Existence Conditions in Symmetric Multivalued Vector Quasiequilibrium Problems[1]

by

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Abstract: We consider symmetric multivalued vector quasiequilibrium problems in topological vector spaces. Sufficient conditions for the solution existence are established under relaxed assumptions, which are shown by examples to be essential, easy to check and more advantageous than recent known results. Some typical applications are given for particular cases as lower and upper bounded quasiequilibrium problems and coincidence point problems.

Keywords: Symmetric Quasiequilibrium Problems, Upper Semicontinuity, 0-Level C-Quasiconvexity Relative to a Set, Lower and Upper Bounded Quasiequilibrium Problems, Coincidence Points

1 Introduction

The equilibrium problem was proposed in Blum and Oettli (1994) and has been intensively studied so far. This problem setting proves to be general and convenient for applying various mathematical tools in investigation. It contains many optimization− related problems such as variational inequalities, complementarity problems, vector optimization, fixed point and coincidence point problems, the Nash equilibrium problem, etc. As usual for various fields of research, the solution existence is one of the most important issues and so is the aim of numerous papers (see e.g., Bianchi et al., 1997; Ansari et al., 2001; Lin and Chen, 2005; Hai and Khanh, in press) and the references therein. To include more practical problems in a unified framework, a number of extended problem settings have been considered: variational inclusion problems (Luc and Tan, 2004; Tan, 2004; Hai and Khanh, 2006 in press), systems of equilibrium or quasiequilibrium problems (Ansari et al., 2000 and 2002; Hai and Khanh, 2006; Lin, 2006), systems of variational inclusion problems (Hai and Khanh, in press). Noor and Oettli (1994) introduced a symmetric quasiequilibrium problem, which proved to be more suitable in modeling several practical situations. In Fu (2003) this result was extended from the scalar case to the vector case in Hausdorff locally convex spaces. Farajzadeh (in press) supplied a further extension to Hausdorff topological vector spaces with several assumptions being relaxed.

Our goal is extending the problem considered in Noor and Oettli (1994); Fu (2003) and Farajzadeh (in press) from the single−valued case to the multivalued case, in Hausdorff topological vector spaces. Since we use mathematical tools other than that employed in Noor and Oettli (1994); Fu (2003) and Farajzadeh (in press), the results here for this more general problems are different from the ones in these references, when applied to their particular cases. However, our several assumptions are more relaxed than the corresponding ones in Noor and Oettli (1994); Fu (2003) and Farajzadeh (in press).

In the sequel, if not otherwise specified, let X and Y be Hausdorff topological vector spaces, Z be a topological vector space. Let K, D and C be nonempty closed convex subsets of X, Y and Z, respectively, with the interior int$C$ being nonempty. Let $S, A : K\times D \to 2^K$, $T, B : K\times D \to 2^D$, $F : K\times D\times K \to 2^Z$ and $G : D \times K \times D \to 2^Z$ be multivalued mappings , with $S(x, y)$ and $T(x, y)$ being nonempty and convex, $\forall(x, y) \in K \times D$. The two symmetric quasiequilibrium problems under our consideration are as follows

(SVQEP$_1$) find $(\bar{x}, \bar{y}) \in K \times D$ such that $\bar{x} \in S(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x}, \bar{y})$ and

$$F(x, \bar{y}, x^*) \cap (Y\setminus -\mathrm{int}C) \neq \emptyset, \forall x \in S(\bar{x}, \bar{y}), \forall x^* \in A(\bar{x}, \bar{y}),$$

$$G(y, \bar{x}, y^*) \cap (Y\setminus -\mathrm{int}C) \neq \emptyset, \forall y \in T(\bar{x}, \bar{y}), \forall y^* \in B(\bar{x}, \bar{y});$$

(SVQEP$_2$) find $(\bar{x},\bar{y}) \in K \times D$ such that $\bar{x} \in S(\bar{x},\bar{y})$, $\bar{y} \in T(\bar{x},\bar{y})$ and

$$F(x,\bar{y},x^*) \subseteq Y \setminus -\text{int}C, \forall x \in S(\bar{x},\bar{y}), \forall x^* \in A(\bar{x},\bar{y}),$$

$$G(y,\bar{x},y^*) \subseteq Y \setminus -\text{int}C, \forall y \in T(\bar{x},\bar{y}), \forall y^* \in B(\bar{x},\bar{y}).$$

If F and G are single$-$valued, C is a closed convex cone with int$C \neq \emptyset$ and $A(x,y) = \{x\}$ and $B(x,y) = \{y\}$, for all $(x,y) \in K \times D$, then our two problems collapse to problem (SVQEP) investigated in Farajzadeh (in press). If in addition X and Y are locally convex and C and D are compact, the two problems become problem (SVQEP) of Fu (2003). If furthermore $Z = R$ and $C = R_+$, these problems coincide with the scalar problem studied in Noor and Oettli (1994).

If $Y \equiv \{y\} \equiv \{0\}, G(y,\bar{x},y^*) \equiv C, B(x,y) \equiv \{y\}$ and $T(x,y) \equiv \{y\}$ then these problems are reduced to multivalued vector quasiequilibrium problems considered by many authors. If, more specifically, $Y \equiv \{y\} \equiv \{0\}$, $A(x,y) = S(x,y), \forall x \in K$, and $F(x,y,x^*) = \big(H(x), x^* - x\big)$ where $H : X \to 2^{L(X,Z)}$ and (h,x) is the value of linear mapping h at x, then the two problems become a multivalued vector quasivariational inequality.

The layout of this paper is as follows. In the remaining part of this section we recall some definitions and preliminaries needed in the sequel. Section 2 is devoted to the main existence results for our problems. Examples are also provided here to see that the imposed assumptions are essential, relaxed and not hard to be checked, and hence the results are more advantageous than that of recent works in many situations. In Section 3, applications of the main results in some typical situations are presented.

