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Higher-order variational sets and higher-order optimality conditions for proper efficiency in set-valued nonsmooth vector optimization

Abstract Higher-order variational sets are proposed for set-valued mappings, which are shown to be more convenient than generalized derivatives in approximating mappings at a considered point. Both higher-order necessary and sufficient conditions for local Henig-proper efficiency, local strong Henig-proper efficiency and local λ-proper efficiency in set-valued nonsmooth vector optimization are established using these sets. The technique is simple but the results help to unify first and higher-order conditions. As consequences recent existing results are derived. Examples are provided to shown some advantages of our notions and results.

Key words Higher-order variational sets · Higher-order optimality conditions · Set-valued nonsmooth vector optimization · local Henig-proper efficiency · local strong Henig-proper efficiency · local λ-proper efficiency

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1 Introduction

In the last several decades, nonsmooth set-valued vector optimization has been attracted increasing attentions, since it has a wide range of applications

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in real-world problems. One of the central issues is considering optimality conditions. In developing the basic idea of the pioneering optimality condition due to Fermat, which asserts that the derivative of a real function is equal to zero at a local extremum, to tackle the generality of nonsmoothness and set-valuedness many generalized derivatives have been introduced with fruitful applications: the contingent derivative (Aubin 1981), the upper and lower Dini derivative (Penot 1984), the contingent coderivative (Ioffe 1984), the Clarke and adjacent derivatives (Aubin and Frankowska 1990), the S-derivative (Shi 1991), the contingent epiderivatives (Jahn and Rauh 1997; Gong et al. 2003), the radial derivative (Taa 1998), the generalized contingent epiderivative (Chen and Jahn 1998), the radial epiderivative (Flores-Bazán 2001), the K-epiderivative (Bigi and Castellani 2002). For nonsmooth optimization in general and some other generalized derivatives in particular refer also to excellent books (Rockafellar and Wets 1998; Bonnans and Shapiro 2000; Mordukhovich 2006).

For many of the mentioned notions, second-order derivatives have also been proposed and applied in various situations. Furthermore, higher-order derivatives along with higher-order optimality conditions have also developed, although at a lower level. Two kinds of higher-order variational derivatives were introduced in Hoffmann and Kornstaedt (1978) for (single-valued) vector optimization. The upper and lower higher-order Dini directional derivatives were used to establish higher-order optimality conditions in Studniarski (1986). In Studniarski (2001) the higher-order Neustadt derivative was employed to extend the classical Dubovitski-Milyutin scheme (Dubovitski and Milyutin 1965). The higher-order lower Hadamard directional derivative was the tool for studying higher-order optimality conditions for single-valued scalar optimization in Ginchev (2002a) and set-valued vector optimization in Ginchev (2002b). The generalized contingent epiderivative introduced in Chen and Jahn (1998) was extended to the higher-order and used together with the higher-order generalized adjacent epiderivative to establish higher-order optimality conditions for the Henig-proper efficiency in set-valued vector optimization in Li and Chen (2006).

The main idea of using generalized derivatives to establish optimality conditions can be highlighted clearly in the first-order case: the intersection of the cone of directions of decrease of the objective and the cone of feasible directions at the optimal point must be empty and these cones are expressed in terms of generalized derivatives of the mappings involved in the problem. For the higher-order case the sets of such directions are of the corresponding higher-order, not cones. However, all the encountered first and higher-order derivatives are defined in directions. The first aim of this note is to propose notions of higher-order variational sets for approximating the mappings (involved in the optimization problem) at a point under consideration, to replace the role of generalized derivatives. Our variational sets are bigger than the corresponding sets defined by the mentioned generalized derivatives and hence the resulting optimality conditions obtained by separations are stronger than many known ones.

On the other hand, since sets of efficient and weakly efficient solutions of vector optimization are often too large. To eliminate anomalous solutions, a number of notions of proper efficiency have been introduced, beginning with the Kuhn-Tucker one (Kuhn and Tucker 1951): by Hurwicz, Klinger, Geoffrion, Vogel, Wierzbicki, Hartley, Borwein, Benson, Henig, Borwein-Zhuang, Zaffaroni, etc. For systematical treatments and comparisons of these notions the reader is referred to Sawaragi et al. (1985), Khanh (1992), Guerraggio et al. (1994) and Zaffaroni (2003). Recently many of these kinds of properness still attracted attentions: supper efficiency (Gong et al. 2003; Zaffaroni 2003; Rong and Wu 1998; Wantao and Yonghong 2001; Mehra 2002), Benson properness (Sach 2003, 2005), Hartley properness (Lee et al. 2005), Henig properness (Zheng 2000; Gong 2005; Li and Chen 2006; Liu and Gong 2000; Gong et al. 2003), Geoffrion properness (Huang and Yang 2002). Most of these works on properness do not use derivatives. For instance, kinds of (generalized) convexity were used in characterizing various properness in Rong and Wu (1998), Sach (2003, 2005) and Lee et al. (2005), scalarization techniques were main tools to investigate different notions of properness in Khanh (1993), Zheng (2000), Huang and Yang (2002), Mehra (2002) and Zaffaroni (2003).

