moreover,* v_0 *is a signed measure ,* v_1 *a measure and* F_A *a Lévy measure.*

The above representation implies the following

5.Theorem *(cf.Raiput, Rosinski [3], Proposition 2.1) The characteristic function (19) can be rewritten in the unique form:*

$$(20) \qquad Eexp(it\Lambda(A)) = exp(\int_A K(t,s)\lambda(ds)),$$

, where $t \in \mathbb{R}, A \in S$ *and*

$$(21) \qquad \begin{cases} K(t,s) = ita(s) - 1/2t^2\sigma^2(s) \\ + \int_A (e^{itx} - 1 - it\tau(x))\rho(s,dx), \end{cases}$$

with

$$(22) \qquad a(s) = \frac{dv_0}{d\lambda}(s),$$

and

$$(23) \qquad \sigma^2(s) = \frac{dv_1}{d\lambda}(s)$$

and ρ *is given by Lemma 2.3 in [3]. Moreover, we have*

$$|a(s)| + \int_{\mathbb{R}} min\{1, x^2\}\rho(s,dx) = 1 \quad a.e.[\lambda].$$

4.Definition*(cf.Urbanik and Woyczynski [8])* *(a) If f is a simple function on* $S, f = \Sigma_j x_j \chi_{A_j}, A_j \in S$ *then we put, for each* $A \in \sigma(S)$

$$\int_A f d\Lambda = \Sigma_j \lambda(A \cap A_j);$$

(b) A measurable function $f : (S, \sigma(S)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})$ *is said to be* $\Lambda-$*integrable if there exists a sequence* $\{f_n\}$ *of simple functions as defined in (a) such that*

(i) $f_n \to f \quad a.e.[\lambda]$,

(ii) For every $A \in \sigma(S)$, *the sequence* $\{\int_A f_n d\Lambda\}$ *converges in prob., as* $n \to \infty$. *if f is* $\Lambda-$*integrable, then we put*

$$\{\int_A f d\Lambda = P - lim_{n\to\infty} \int_A f_n d\Lambda,$$

where $\{f_n\}$ *satisfies (i) and (ii).*

Now, combining Theorems 3,4,5 we get the following:

6.Theorem *Given* $\alpha > 0$ *let* $\Lambda(A), A \in \mathcal{S}$ *be a* $\alpha-$ *s.d.r.m. Then, the characteristic function of* $\Lambda(A)$ *is of the unique form (20) where*

$$\left\{ \begin{array}{l} K(t,s) = ita(s) - 1/2t^2\sigma^2(s) \\ + \int_A (e^{itx} - 1 - it\tau(x))\rho(s,dx), \end{array} \right. \tag{24}$$

with

$$a(s) = \frac{dv_0}{d\lambda}(s), \tag{25}$$

and

$$\sigma^2(s) = \frac{dv_1}{d\lambda}(s) \tag{26}$$

and ρ *is given by Lemma 2.3 in [3]. Moreover, we have*

$$|a(s)| + \int_{\mathbb{R}} min\{1, x^2\}\rho(s, dx) = 1 \quad a.e.[\lambda].$$

Proof *By virtue of (13) it follows that for any* $A \in S$ *and* $t \in \mathbb{R}$ $\Lambda(A)$ *has the representation*

$$\begin{cases} -logEexp(it\Lambda(A)) = it\nu_0(A) + \frac{1}{2}t^2\nu_1(A) - \\ \int_{-\infty}^{\infty} v_\alpha(x)(\int_0^\infty k(e^{-u}x, t)u^{\alpha-1}du)m(A, dx), \end{cases} \tag{27}$$

which, by a similar argument of Proposition 2.1 in [3], implies that there exists a unique finite measure ν *on* $\sigma(\mathcal{S}) \times \mathcal{B}(\mathcal{R})$ *such that*

$$\nu(A \times B) = m(A, B), \quad for\ any \quad A \in \mathcal{S}, B \in \mathcal{B}(\mathcal{R}).$$

Moreover, for every $A \in \sigma(\mathcal{S})$ *we have* $\nu(A, \{0\}) = 0$.

Now,we are in the position to present the following theorem whose proof is a simple combination of Theorem 6 and the Komogorov extension theorem and Threorem 5.2 in [3].

7.Theorem *Given $0 < \alpha \leqslant \infty$ let $\{X_t : t \in T\}$ be an $\alpha -$ s.d. stochastic process defined on a probability space $(\Omega^{'}, \mathcal{P}^{'})$. Then there exists an $\alpha -$ s.d.r.m., say Λ, defined on the probability space $(\Omega, \mathcal{P})$ such that*

$$\big(\Omega = \Omega^{'} \times [0,1], \mathcal{P} = \mathcal{P}^{'} \times Leb\big),$$

Leb being the Lebesgue measure on [0,1] and

$$\{X_t : t \in T\} = \{\int_{\mathbb{S}} f_t(s) d\Lambda(s) : t \in T\},$$

where $f_t(s) : t \in T, s \in S$ denote some measurable functions on $\mathbb{S}$.

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