Recall first some notions. Let X and Y be topological spaces and $G : X \to 2^Y$ be a multifunction. G is called upper semicontinuous (usc) at x_0 if for each open set $U \supseteq G(x_0)$, there is a neighborhood N of x_0 such that $U \supseteq G(N)$. We say that G satisfies a certain property in a subset $A \subseteq X$ if G satisfies it at every point of A. If $A = X$ we omit "in X" in the statement.

A multifunction H of a subset A of a topological vector space X into X is said to be KKM in A if for each $\{x_1, ..., x_n\} \subseteq A$, one has $\text{co}\{x_1, ..., x_n\} \subseteq \bigcup_{i=1}^n H(x_i)$, where $\text{co}\{.\}$ stands for the convex hull.

The main machinery for proving existence results in this paper is the following well-known KKM-Fan Theorem.

Lemma 1.1 (Ky Fan, 1984) *Assume that X is a topological vector space, $A \subseteq X$ is nonempty and $H : A \to 2^X$ is a KKM mapping with closed values. If there is a subset A_0 contained in a compact convex subset of A such that $\bigcap_{x \in A_0} H(x)$ is compact, then $\bigcap_{x \in A} H(x) \neq \emptyset$.*

The following fixed point theorem is a slightly weaker version (suitable for our use) of Tarafdar's Theorem, which is equivalent to Lemma 1.1.

LEMMA 1.2 (Tarafdar, 1987) *Assume that X is a Hausdorff topological vector space, $A \subseteq X$ is nonempty and convex and $h : A \to 2^A$ is a multifunction with nonempty convex values. Assume that*

(i) *$h^{-1}(y)$ is open in A for each $y \in A$;*

(ii) *there exists a nonempty subset A_0 contained in a compact convex subset of A such that $A \setminus \bigcup_{y \in A_0} h^{-1}(y)$ is compact or empty.*

Then, there exists $\bar{x} \in A$ such that $\bar{x} \in h(\bar{x})$.

2 Main results

The following very relaxed quasiconvexity, will be assumed in our main existence theorems.

DEFINITION 2.1 Let X and Z be vector spaces, let $B \subseteq X$ and $C \subseteq Z$ be nonempty and convex, with $\mathrm{int}C \neq \emptyset$ and let $F : X \times B \to 2^Z$.

(i) F is said to be $0-$level $C-$quasiconvex relative to B of type 1 if for any subset $\{x_1, ..., x_n\} \subseteq X$, any $\{\alpha_1, ..., \alpha_n\} \subseteq R_+$ with $\sum_{i=1}^n \alpha_i = 1$,

$$\left[\exists x_i^* \in B, i = 1, ..., n : F(x_i, x_i^*) \subseteq -\mathrm{int}C\right]$$

$$\Longrightarrow \left[\exists x^* \in B : F\Big(\sum_{i=1}^n \alpha_i x_i, x^*\Big) \subseteq -\mathrm{int}C\right]. \qquad (1)$$

(ii) F is called $0-$level $C-$quasiconvex relative to B of type 2 if (1) is replaced by

$$\Big[\exists x_i^* \in B, i = 1, ..., n : F(x_i, x_i^*) \cap -\mathrm{int}C \neq \emptyset\Big]$$

$$\Longrightarrow \Big[\exists x^* \in B : F\Big(\sum_{i=1}^{n} \alpha_i x_i, x^*\Big) \cap -\mathrm{int}C \neq \emptyset\Big].$$

To see the nature of the above generalized convexity, let us consider the simplest case, where B is a singleton, $Z = R$ and F is single-valued depending only on $x \in X$. Then (i) and (ii) coincide and become: if $F(x_i) < 0, i = 1, \cdots, n$, then $\forall \alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1, F\left(\sum_{i=1}^{n} \alpha_i x_i\right) < 0$. This property is a relaxed 0-level quasiconvexity, since F is called quasiconvex if $F\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \max\limits_{1\leq i\leq n} F(x_i)$.

In the sequel let $E(x, y) = \{(x, y) \in K \times D : x \in S(x, y), y \in T(x, y)\}$.

A sufficient condition for the solution existence of problem (SVQEP$_1$) is

Theorem 2.1 *Assume that*

(i) $\forall (x, y) \in K \times D, \forall (x^*, y^*) \in A(x, y) \times B(x, y), F(x, y, x^*) \nsubseteq -\mathrm{int}C$ *and* $G(y, x, y^*) \nsubseteq -\mathrm{int}C$;

(ii) $\forall (x, y) \in K \times D$, $F(., y, .)$ *and* $G(., x, .)$ *are* $0-$*level* $C-$*quasiconvex relative to* $A(x, y)$ *and* $B(x, y)$, *respectively, of type* 1;

(iii) $\forall (x, y) \in K \times D$, *the set* $\{(\bar{x}, \bar{y}) \in K \times D \mid F(x, \bar{y}, x^*) \nsubseteq -\mathrm{int}C$ *and* $G(y, \bar{x}, y^*) \nsubseteq -\mathrm{int}C, \forall (x^*, y^*) \in A(\bar{x}, \bar{y}) \times B(\bar{x}, \bar{y})\}$ *is closed in* $K \times D$;

(iv) $S(.,.)$ *and* $T(.,.)$ *are usc in* $K \times D$ *and,* $\forall (x,y) \in K \times D$*,* $S^{-1}(x)$ *and* $T^{-1}(y)$ *are open in* $K \times D$*;*