Our second aim is to apply higher-order variational sets in establishing both higher-order necessary and sufficient conditions for several kinds of proper efficiency. The results help to unify first and higher-order optimality conditions. There are almost no assumptions on the data of the problem. Our results imply the corresponding ones in recent papers. In many cases, the proofs of our results and the deriving known ones together are still shorter than the original proofs of the latter. Examples are provided to show their advantages over the recent known results. Higher-order optimality conditions for efficiency and weak efficiency using higher-order variational sets are the subject of another works of ours.

The organization of the paper is as follows. In the rest of this section the notations which are almost standard, are specified. Section 2 is devoted to higher-order variational sets and comparisons with other approximations for mappings related to generalized derivatives. Higher-order necessary and sufficient conditions for proper efficiency, the main results, are established in Section 3 followed by derivations of and comparisons with recent known results.

Throughout the paper, if not otherwise stated, let X, Y and Z be real normed spaces and let $C \subseteq Y$ and $D \subseteq Z$ be closed convex cones with nonempty interiors. For $H : X \to 2^Y$, the domain, graph and epigraph of H are

$$\mathrm{dom}H = \{x \in X |\ H(x) \neq \emptyset\},\ \mathrm{gr}H = \{(x, y) \in X \times Y |\ y \in H(x)\},$$

$$\mathrm{epi}H = \{(x, y) \in X \times Y |\ y \in H(x) + C\}.$$

If $\mathrm{dom}H = Q$, we also write $H : Q \to 2^Y$ instead of saying $H : X \to 2^Y$ with $\mathrm{dom}H = Q$. For $Q \subseteq X$, int Q, cl Q, bd Q denote its interior, closure and boundary, respectively. Furthermore,

$$\mathrm{cone}\ Q = \{\lambda q |\ \lambda \geq 0, q \in Q\},$$

$$\mathrm{cone}_+ Q = \{\lambda q |\ \lambda > 0, q \in Q\}.$$

B_X stands for the closed unit ball in X and $B_X(u, \delta)$ for the ball centered at $u \in X$ and of radius δ. For $H : X \to 2^Y$, the so-called profile mapping of H is H_+ defined by $H_+(x) = H(x) + C, \forall x \in X$ (then clearly $\mathrm{gr}H_+ = \mathrm{epi}H$). For a cone $C \subseteq Y, C^*$ is the (positive) polar cone:

$$C^* = \{y^* \in Y^* |\ \langle y^*, c\rangle \geq 0, \forall c \in C\}$$

and, for $u \in X$,

$$C(u) = \mathrm{cone}\ (C + u).$$

A nonempty convex subset Q of a convex cone C is called a base of C if $C = \mathrm{cone}\ Q$ and $0 \notin \mathrm{cl}\ Q$. $\mathcal{U}(x_0)$ is used to denote the set of all neighborhoods of $x_0 \in X$.

A subset $Q \subseteq X$ is called star-shaped at x_0 if $\forall x \in Q, \forall \alpha \in [0, 1], (1 - \alpha)x_0 + \alpha x \in Q$. A set-valued mapping $H : X \to 2^Y$ is said to be C-convex-along-rays at $(x_0, y_0) \in \mathrm{gr}H$ on a star-shaped set Q if $\forall x \in Q, \forall \alpha \in [0, 1]$,

$$(1 - \alpha)H(x_0) + \alpha H(x) \subseteq H((1 - \alpha)x_0 + \alpha x) + C.$$

$H : X \to 2^Y$ is called pseudoconvex at $(x_0, y_0) \in \mathrm{gr}H$ if

$$\mathrm{epi}H \subseteq (x_0, y_0) + T_{epiH}(x_0, y_0),$$

where, for a subset $Q \subseteq X$, the contingent cone of Q at $\bar{x} \in X$ is

$$T_Q(\bar{x}) = \{u \in X |\ \exists t_n \to 0^+, \exists u_n \to u, \forall n, \bar{x} + t_n u_n \in Q\}.$$

The interior tangent cone of S at x_0 is (Dubovitski and Milyutin 1965):

$$IT(S, x_0) = \{u \in X |\ \exists \delta > 0, \forall t \in (0, \delta), \forall u' \in B_X(u, \delta), x_0 + tu' \in S\}.$$

The Painlevé-Kuratowski sequential upper limit is defined by

$$\limsup_{x \xrightarrow{H} x_0} H(x) = \{y \in Y |\ \exists x_n \in \mathrm{dom}H : x_n \to x_0, \exists y_n \in H(x_n) : y_n \to y\},$$

where $x \xrightarrow{\mathrm{H}} x_0$ means that $x \in \mathrm{dom}H$ and $x \to x_0$.

2 Higher-order variational sets

Instead of a generalized derivative, to approximate set-valued mapping $F : X \to 2^Y$ at $(x_0, y_0) \in \mathrm{gr}F$ we propose the following two kinds of higher-order variational sets, where $u_1, ..., u_{m-1} \in X$.