(v) *if* $K \times D$ *is not compact, there exist a nonempty compact subset* $\overline{K} \times \overline{D}$ *of* $K \times D$ *and a subset* $K_0 \times D_0$ *of a compact convex subset of* $K \times D$ *such that,* $\forall (x,y) \in (K \times D) \setminus (\overline{K} \times \overline{D})$*,* $\exists (\bar{x},\bar{y}) \in [K_0 \times D_0] \cap [S(x,y) \times T(x,y)]$*,* $\exists (x^*,y^*) \in A(x,y) \times B(x,y)$*,* $F(\bar{x},y,x^*) \subseteq -\mathrm{int}C$ *or* $G(\bar{y},x,y^*) \subseteq -\mathrm{int}C$*.*

Then problem (SVQEP$_1$) *is solvable.*

Proof. For $(x,y) \in K \times D$, set

$$P(x,y) = \{(\hat{x},\hat{y}) \in K \times D \mid \exists (x^*,y^*) \in A(x,y) \times B(x,y),$$
$$F(\hat{x},y,x^*) \subseteq -\mathrm{int}C \text{ or } G(\hat{y},x,y^*) \subseteq -\mathrm{int}C\},$$

$$\Phi(x,y) = \begin{cases} \big(S(x,y) \times T(x,y)\big) \cap P(x,y) & \text{if} \quad (x,y) \in E, \\ S(x,y) \times T(x,y) & \text{otherwise,} \end{cases}$$

$$Q(x,y) = (K \times D) \setminus \Phi^{-1}(x,y).$$

We claim that $Q(.,.)$ is a KKM mapping in $K \times D$. Indeed, suppose there is a convex combination $(\bar{x},\bar{y}) := \sum_{j=1}^{n} \alpha_j (x_j,y_j)$ in $K \times D$ such that $(\bar{x},\bar{y}) \notin \bigcup_{j=1}^{n} Q(x_j,y_j)$. Then, $(\bar{x},\bar{y}) \notin Q(x_j,y_j)$, i.e., $(\bar{x},\bar{y}) \in \Phi^{-1}(x_j,y_j)$, $\forall j = 1,...,.n$. Thus, $(x_j,y_j) \in \Phi(\bar{x},\bar{y})$, $\forall j = 1,...,n$. If $(\bar{x},\bar{y}) \in E$, one has $(x_j,y_j) \in P(\bar{x},\bar{y})$, $\forall j = 1,...,n$, i.e., $\exists (x_j^*,y_j^*) \in A(\bar{x},\bar{y}) \times B(\bar{x},\bar{y})$ such that $F(x_j,\bar{y},x_j^*) \subseteq -\mathrm{int}C$ or $G(y_j,\bar{x},y_j^*) \subseteq -\mathrm{int}C$. Due to the fact that $F(.,\bar{y},.)$ and $G(.,\bar{x},.)$ are 0 - level

C-quasiconvex relative to $A(\bar{x},\bar{y})$ and $B(\bar{x},\bar{y})$ of type 1, respectively, there is $(\bar{x}^*,\bar{y}^*) \in A(\bar{x},\bar{y}) \times B(\bar{x},\bar{y})$ such that $F(\bar{x},\bar{y},\bar{x}^*) \subseteq -\mathrm{int}C$ or $G(\bar{y},\bar{x},\bar{y}^*) \subseteq -\mathrm{int}C$, contradicting (i). On the other hand, if $(\bar{x},\bar{y}) \in (K \times D) \setminus E$ (i.e., $(\bar{x},\bar{y}) \notin S(\bar{x},\bar{y}) \times T(\bar{x},\bar{y})$), then $(x_j,y_j) \in \Phi(\bar{x},\bar{y}) = S(\bar{x},\bar{y}) \times T(\bar{x},\bar{y})$. So $(\bar{x},\bar{y}) \in S(\bar{x},\bar{y}) \times T(\bar{x},\bar{y})$, another contradiction. Thus, $Q(.,.)$ must be a KKM mapping.

Next we verify the closedness of $Q(x,y)$, $\forall (x,y) \in K \times D$. One has

$$\begin{aligned}
\Phi^{-1}(x,y) &= \{(\bar{x},\bar{y}) \in E \mid (x,y) \in [S(\bar{x},\bar{y}) \times T(\bar{x},\bar{y})] \cap P(\bar{x},\bar{y})\} \\
&\quad \cup \{(\bar{x},\bar{y}) \in (K \times D) \setminus E \mid (x,y) \in S(\bar{x},\bar{y}) \times T(\bar{x},\bar{y})\} \\
&= \{(\bar{x},\bar{y}) \in E \mid (\bar{x},\bar{y}) \in [S^{-1}(x) \cap T^{-1}(y)] \cap P^{-1}(x,y)\} \\
&\quad \cup \{(\bar{x},\bar{y}) \in (K \times D) \setminus E \mid (\bar{x},\bar{y}) \in S^{-1}(x) \cap T^{-1}(y)\} \\
&= \{E \cap S^{-1}(x) \cap T^{-1}(y) \cap P^{-1}(x,y)\} \\
&\quad \cup \{[(K \times D) \setminus E] \cap [S^{-1}(x) \cap T^{-1}(y)]\} \\
&= \{[(K \times D) \setminus E] \cup P^{-1}(x,y)\} \cap S^{-1}(x) \cap T^{-1}(y).
\end{aligned}$$

Therefore,

$$\begin{aligned}
Q(x,y) &= \{K \times D\} \setminus \{[((K \times D) \setminus E) \cup P^{-1}(x,y)] \cap S^{-1}(x) \cap T^{-1}(y)\} \\
&= \{[K \times D] \setminus [((K \times D) \setminus E) \cup P^{-1}(x,y)]\} \\
&\quad \cup \{[K \times D] \setminus [S^{-1}(x) \cap T^{-1}(y)]\} \\
&= \{E \cap [(K \times D) \setminus P^{-1}(x,y)]\} \cup \{[K \times D] \setminus [S^{-1}(x) \cap T^{-1}(y)]\}. \qquad (2)
\end{aligned}$$

Since $S(x,y)\times T(x,y)\neq\emptyset$, $\forall(x,y)\in K\times D$, we have $\bigcup_{(x,y)\in K\times D} S^{-1}(x)\cap T^{-1}(y) = K\times D$. Lemma 1.2 in turn assures that $S(.,.)\times T(.,.)$ has a fixed point in $K\times D$ (hence $E\neq\emptyset$). Indeed, only (ii) of Lemma 1.2 is to be checked. By assumption (v), $(K\times D)\setminus(\overline{K}\times\overline{D})\subseteq\bigcup_{(x,y)\in K_0\times D_0} S^{-1}(x)\cap T^{-1}(y)\subseteq K\times D$ and then, $(K\times D)\setminus\bigcup_{(x,y)\in K_0\times D_0} S^{-1}(x)\cap T^{-1}(y)\subseteq\overline{K}\times\overline{D}$ and is compact, i.e., (ii) of Lemma 1.2 is satisfied. Furthermore, since $S(.,.)$ and $T(.,.)$ are usc and have closed values, E is closed. We also have

$$(K\times D)\setminus P^{-1}(x,y)=\{(\bar{x},\bar{y})\in K\times D\mid\forall(\bar{x}^*,\bar{y}^*)\in A(\bar{x},\bar{y})\times B(\bar{x},\bar{y})$$

$$F(x,\bar{y},\bar{x}^*)\not\subseteq-\mathrm{int}C, G(y,\bar{x},\bar{y}^*)\not\subseteq-\mathrm{int}C\},$$

which is closed by (iii). It follows from (2) that $Q(x,y)$ is closed.

Because of (v), $\forall(x,y)\in(K\times D)\setminus(\overline{K}\times\overline{D})$, $\exists(\bar{x},\bar{y})\in K_0\times D_0$, $(\bar{x},\bar{y})\in\Phi(x,y)$. Therefore, $(K\times D)\setminus(\overline{K}\times\overline{D})\subseteq\bigcup_{(x,y)\in K_0\times D_0}\Phi^{-1}(x,y)\subseteq K\times D$. Hence, $(K\times D)\setminus\bigcup_{(x,y)\in K_0\times D_0}\Phi^{-1}(x,y)\subseteq(\overline{K}\times\overline{D})$, i.e., $\bigcap_{(x,y)\in K_0\times D_0}(K\times D)\setminus\Phi^{-1}(x,y)\subseteq(\overline{K}\times\overline{D})$ and then $\bigcap_{(x,y)\in K_0\times D_0}Q(x,y)$ is compact.

Applying Lemma 1.1 one obtains a point $(\bar{x},\bar{y})$ such that

$$(\bar{x},\bar{y})\in\bigcap_{(x,y)\in K\times D}Q(x,y)=(K\times D)\setminus\bigcup_{(x,y)\in K\times D}\Phi^{-1}(x,y).$$

So, $(\bar{x},\bar{y})\notin\Phi^{-1}(x,y)$, $\forall(x,y)\in K\times D$, i.e., $(x,y)\notin\Phi(\bar{x},\bar{y})$, $\forall(x,y)\in K\times D$. Hence, $\Phi(\bar{x},\bar{y})=\emptyset$. If $(\bar{x},\bar{y})\in(K\times D)\setminus E$, then $\Phi(\bar{x},\bar{y})=S(\bar{x},\bar{y})\times T(\bar{x},\bar{y})$, a contradiction. In the remaining case, $(\bar{x},\bar{y})\in E$, one has $\emptyset=\big(S(\bar{x},\bar{y})\times$

$T(\bar{x},\bar{y})\big) \cap P(\bar{x},\bar{y})$. Thus, for all $(x,y) \in S(\bar{x},\bar{y}) \times T(\bar{x},\bar{y})$, $(x,y) \notin P(\bar{x},\bar{y})$, i.e., $\forall(\bar{x}^*,\bar{y}^*) \in A(\bar{x},\bar{y}) \times B(\bar{x},\bar{y})$ $F(x,\bar{y},\bar{x}^*) \nsubseteq -\mathrm{int}C$, $G(y,\bar{x},\bar{y}^*) \nsubseteq -\mathrm{int}C$, which means that $(\bar{x},\bar{y})$ is a solution. $\square$

The following examples show that either of assumptions of Theorem 2.1 cannot be dropped.

Example 2.1 (Assumption (i) is essential). Let $X = Y = Z = R, K = D = [0,1], C = R_+, S(x,y) \equiv T(x,y) \equiv [0,1], A(x,y) = \{x\}, B(x,y) = \{y\}$, $F(x,\bar{y},x^*) = \{x^* - 2\}$ and $G(y,\bar{x},y^*) = \{y^* - 2\}$.

We check assumptions (ii) - (v). To see (ii), for given $x_i, x_i^* \in A(x,y) = \{x\}$ and $y_i, y_i^* \in B(x,y) = \{y\}$, we simply take $x^* = x_i^*, y^* = y_i^*$. Assumption (iii) is satisfied since the mentioned set is empty. (iv) is clearly fulfilled and (v) is satisfied as K and D are compact. However, problem (SVQEP$_1$) has no solutions, since $\forall(\bar{x},\bar{y}) \in K \times D, \forall(x,x^*) \in S(\bar{x},\bar{y}) \times A(\bar{x},\bar{y}), \forall(y,y^*) \in T(\bar{x},\bar{y}) \times B(\bar{x},\bar{y})$,

$$F(x,\bar{y},x^*) = x^* - 2 < 0,$$

$$G(y,\bar{x},y^*) = y^* - 2 < 0.$$

The reason is that assumption (i) is violated.