Definition 1 *The first, second and higher-order variational sets of type 1 defined as follows*

$$V^1(F, x_0, y_0) = \limsup_{x \xrightarrow{\mathrm{F}} x_0, t \to 0^+} \frac{1}{t}(F(x) - y_0),$$

$$V^2(F, x_0, y_0, u_1) = \limsup_{x \xrightarrow{\mathrm{F}} x_0, t \to 0^+} \frac{1}{t^2}(F(x) - y_0 - tu_1),$$

$$V^m(F, x_0, y_0, u_1, ..., u_{m-1}) = \limsup_{x \xrightarrow{\mathrm{F}} x_0, t \to 0^+} \frac{1}{t^m}(F(x) - y_0 - tu_1 - ... - t^{m-1}u_{m-1}).$$

Definition 2 *The first, second and higher-order variational sets of type 2 are defined as*

$$W^1(F, x_0, y_0) = \limsup_{x \xrightarrow{\mathrm{F}} x_0} \mathrm{cone}_+(F(x) - y_0),$$

$$W^2(F, x_0, y_0, u_1) = \limsup_{x \xrightarrow{\mathrm{F}} x_0, t \to 0^+} \frac{1}{t}(\mathrm{cone}_+(F(x) - y_0) - u_1),$$

$$W^m(F, x_0, y_0, u_1, ..., u_{m-1}) = \limsup_{x \xrightarrow{\mathrm{F}} x_0, t \to 0^+} \frac{1}{t^{m-1}}(\mathrm{cone}_+(F(x) - y_0) - u_1 - ... - t^{m-2}u_{m-1}).$$

Remark 1 $0 \in V^1(F, x_0, y_0)$ and, for all $m \geq 1$, we have

(a) $V^m(F, x_0, y_0, u_1, ..., u_{m-1}) \subseteq W^m(F, x_0, y_0, u_1, ..., u_{m-1})$;

(b) $V^m(F, x_0, y_0, 0, ..., 0) = V^1(F, x_0, y_0)$,
$W^m(F, x_0, y_0, 0, ..., 0) = W^1(F, x_0, y_0)$.

(c) If $u_1 \notin V^1(F, x_0, y_0)$ then $V^2(F, x_0, y_0, u_1) = \emptyset$. If one of the conditions $u_1 \in V^1(F, x_0, y_0), ..., u_{m-1} \in V^{m-1}(F, x_0, y_0, u_1, ..., u_{m-2})$ is violated, then $V^m(F, x_0, y_0, u_1, ..., u_{m-1}) = \emptyset$. The variational sets of type 2 have the same property.

The following example shows that the inclusions in Remark 1(a) may be strict and may also become equalities.

Example 1 (a) Let $X=\mathbb{R}, Y=\mathbb{R}^2, S=X$ and, for $n=1,2,...,$

$$F(x)=\begin{cases}\{(0,0)\} & \text{if } x=0,\\ \{(1,0)\} & \text{if } x=\frac{1}{n},\\ \left\{\left(-\frac{1}{n},\frac{2}{n}\right)\right\} & \text{if } x=\sin\frac{1}{n},\\ \left\{\left(-1+\frac{1}{n},2\right)\right\} & \text{if } x=\tan\frac{1}{n},\\ \emptyset & \text{otherwise.}\end{cases}$$

Then, for $(x_0,y_0)=(0,(0,0))\in \mathrm{gr}F$ and $u_1=(-1,2)\in Y$ one has

$$V^1(F,x_0,y_0)=\{(-x,2x)\in Y|\ x\geq 0\},$$

$$W^1(F,x_0,y_0)=\{(x,0)\in Y|\ x\geq 0\}\cup\{(-x,2x)\in Y|\ x\geq 0\},$$

$$V^2(F,x_0,y_0,u_1)=\{(-x,2x)|\ x\in\mathbb{R}\},$$

$$W^2(F,x_0,y_0,u_1)=\{(x,y)\in Y|\ 2x+y\geq 0\}.$$

(b) Let $S=X=\mathbb{R}, Y=\mathbb{R}^2$ and F is defined by

$$F(x)=\begin{cases}\{(0,0)\} & \text{if } x=0,\\ \{(1,0)\} & \text{if } x=\frac{1}{n}, n=1,2,...,\\ \left\{\left(-\frac{1}{n},\frac{2}{n}\right)\right\} & \text{if } x=\sin\frac{1}{n}, n=1,2,...,\\ \emptyset & \text{otherwise.}\end{cases}$$

For $(x_0,y_0)=(0,(0,0))\in \mathrm{gr}F$ and $u_1=(-1,2)\in Y$ one has

$$V^1(F,x_0,y_0)=\{(-x,2x)\in Y|\ x\geq 0\},$$

$$W^1(F,x_0,y_0)=\{(x,0)\in Y|\ x\geq 0\}\cup\{(-x,2x)\in Y|\ x\geq 0\},$$

$$V^2(F,x_0,y_0,u_1)=W^2(F,x_0,y_0,u_1)=\{(-x,2x)|\ x\in\mathbb{R}\}.$$

Proposition 1 *Let $x_0\in S\subseteq X$ and $y_0\in F(x_0)$. Let one of the following two conditions hold*

(a) *S is star-shaped at x_0 and F is C-convex-along-rays at (x_0,y_0);*

(b) *F is pseudoconvex at (x_0,y_0).*

Then, $\forall x\in S$,

$$F(x)-y_0\subseteq V^1(F_+,x_0,y_0).$$

Proof Let $(x, y) \in \mathrm{gr}F$ be arbitrary and fixed.