Example 2.2 ((ii) is essential). Let $X, Y, Z, C, A(x,y)$ and $B(x,y)$ be as in Example 2.1. Let $K = D = [0,2]$, $S(x,y) = T(x,y) \equiv [0,2]$ and

$$F(x, \bar{y}, x^*) = \begin{cases} \{1\} & \text{if} \quad x^* = x, \\ \{-1\} & \text{if} \quad x^* \neq x, \end{cases}$$

$$G(y, \bar{x}, y^*) = \begin{cases} \{1\} & \text{if} \quad y^* = y, \\ \{-1\} & \text{if} \quad y^* \neq y. \end{cases}$$

Then assumptions (i), (iii) - (v) are clearly satisfied. However, $\forall(\bar{x}, \bar{y}) \in K \times D$, for $x^* \neq x, y^* \neq y$ one has

$$F(x, \bar{y}, x^*) = G(y, \bar{x}, y^*) = -1 < 0,$$

i.e. problem (SVQEP$_1$) is not solvable. To see the reason we check assumption (ii) by picking $x = 1, y = 1, x_1 = x_2 = y_1 = y_2 = \frac{1}{2}, \alpha_1 = \alpha_2 = \frac{1}{2}, x_1^* = x_2^* = 1 \in A(1,y), y_1^* = y_2^* = 1 \in B(y,1)$. Then, $\forall x^* \in A(1,1)$, $\forall y^* \in B(1,1)$ and $i = 1,2$,

$$F(x_i, y, x_i^*) = F(\frac{1}{2}, 1, 1) = -1 < 0,$$

$$G(y_i, x, y_i^*) = G(\frac{1}{2}, 1, 1) = -1 < 0,$$

but

$$F(\alpha_1 x_1 + \alpha_2 x_2, y, x^*) = F(1,1,1) = 1 > 0,$$

$$G(\alpha_1 y_1 + \alpha_2 y_2, x, y^*) = G(1,1,1) = 1 > 0,$$

i.e. assumption (ii) is not satisfied.

Example 2.3 ((iii) is essential). Let X, Y, Z, K, D and C be as in Example 2.1. Let

$$S(x,y) \equiv [0\frac{1}{4}] \cup [\frac{3}{4}, 1],$$

$$T(x,y) = [0,1],$$

$$A(x,y) = \begin{cases} \{x\} & \text{if} \quad x \neq \frac{1}{2}, \\ \{\frac{x}{2}\} & \text{if} \quad x = \frac{1}{2}, \end{cases}$$

$$B(x,y) = \{y\},$$

$$F(x,\bar{y},x^*) = \begin{cases} \{-1\} & \text{if} \quad x + x^* = 1, \\ \{1\} & \text{otherwise}, \end{cases}$$

$$G(y,x,y^*) \equiv \{1\}.$$

Then, assumption (i), (iv) and (v) are easy to check. $G(.,x,.)$ clearly satisfied (ii). Let $(x,y) \in K \times D$ and $\{x_1,...,x_n\} \subseteq K$ be arbitrary. If $x \neq \frac{1}{2}$, then $A(x,y) = \{x\}$ and if $F(x_i,y,x_i^*) = F(x_i,y,x) < 0$ then $x_i + x = 1$ and for $\alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1$, $\sum_{i=1}^n \alpha_i x_i + x = 1$. Hence $F\big(\sum_{i=1}^n \alpha_i x_i, y, x\big) = -1 < 0$. If $x = \frac{1}{2}$, then $A(x,y) = \{\frac{x}{2}\}$. From $F(x_i,y,x_i^*) = F(x_i,y,\frac{x}{2}) < 0$ it follows that $x_i + \frac{x}{2} = 1$ and, for $\alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1$, $\sum_{i=1}^n \alpha_i x_i + \frac{x}{2} = 1$. Therefore, $F\big(\sum_{i=1}^n \alpha_i x_i, y, \frac{x}{2}\big) = -1 < 0$. Thus $F(.,y,.)$ satisfied (ii). However, assumption (iii) is violated, since for $(0,0) \in K \times D$, the set

$$\{(\bar{x},\bar{y}) \in K \times D \mid F(0,\bar{y},x^*) \geq 0, G(0,\bar{x},y^*) \geq 0, \forall (x^*,y^*) \in A(\bar{x},\bar{y}) \times B(\bar{x},\bar{y})\} = [0,1) \times D$$

is not closed in $K \times D$.

We verify that problem (SVQEP_1) is not solvable. Indeed, $\forall (\bar{x},\bar{y}) \in S(\bar{x},\bar{y}) \times T(\bar{x},\bar{y}) = \big([0,\frac{1}{4}] \cup [\frac{3}{4},1]\big) \times [0,1]$, $\exists x \in S(\bar{x},\bar{y}), \exists x^* \in A(\bar{x},\bar{y}) = \{\bar{x}\}$, such that $x + x^* = 1$ and hence $F(x, \bar{y}, x^*) = -1 < 0$.

Example 2.4 ((iv) cannot be dropped). Let X, Y, Z, C, K and D be as in Example 2.2. Let, for $x, \bar{x}, x^* \in K$ and $y, \bar{y}, y^* \in D$,

$$S(x, y) = [0, x + \tfrac{3}{2}] \cap K,$$

$$T(x, y) = B(x, y) \equiv D,$$

$$A(x, y) = [x - 1, x] \cap K,$$

$$F(x, \bar{y}, x^*) = \begin{cases} \{1\} & \text{if} \quad |x - x^*| \leq 1, \\ \{-1\} & \text{if} \quad |x - x^*| > 1, \end{cases}$$

$$G(y, \bar{x}, y^*) \equiv \{1\}.$$

Then, (i) is satisfied since $\forall (x, y) \in K \times D$, $\forall (x^*, y^*) \in A(x, y) \times B(x, y)$, $F(x, y, x^*) = G(y, x, y^*) = \{1\}$. Assumption (ii) is clearly satisfied for $G(., x, .)$. Now assume that, for $\{x_1, ..., x_n\} \subseteq K$ and $x_i^* \in A(x, y)$, $F(x_i, y, x_i^*) < 0$, $i = 1, ..., n$. Then $|x_i - x_i^*| > 1$. For any $\alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1$, choose $x^* = \sum_{i=1}^n \alpha_i x_i^* \in A(x, y)$ we see that $\left| \sum_{i=1}^n \alpha_i x_i - x^* \right| = \sum_{i=1}^n \alpha_i |x_i - x_i^*| > 1$. Hence, $F\left(\sum_{i=1}^n \alpha_i x_i, y, x^* \right) < 0$, i.e. $F(., y, .)$ satisfied (ii). To check (iii) we have, $\forall (x, y) \in K \times D$,

$$U := \{(\bar{x}, \bar{y}) \in K \times D \mid F(x, \bar{y}, x^*) \geq 0, G(y, \bar{x}, y^*) \geq 0, \forall (x^*, y^*) \in A(\bar{x}, \bar{y}) \times B(\bar{x}, \bar{y})\}$$

$$= \{(\bar{x}, \bar{y}) \in K \times D \,\big|\, |x - x^*| \leq 1, \forall (x^*, y^*) \in \big([\bar{x} - 1, \bar{x}] \cap K\big) \times D\}$$

$$= \{(\bar{x}, \bar{y}) \in K \times D \,\big|\, \max\{\bar{x} - 1, 0\} \geq x - 1, \bar{x} \leq x + 1\}.$$

If $x < 1$, then $\max\{\bar{x}-1, 0\} \geq x-1, \forall \bar{x} \in K$. Therefore, $U = \big((-\infty, x+1] \times K\big) \times D$ is closed in $K \times D$. If $x = 1$, then $\max\{\bar{x}-1, 0\} \geq x-1$ and $\bar{x} \leq x+1, \forall \bar{x} \in K$ and hence $U = K \times D$ is closed in $K \times D$. If $x > 1$ then $\bar{x} \leq x+1, \forall \bar{x} \in K$ and $\max\{\bar{x}-1, 0\} \geq x-1$ means that $\bar{x} \geq x$. Therefore $U = \big([x, +\infty) \cap K\big) \times D$ is closed in $K \times D$. Finally, as K and D are compact, assumption (v) is obviously fulfilled. However, problem (SVQEP_1) has no solution, since $\forall (\bar{x}, \bar{y}) \in K \times D$ we can choose $x \in S(\bar{x}, \bar{y}), x^* \in A(\bar{x}, \bar{y})$ such that $F(x, \bar{y}, x^*) < 0$ as follows: if $\bar{x} < 1$, pick $x^* = \bar{x}$ and $x = \min\{\bar{x} + \frac{3}{2}, 2\}$; if $\bar{x} = 1$, pick $x^* = 0$, $x = 2$ and if $1 < \bar{x} \leq 2$, take $x^* = \bar{x}$ and $x = 0$, then in all cases $F(x, \bar{y}, x^*) = -1$. The reason is that assumption (iv) is violated, since (although $S(.,.)$ and $T(.,.)$ are continuous and $T^{-1}(y) = K \times D$ is open in $K \times D$ for $\forall y \in D$)

$$S^{-1}\left(\frac{7}{4}\right) = \left\{(x, y) \in K \times D \mid x + \frac{3}{2} \geq \frac{7}{4}\right\} = \left[\frac{1}{4}, 2\right] \times D_0$$

is not open in $K \times D$.

Example 2.5 ((v) cannot be omitted). Let $X = Y = Z = K = D = R, C = R_+, S(x, y) = T(x, y) \equiv R, A(x, y) = \{x\}, B(x, y) = \{y\}$ and

$$F(x, \bar{y}, x^*) = \{x - x^*\},$$

$$G(y, \bar{x}, y^*) = \{y - y^*\}.$$

Then it is easy to see that assumptions (i)-(iv) are fulfilled. However, problem (SVQEP_1) has no solutions, since $\forall(\bar{x},\bar{y}) \in K \times D, \exists(x,y) \in S(\bar{x},\bar{y}) \times T(\bar{x},\bar{y}), \exists(x^*,y^*) \in A(\bar{x},\bar{y}) \times B(\bar{x},\bar{y}) = \{(\bar{x},\bar{y})\}$,

$$F(x,\bar{y},x^*) = x - x^* < 0,$$

$$G(y,\bar{x},y^*) = y - y^* < 0.$$

To see that assumption (v) is violated let $\overline{K} \times \overline{D} \subseteq K \times D$ and $K_0 \times D_0 \subseteq K \times D$ be compact. Then, there is $(x,y) \in R^2 \setminus \overline{K} \times \overline{D}$ such that $\forall(\bar{x},\bar{y}) \in K_0 \times D_0, \forall(x^*,y^*) \in A(x,y) \times B(x,y) = \{(x,y)\}$,

$$F(\bar{x},y,x^*) = \bar{x} - x^* = \bar{x} - x \geq 0,$$

$$G(\bar{y},x,y^*) = \bar{y} - y^* = \bar{y} - y \geq 0.$$

i.e. (v) is not fulfilled.

Passing to problem (SVQEP_2) we have

THEOREM 2.2 *Assume five conditions corresponding to that of Theorem 2.1: in* (i) *and* (iii) *" $\not\subseteq -\mathrm{int}C$ " is replaced by " $\subseteq Y \setminus -\mathrm{int}C$ "; in* (ii) *" type 1 " is replaced by " type 2 ";* (iv) *remains the same; and in* (v) *" $\subseteq -\mathrm{int}C$ " is replaced by " $\not\subseteq Y \setminus -\mathrm{int}C$ ".*

Then problem (SVQEP_2) *has solutions.*

Proof. We can adopt the same lines of the proof of Theorem 2.1 with a new multifunction $P(x,y)$ defined as

$$P(x,y) = \{(\hat{x},\hat{y}) \in K \times D : \exists (x^*,y^*) \in A(x,y) \times B(x,y),$$

$$F(\hat{x},y,x^*) \not\subseteq Y \setminus -\mathrm{int}C \text{ or } G(\hat{y},x,y^*) \not\subseteq Y \setminus -\mathrm{int}C\}. \qquad \square$$

Remark 2.1 Since our two problems coincide if F and G are single-valued, Examples 2.1 - 2.5 indicate also that each of the five assumptions of Theorem 2.2 is essential. They explain also that in general it is not hard to check the assumptions. The following example shows that our assumptions are very relaxed by proving a case of the problem considered in Fu (2003) and Farajzadeh (in press) but the results there cannot be applied while ours can.