(a) Let $t_n \to 0^+$ with $t_n \in (0,1), \forall n$. By the assumed generalized convexity, we have

$$x_n := x_0 + t_n(x - x_0) \in S,$$
$$y_n := y_0 + t_n(y - y_0) \in F_+(x_0 + t_n(x - x_0)).$$

From $x \xrightarrow{\mathrm{F}} x_0$ and $\frac{1}{t_n}(y_n - y_0) = y - y_0$ it follows that $y - y_0 \in V^1(F_+, x_0, y_0)$.

(b) Now assume that F is pseudoconvex at (x_0, y_0). Then

$$(x - x_0, y - y_0) \in T_{epiF}(x_0, y_0),$$

i.e. there exist $t_n \to 0^+$ and $(x_n, y_n) \in \mathrm{epi}F$ such that

$$\frac{1}{t_n}\Big((x_n, y_n) - (x_0, y_0)\Big) \to (x - x_0, y - y_0).$$

Hence, $x_n \to x_0$, $y_n \in F_+(x_n)$ and

$$\frac{1}{t_n}(y_n - y_0) \to y - y_0.$$

Thus, $y - y_0 \in V^1(F_+, x_0, y_0)$. □

To compare our variational sets with other approximations of set-valued mappings defined by generalized derivatives we recall some notions.

Definition 3 (Aubin and Frankowska (1990)) *Assume that* $S \subseteq X$ *and* $u_1, ..., u_{m-1} \in X, m \geq 1$.

(a) *The mth-order contingent set of* S *at* $(x, u_1, ..., u_{m-1})$ *is*

$$T_S^m(x, u_1, ..., u_{m-1}) = \limsup_{t \to 0^+} \frac{1}{t^m}(S - x - tu_1 - ... - t^{m-1}u_{m-1}).$$

(b) *The mth-order adjacent set of* S *at* $(x, u_1, ..., u_{m-1})$ *is*

$$T_S^{bm}(x, u_1, ..., u_{m-1}) = \liminf_{t \to 0^+} \frac{1}{t^m}(S - x - tu_1 - ... - t^{m-1}u_{m-1}).$$

(c) *The mth-order Clarke (or circatangent) set of* S *at* $(x, u_1, ..., u_{m-1})$ *is*

$$C_S^m(x, u_1, ..., u_{m-1}) = \liminf_{t \to 0^+, z \xrightarrow{\mathrm{S}} x} \frac{1}{t^m}(S - z - tu_1 - ... - t^{m-1}u_{m-1}).$$

(d) **(Penot (2000))** *The asymptotic second-order tangent cone of* S *at* (x_0, v) *is*

$$T''(S, x_0, v) = \{w \in X |\ \exists (t_n, r_n) \to (0^+, 0^+) : \frac{t_n}{r_n} \to 0, \exists w_n \to w,$$

$$\forall n \in \mathbb{N}, x_0 + t_n v + \tfrac{1}{2} t_n r_n w_n \in S\}.$$

Definition 4 (Aubin and Frankowska (1990)) *Let* $F : S \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$ *and* $(u_1, v_1), ..., (u_{m-1}, v_{m-1}) \in X \times Y, m \geq 1$.

(a) *The mth-order contingent derivative of* F *at* (x_0, y_0) *with respect to (wrt)* $(u_1, v_1), ..., (u_{m-1}, v_{m-1})$ *is the set-valued mapping* $D^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})$ *whose graph is*

$$\mathrm{gr}D^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}) = T^m_{grF}(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}).$$

(b) *The mth-order adjacent derivative of* F *at* (x_0, y_0) *wrt* $(u_1, v_1), ..., (u_{m-1}, v_{m-1})$ *is the set-valued mapping* $D^{bm} F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})$ *defined by the following graph*

$$\mathrm{gr}D^{bm} F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}) = T^{bm}_{grF}(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}).$$

(c) *The mth-order Clarke (or circatangent) derivative of* F *at* (x_0, y_0) *wrt* $(u_1, v_1), ..., (u_{m-1}, v_{m-1})$ *is the set-valued mapping* $C^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})$ *with the graph*

$$\mathrm{gr}C^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}) = C^m_{grF}(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}).$$

Definition 5 *Let* $F : S \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$ *and* $(u_1, v_1), ..., (u_{m-1}, v_{m-1}) \in X \times Y, m \geq 1$.

(a) *The mth-order contingent epiderivative of* F *at* (x_0, y_0) *wrt* $(u_1, v_1), ..., (u_{m-1}, v_{m-1})$ *is the single-valued mapping* $ED^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})$ *whose epigraph is*

$$\mathrm{epi}ED^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}) = T^m_{epiF}(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}).$$

(b) *The mth-order adjacent epiderivative* $ED^{bm} F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})$ *and mth-order Clarke epiderivative are defined similarly from the corresponding tangent sets.*

The 1st-order contingent epiderivative was introduced in Jahn and Rauh (1997) and the 2nd-order one in Jahn et al. (2005). We define the other and higher-order epiderivatives in a natural way.

Definition 6 (Li and Chen (2006)) *Let* $F : S \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$ *and* $(u_1, v_1), ..., (u_{m-1}, v_{m-1}) \in X \times Y, m \geq 1$.

(a) *The mth-order generalized contingent epiderivative of* F *at* (x_0, y_0) *wrt* $(u_1, v_1), ..., (u_{m-1}, v_{m-1})$ *is the set-valued mapping* $ED^m_g F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})$ *defined by, for* $x \in X$,

$$\begin{aligned} &ED^m_g F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x) \\ &\quad = \mathrm{Min}_C\{y \in Y |\ y \in D^m F_+(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x)\}. \end{aligned}$$

(b) *The mth-order generalized adjacent epiderivative of* F *at* (x_0, y_0) *wrt*

$(u_1, v_1), ..., (u_{m-1}, v_{m-1})$ *is the set-valued mapping* $ED_g^{bm}F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})$ *defined by, for* $x \in X$,

$$ED_g^{bm}F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x)$$
$$= \mathrm{Min}_C\{y \in Y |\ y \in D^{bm}F_+(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x)\}.$$

Of course we can define similarly the mth-order generalized Clarke epiderivative. Here $\mathrm{Min}_C\{.\}$ denotes the set of efficient points of the set $\{.\}$ with respect to the ordering cone C.

The following immediate consequence of the definitions constitutes a base for the coming comparisons showing the generality of our simple results.

Proposition 2 *Let* $F : S \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$, $(u_1, v_1), ..., (u_{m-1}, v_{m-1}) \in X \times Y, m \geq 1$, *and* $x \in X$.

(a) $ED^mF(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x)$
$\subseteq D^mF_+(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x) \subseteq V^m(F_+, x_0, y_0, v_1, ..., v_{m-1})$.

(b) $ED^{bm}F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x)$
$\subseteq D^{bm}F_+(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x) \subseteq V^m(F_+, x_0, y_0, v_1, ..., v_{m-1})$.

(c) $ED_g^mF(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x)$
$\subseteq D^mF_+(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x) \subseteq V^m(F_+, x_0, y_0, v_1, ..., v_{m-1})$.

(d) $ED_g^{bm}F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x)$
$\subseteq D^{bm}F_+(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x) \subseteq V^m(F_+, x_0, y_0, v_1, ..., v_{m-1})$.

The inclusions in Proposition 2 may be strict as indicated by the following examples.

Example 2 Let $X = Y = \mathbb{R}, S = X, C = \mathbb{R}_+, F(x) = \{y \in \mathbb{R} |\ y \geq x^3\}$ $\forall x \in \mathbb{R}, (x_0, y_0) = (0, 0)$ and $(u, v) = (1, 0)$. Then $T^1_{epiF}(x_0, y_0) = T^2_{epiF}(x_0, y_0, u, v) = \mathbb{R} \times \mathbb{R}_+$. Hence, $\forall x \in \mathbb{R}$,

$$D^1F_+(x_0, y_0)(x) = \mathbb{R}_+,$$
$$ED^1F(x_0, y_0)(x) = 0, ED_g^1F(x_0, y_0)(x) = \{0\},$$
$$D^2F_+(x_0, y_0, u, v)(x) = \mathbb{R}_+,$$
$$ED^2F(x_0, y_0, u, v)(x) = 0, ED_g^2F(x_0, y_0, u, v)(x) = \{0\}.$$

On the other hand

$$V^1(F_+, x_0, y_0) = W^1(F_+, x_0, y_0) = \mathbb{R},$$
$$V^2(F_+, x_0, y_0, v) = W^2(F_+, x_0, y_0, v) = \mathbb{R}.$$

Example 3 Let $X, Y, S, C, (x_0, y_0)$ and (u, v) be as in Example 2. Let $F(x) =$

$\{y \in \mathbb{R}|\ y \geq |x|^{\frac{5}{4}}\}, \forall x \in \mathbb{R}$. Then

$$T^2_{epiF}(x_0, y_0, u, v) = \emptyset.$$

Therefore all $D^2F_+(x_0, y_0, u, v)$, $ED^2_gF(x_0, y_0, u, v)$ and $ED^2F(x_0, y_0, u, v)$ do not exist. On the other hand

$$V^2(F_+, x_0, y_0, v) = W^2(F_+, x_0, y_0, v) = \mathbb{R}_+.$$

In general the calculation of the upper limit to evaluate our variational sets is not so difficult. To compute several existing generalized derivatives or the corresponding approximating sets, we need to find sets of efficient points, i.e. to solve a vector optimization problem, which is more difficult.

3 Higher-order optimality conditions for local proper efficiency

In this section we restrict ourselves to dealing with three kinds of properly efficient points, leaving other properness to possible further considerations. For each kind of proper efficiency we establish necessary conditions using both types of our variational sets and also sufficient conditions. These main results are shown to include recent known results by corollaries. The set-valued vector problem under our consideration is

$$\text{(P)} \qquad \min F(x), \text{ s.t. } x \in S,\ G(x) \cap -D \neq \emptyset,$$

where $S \subseteq X, F : S \to 2^Y$ and $G : S \to 2^Z$.