Example 2.6 Let $X, Y, Z, C, K, D, S, T, A$ and B be as in Example 2.1. Let

$$F(x,\bar{y},x^*) = f(x,\bar{y}) - f(x^*,\bar{y}),$$

$$G(y,\bar{x},y^*) = g(\bar{x},y) - g(\bar{x},y^*),$$

where

$$f(x,y) = \begin{cases} 1 & \text{if } x < \frac{1}{2}, \\ -1 & \text{if } x \geq \frac{1}{2}, \end{cases}$$

$$g(x,y) = \begin{cases} 1 & \text{if } y < \frac{1}{2}, \\ -1 & \text{if } y \geq \frac{1}{2}. \end{cases}$$

Then assumptions (i), (iv) and (v) are clearly fulfilled (Theorems 2.1 and 2.2

coincide in this case). To check (ii), we have

$$F(x,\bar{y},x^*) = \begin{cases} 2 & \text{if } x < \frac{1}{2}, x^* \geq \frac{1}{2}; \\ 0 & \text{if } x,^* x < \frac{1}{2} \text{ or } x, x^* \geq \frac{1}{2}; \\ -2 & \text{if } x \geq \frac{1}{2}, x^* < \frac{1}{2}. \end{cases}$$

For $x, y \in [0,1], \{x_1, \cdots, x_n\} \subseteq R$ and $\{x_1^*, \cdots, x_n^*\} \subseteq A(x,y) = \{x\}$, if $F(x_i, y, x_i^*) < 0$, then $x_i \geq \frac{1}{2}$ and $x_i^* = x < \frac{1}{2}$. Hence, for $\alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1$, taking $x^* = x$ we have

$$F\left(\sum_{i=1}^n \alpha_i x_i, y, x^*\right) = -2 < 0$$

as $\sum_{i=1}^n \alpha_i x_i \geq \frac{1}{2}$, the same argument is valid for G. Therefore (ii) is satisfied.

To see (iii) being fulfilled consider any $x, y \in [0,1]$. If $x \geq \frac{1}{2}$, then $F(x,\bar{y},x^*) \geq 0, \forall \bar{y} \in [0,1]$ and for $x^* \geq \frac{1}{2}$. If $x < \frac{1}{2}$, then $F(x,\bar{y},x^*) \geq 0, \forall \bar{y}, x^* \in [0,1]$. The argument for G is similar. Hence, setting

$$U = \{(\bar{x},\bar{y}) \in K \times D \mid F(x,\bar{y},x^*) \geq 0, G(y,\bar{x},y^*) \geq 0, \text{ for } (x^*,y^*) = (\bar{x},\bar{y})\}$$

we see that

$U = [\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$, if $x \geq \frac{1}{2}, y \geq \frac{1}{2}$,

$U = [\frac{1}{2}, 1] \times D$, if $x \geq \frac{1}{2}, y < \frac{1}{2}$,

$U = K \times [\frac{1}{2}, 1]$, if $x < \frac{1}{2}, y \geq \frac{1}{2}$,

$U = K \times D$, if $x < \frac{1}{2}, y < \frac{1}{2}$.

Thus, $\forall (x,y) \in K \times D, U$ is closed in $K \times D$. By Theorem 2.1 (or, the same, Theorem 2.2) problem ($SVQEP_1$) has solutions. However, since

$$f^{-1}\big([0,+\infty)\big) = (-\infty, \frac{1}{2}) \times R,$$

$$g^{-1}\big([0,+\infty)\big) = R \times (-\infty, \frac{1}{2})$$

are not closed in R^2. Hence f and g are not demicontinuous and the results in Fu (2003) and Farajzadeh (in press) cannot be employed. Recall here that a mapping $f : X \to Z$ is said to be demicontinuous if $f^{-1}(M)$ is closed in X for each closed half space M in Z. Checking directly we see that the solution set is $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$.

3 Applications

Since our symmetric quasiequilibrium problems include many rather general problems as particular cases as mentioned in Section 1, Theorem 2.1 and 2.2 imply directly new results for these problems. In this section we present only several typical applications showing clearly advantages of the symmetric structure of the problem setting.

3.1 A lower and upper bounded quasiequilibrium problem

Let X and K be as in Section 1. Let $S : K \to 2^K, f : K \times K \to R, \alpha, \beta \in R$. The lower and upper bounded quasiequilibrium problem consists of

(BQEP) finding $\bar{x} \in K$ such that $\bar{x} \in S(\bar{x}), \forall x \in S(\bar{x})$,

$$\alpha \leq f(\bar{x}, x) \leq \beta.$$

COROLLARY 3.1 *Assume that*

(i) $\forall x \in K, \alpha \leq f(x, x) \leq \beta$;

(ii) $f(.,.)$ $\big($*and* $-f(.,.)\big)$ *is* $\alpha-$*level* ($\beta-$*level, respectively*) R_+-*quasiconvex relative to* K *of type 1*;

(iii) $\forall (x, y) \in K \times K$, $\{(\bar{x}, \bar{y}) \in K \times K \mid f(\bar{x}, x) \geq \alpha, f(\bar{y}, y) \leq \beta\}$ *is closed in* $K \times K$;