Let $A := \{x \in S|\ G(x) \cap -D \neq \emptyset\}$ and $F(A) := \bigcup_{x \in A} F(x)$.

Definition 7 *Consider problem* (P). *Let $x_0 \in A$ and $y_0 \in F(x_0)$.*

(a) **(Henig (1982))** *A pair (x_0, y_0) is called a local Henig-properly efficient pair if there is $U \in \mathcal{U}(x_0)$ and a pointed convex cone $H \subseteq Y$ with $C \setminus \{0\} \subseteq$ int H such that*

$$(F(A \cap U) - y_0) \cap -H = \{0\}. \tag{1}$$

(b) **(e.g. Khanh (1992))** *Let $\lambda \in C^* \setminus \{0\}$. A pair (x_0, y_0) is said to be a local λ-properly efficient pair if there is $U \in \mathcal{U}(x_0)$ such that, $\forall x \in A \cap U$,*

$$\langle \lambda, F(x) - y_0 \rangle \geq 0. \tag{2}$$

Of course if $U = X$ in (a) or (b), then the word "local" is omitted.

3.1 Henig-proper efficiency

To prove Lemma 1 below, which is needed for the main results, we use the following fact.

Proposition 3 (Jiménez and Novo (2003)) *If* $S \subseteq X$ *is convex,* $x_0 \in \mathrm{cl} S$ *and* $\mathrm{int}\ S \neq \emptyset$, *then*

$$IT(\mathrm{int}\ S, x_0) = \mathrm{int\ cone}(S - x_0).$$

Lemma 1 *If* $K \subseteq X$ *is a closed convex cone with nonempty interior,* $z_0 \in -K$, $z \in -\mathrm{int\ cone}(K + z_0)$ *and* $\frac{1}{t_n}(z_n - z_0) \to z$ *and* $t_n \to 0^+$ *then* $z_n \in -\mathrm{int}\ K$ *for large* n.

Proof By Proposition 3, $-z \in IT(\ \mathrm{int}\ K, -z_0)$. The definition of $IT(\ \mathrm{int}\ K, -z_0)$ implies that $\exists \delta > 0, \forall t \in (0, \delta), \forall u' \in B_X(-z, \delta), -z_0 + tu' \in \mathrm{int}\ K$. Hence, for n large enough,

$$-z_0 + t_n\Big(-\frac{1}{t_n}(z_n - z_0)\Big) \in \mathrm{int}\ K,$$

i.e. $z_n \in -\ \mathrm{int}\ K$. □

Using the two types of variational sets we can establish the following two necessary conditions for the Henig properness.

Theorem 1 *Assume that* (x_0, y_0) *is a local Henig-properly efficient pair (satisfying* (1)) *of problem* (P) *and* $z_0 \in G(x_0) \cap -D$. *Then*

(a) $V^1((F,G)_+, x_0, (y_0, z_0)) \bigcap -\ \mathrm{int}\ ((\mathrm{cl}\ H) \times D(z_0)) = \emptyset$;

(b) *if* $(u_1, v_1) \in V^1((F,G)_+, x_0, (y_0, z_0)) \bigcap -\mathrm{bd}\ ((\mathrm{cl}\ H) \times D(z_0))$, $(u_2, v_2) \in V^2((F,G)_+, x_0, (y_0, z_0), (u_1, v_1)) \bigcap -\mathrm{bd}\ ((\mathrm{cl}\ H)(u_1) \times D(z_0))$,..., $(u_{m-1}, v_{m-1}) \in V^{m-1}((F,G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-2}, v_{m-2})) \bigcap -\mathrm{bd}\ ((\mathrm{cl}\ H)(u_1) \times D(z_0)), m \geq 2$, *then*

$$V^m((F,G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-1}, v_{m-1})) \bigcap -\mathrm{int}\ ((\mathrm{cl}\ H)(u_1) \times D(z_0)) = \emptyset. \tag{3}$$

Proof By Remark 1(b), with $(u_2, v_2) = ... = (u_{m-1}, v_{m-1}) = (0, 0)$, assertion (b) becomes (a). Hence, it suffices to demonstrate (b). Suppose, with $(u_i, v_i), i = 1, ..., m-1$, as given in (b), there exists (y, z) in the intersection in (3). By Definition 1, there are $x_n \xrightarrow{(F,\,G)} x_0$, $t_n \to 0^+$ and $(y_n, z_n) \in (F,G)(x_n) + C \times D$ such that

$$\frac{1}{t_n}\Big((y_n, z_n) - (y_0, z_0) - t_n(u_1, v_1) - ... - t_n^{m-1}(u_{m-1}, v_{m-1})\Big) \to (y, z),$$

where $y \in -\mathrm{int}\ [(\mathrm{cl} H)(u_1)]$ and $z \in -\mathrm{int}\ D(z_0)$. For $i = 2, ..., m-1$, $u_i \in -\mathrm{cone}\ [(\mathrm{cl} H) + u_1]$. Hence there are $\alpha_i \geq 0$ and $h_i \in \mathrm{cl} H$ such that $u_i = -\alpha_i(h_i + u_1)$. Therefore,

$$\frac{1}{t_n^m}(y_n - y_0 - t_n u_1 - ... - t_n^{m-1} u_{m-1}) = \frac{1}{t_n^m}\Big(y_n - y_0 - t_n u_1 + \textstyle\sum_{i=2}^{m-1} \alpha_i t_n^i (h_i + u_1)\Big)$$

$$= \left(\frac{y_n - y_0 + \sum_{i=2}^{m-1} \alpha_i t_n^i h_i}{t_n(1 - \sum_{i=2}^{m-1} \alpha_i t_n^{i-1})} - u_1\right) \frac{1 - \sum_{i=2}^{m-1} \alpha_i t_n^{i-1}}{t_n^{m-1}} \to y.$$

By virtue of Lemma 1, for n large enough we have

$$y_n - y_0 + \textstyle\sum_{i=2}^{m-1} \alpha_i t_n^i h_i \in -\text{int } (\text{cl}H).$$

Then

$$y_n - y_0 \in -\text{int cl}(H) = -\text{ int } H. \tag{4}$$

Similarly, for $i = 1, ..., m-1$, as $v_i \in -\text{cone } (D + z_0)$ there are $\beta_i \geq 0$ and $d_i \in D$ with $v_i = -\beta_i(d_i + z_0)$. Consequently,

$$\frac{1}{t_n^m}(z_n - z_0 - t_n v_1 - ... - t_n^{m-1} v_{m-1})$$

$$= \left(\frac{z_n + \sum_{i=1}^{m-1} \beta_i t_n^i d_i}{1 - \sum_{i=1}^{m-1} \beta_i t_n^i} - z_0\right) \frac{1 - \sum_{i=1}^{m-1} \beta_i t_n^i}{t_n^m} \to z.$$

Again Lemma 1 implies that

$$z_n \in -\text{int } D. \tag{5}$$

On the other hand, there exist $(\bar{y}_n, \bar{z}_n) \in (F, G)(x_n)$ and $(\bar{c}_n, \bar{d}_n) \in C \times D$ such that

$$(y_n, z_n) = (\bar{y}_n, \bar{z}_n) + (\bar{c}_n, \bar{d}_n).$$

Hence, (4) and (5) together imply, for sufficiently large n, that

$$\bar{y}_n + \bar{c}_n - y_0 \in -\text{int } H, \; \bar{z}_n + \bar{d}_n \in -\text{int } D.$$

Therefore,

$$\bar{y}_n - y_0 \in -\text{int } H, \; \bar{z}_n \in -\text{int } D,$$

contradicting the fact that (x_0, y_0) is a local Henig-properly efficient pair. □

By a similar proof we have

Theorem 2 *Assume the same as for Theorem 1. Then*

(a) $W^1((F,G)_+, x_0, (y_0, z_0)) \bigcap -\text{ int } ((\text{cl } H) \times D) = \emptyset$;

(b) *if* $(u_1, v_1) \in W^1((F,G)_+, x_0, (y_0, z_0)) \bigcap -\text{bd } ((\text{cl } H) \times D)$, $(u_2, v_2) \in W^2((F,G)_+, x_0, (y_0, z_0), (u_1, v_1)) \bigcap -\text{bd } ((\text{cl } H)(u_1) \times D(v_1))$,..., (u_{m-1}, v_{m-1})

$\in W^{m-1}((F,G)_+, x_0, (y_0,z_0), (u_1,v_1), ..., (u_{m-2}, v_{m-2})) \bigcap -\text{bd } ((\text{cl } H)(u_1) \times D(v_1)), m \geq 2$, *then*

$$W^m((F,G)_+, x_0, (y_0,z_0), (u_1,v_1), ..., (u_{m-1}, v_{m-1})) \\ \bigcap -\text{int } ((\text{cl } H)(u_1) \times D(v_1)) = \emptyset.$$

With relaxed convexity assumptions we establish the following sufficient condition

Theorem 3 *For problem* (P) *assume that* $x_0 \in A, y_0 \in F(x_0), z_0 \in G(x_0) \cap -D$ *and* $S \subseteq \text{dom}F \cap \text{dom}G$. *Assume either* S *is star-shaped at* x_0, F *is* C*-convex-along-rays at* (x_0, y_0) *and* G *is* D*-convex-along-rays at* (x_0, z_0) *or* (F, G) *is pseudoconvex at* $(x_0, (y_0, z_0))$. *Assume further that* H *is a pointed convex cone in* Y *with* $C \setminus \{0\} \subseteq \text{int } H$. *Then* (x_0, y_0) *is a Henig-properly efficient pair if one of the following conditions is satisfied*

(a) $V^1((F,G)_+, x_0, (y_0,z_0)) \bigcap -(H \times D(z_0)) = \{(0,0)\}$;