(iv) $S(.)$ *is usc in* K *and,* $\forall x \in K, S^{-1}(x)$ *is open in* K;

(v) *if* K *is not compact, there exist a nonempty compact subset* $\overline{K}$ *of* K *and a subset* K_0 *of a compact convex subset of* K *such that* $\forall x \in K \setminus \overline{K}, \exists \bar{x}, \bar{y} \in K_0 \cap S(x)$,

$$f(x, \bar{x}) < \alpha,$$

$$f(x, \bar{y}) > \beta.$$

Then (BQEP) *has solutions.*

Proof. Setting $Y = X, D = K, Z = R, C = R_+, S(x, y) = T(x, y) = S(x), A(x,$

$y) = \{x\}, B(x,y) = \{y\}, F(x,\bar{y},x^*) = f(x^*,x) - \alpha$ and $G(y,\bar{x},y^*) = \beta - f(\bar{x},y)$, problem (BQEP) becomes a particular case of (SVQEP_1) and the corollary is a direct consequence of Theorem 2.1. $\square$

3.2 A coincidence point problem

Let X, Y, K and D be as in Section 1. Let $U : D \to 2^K$ and $V : K \to 2^D$ be multifunctions with nonempty convex images. We consider the following coincidence point problem

(CP) find $(\bar{x},\bar{y}) \in K \times D$ such that $\bar{x} \in U(\bar{y}), \bar{y} \in V(\bar{x})$.

COROLLARY 3.2 *Assume that*

(a) *$U(.)$ and $V(.)$ are usc and, $\forall (x,y) \in K \times D$, $V^{-1}(y)$ and $U^{-1}(x)$ are open in K and D, respectively*;

(b) *K and D are compact.*

Then problem (CP) *has solutions.*

Proof. We set $Z = R, C = R_+, S(x,y) = U(y), T(x,y) = V(x), A(x,y) = \{x\}$, $B(x,y) = \{y\}, F(x,\bar{y},x^*) \equiv G(y,\bar{x},y^*) \equiv \{1\}$. Then (CP) becomes a special case of (SVQEP_1).

To apply Theorem 2.1 we see that assumptions (i)-(iii) are obviously satisfied. Assumption (iv) is fulfilled by (a) and (v) - by (b). Hence Theorem 2.1

yields the solvability of (CP). $\square$

References

Ansari, Q.H., Konnov, I.V. and Yao, J.C. (2001) Existence of a Solution and Variational Principles for Vector Equilibrium Problems. *Journal of Opptimization Theory and Applications*, **110**, 481 - 492.

Ansari, Q.H., Schaible, S. and Yao, J.C. (2000) The System of Vector Equilibrium Problems and Its Applications. *Journal of Optimization Theory and Applications*, **107**, 547 - 557.

Ansari, Q.H., Schaible, S. and Yao, J.C. (2002) The System of Generalized Vector Equilibrium Problems with Applications. *Journal of Global Optimization*, **22**, 3 - 16.

Bianchi,M., Hadjisavvas, N. and Schaible, S. (1997) Vector Equilibrium Problems with Generalized Monotone Bifunctions. *Journal of Optimization Theory and Applications*, **92**, 527 - 542.

Blum, E. and Oettli, W. (1994) From Optimization and Variational Inequalities to Equilibrium Problems. *Mathematics Student*, **63**, 123 - 145.

Fan, K. (1984) Some Properties of Convex Sets Related to Fixed Point Theorems. *Mathematische Annalen*, **266**, 519 - 537.

Farajzadeh, A.P. On the Symmetric Vector Quasiequilibrium Problems. *Journal* *of Mathematical Analysis and Applications*, in press (online: November 4, 2005).

Fu, J.Y. (2003) Symmetric Vector Quasiequilibrium Problems. *Journal of Mathematical Analysis and Applications*, **285**, 708 - 713.

Hai, N.X. and Khanh, P.Q. (2006) Systems of Multivalued Quasiequilibrium Problems. *Advances in Nonlinear Variational Inequalities*, **9**, 97 - 108.

Hai, N.X. and Khanh, P.Q. (2006) The Solution Existence of General Variational Inclusion Problems. *Journal of Mathematical Analysis and Applications*, in press, (online: July 13, 2006).

Hai, N.X. and Khanh, P.Q. Existence of Solutions to General Quasiequilibrium Problems and Applications. *Journal of Optimization Theory and Applications*, in press.

Hai, N.X. and Khanh, P.Q. Systems of Set-Valued Quasivariational Inclusion Problems. *Journal of Optimization Theory and Applications*, in press.

Lin, L.J. (2006) Systems of Generalized Vector Quasiequilibrium Problems with Applications to Fixed Point Theorems for a Family of Nonexpansive Multivalued Mappings. *Journal of Global Optimization*, **34**, 15 - 32.

Lin, L.J. and Chen, H.L. (2005) The Study of KKM Theorems with Applications to Vector Equilibrium Problems and Implicit Vector Variational Inequalities Problems. *Journal of Global Optimization*, **32**, 135 -157.

Luc, D.T. and Tan, N.X. (2004) Existence Conditions in Variational Inclusions with Constraints. *Optimization*, **53**, 505 - 515.

Noor, M.A. and Oettli, W. (1994) On General Nonlinear Complementarity Problems and Quasiequilibria. *Le Matematiche*, **49**, 313 - 331.

Tan, N.X. (2004) On the Existence of Solutions of Quasivariational Inclusion Problem. *Journal of Optimization Theory and Applications*, **123**, 619 - 638.

Tarafdar, E. (1987) A Fixed Point Theorem Equivalent to the Fan-Knaster-Kuratowski-Mazurkiewicz Theorem. *Journal of Mathematical Analysis and Applications*, *128*, 475 - 479.