(b) *if* $(u_1,v_1) \in V^1((F,G)_+, x_0, (y_0,z_0)) \bigcap -\text{bd } (H \times D(z_0))$, $(u_2,v_2) \in V^2((F,G)_+, x_0, (y_0,z_0), (u_1,v_1)) \bigcap -\text{bd } (H(u_1) \times D(z_0))$,..., $(u_{m-1}, v_{m-1}) \in V^{m-1}((F,G)_+, x_0, (y_0,z_0), (u_1,v_1), ..., (u_{m-2}, v_{m-2})) \bigcap -\text{bd } (H(u_1) \times D(z_0))$, $m \geq 2$, *then*

$$V^m((F,G)_+, x_0, (y_0,z_0), (u_1,v_1), ..., (u_{m-1}, v_{m-1})) \\ \bigcap -(H(u_1) \times D(z_0)) \subseteq \{(0,0)\}.$$

Proof Similarly as above we need to prove only (a). By Proposition 1 we have, $\forall x \in S$,

$$(F,G)(x) - (y_0,z_0) \subseteq V^1((F,G)_+, x_0, (y_0,z_0)).$$

Then

$$((F,G)(x)-(y_0,z_0)) \bigcap -(H \times D(z_0)) = \emptyset. \tag{6}$$

Suppose to the contrary that $x \in A$ and $y \in F(x)$ exist such that $y - y_0 \in -H \setminus \{0\}$. Since $x \in A$, there is $z \in G(x) \cap -D$. Then $z - z_0 \in -D - z_0 \subseteq -D(z_0)$. Thus,

$$(y,z) - (y_0,z_0) \in -(H \times D(z_0)) \setminus \{(0,0)\},$$

contradicting (6). □

Remark 2 Using Proposition 2, we can easily derive Theorem 8 of Liu and Gong (2000) and Theorem 2.1 of Jahn and Khan (2002) from Theorem 1(a). Theorem 8 of Liu and Gong (2000) can be deduced also from Theorem 2(a).

Theorem 1 and 2 are more effective than Theorems 8 of Liu and Gong (2000) and 2.1 of Jahn and Khan (2002) in the following example.

Example 4 Let $X = Y = \mathbb{R}, C = \mathbb{R}_+, (x_0, y_0) = (0,0)$ and $F(x) = \{-x^4\}$. Then, $\forall x \in \mathbb{R}$,

$$T^1_{epiF}(x_0, y_0) = \mathbb{R} \times \mathbb{R}_+,$$

$$ED^1F(x_0, y_0)(x) = 0.$$

If H is a pointed convex cone with $C \setminus \{0\} \subseteq \text{int } H$, then $H = \mathbb{R}_+$ and, $\forall x \in \mathbb{R}$,

$$ED^1F(x_0, y_0)(x) \cap -\text{int } H = \emptyset,$$

i.e. the necessary condition stated in Theorems 8 of Liu and Gong (2000) and 2.1 of Jahn and Khan (2002) are fulfilled and these theorems cannot be employed. However, since $V^1(F_+, x_0, y_0) = \mathbb{R}$ intersects $-\text{int } H$, our Theorem 1 says that (x_0, y_0) is not a local Henig-properly efficient pair.

If C has a base B, then there is a slightly stronger notion than the Henig-proper efficiency.

Definition 8 (Borwein and Zhuang (1993)) *Let C have a base B and $\delta = \inf\{\|b\|| \ b \in B\} > 0$. A pair (x_0, y_0) with $x_0 \in A$ and $y_0 \in F(x_0)$ is said to be a local strong Henig-properly efficient pair of problem* (P) *if there are $U \in \mathcal{U}(x_0)$ and $\epsilon \in (0, \delta)$ such that*

$$(F(A \cap U) - y_0) \cap -\text{int } C_\epsilon(B) = \emptyset, \tag{7}$$

where $C_\epsilon(B) = \text{cone } (B + \epsilon B_X)$ with B_X being the unit ball in X.

It is known (Borwein and Zhuang (1993)) that cl (int $C_\epsilon(B)$) is a pointed convex cone and $C \setminus \{0\} \subseteq \text{int } C_\epsilon(B)$. Then taking $H = \text{cl } C_{\frac{\epsilon}{2}}(B)$ we have from (7)

$$(F(A \cap U) - y_0) \cap -H = \{0\},$$

i.e. the above notion is indeed stronger than the local Henig-proper efficiency. If B is compact, then the two notions coincide.

Theorems 1 and 2 are evidently still valid for the local strong Henig-proper efficiency. Moreover, by proofs similar to that of Theorems 1 and 2 we have the following modifications.

Theorem 4 *Assume that C has a base and (x_0, y_0) is a local strong Henig-properly efficient pair of problem* (P) *satisfying* (7). *Then, $\forall z_0 \in G(x_0) \cap -D$,*

(a) $V^1((F,G)_+, x_0, (y_0, z_0)) \bigcap -\text{ int } (C_\epsilon(B) \times D(z_0)) = \emptyset$;

(b) *if* $(u_1, v_1) \in V^1((F,G)_+, x_0, (y_0, z_0)) \bigcap -\text{bd } (C_\epsilon(B) \times D(z_0))$,..., $(u_{m-1}, v_{m-1}) \in V^{m-1}((F,G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-2}, v_{m-2})) \bigcap -\text{bd } (C_\epsilon(B) \times$

$D(z_0)), m \geq 2$, *then*

$$V^m((F,G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-1}, v_{m-1})) \bigcap -\text{int } (C_\epsilon(B) \times D(z_0)) = \emptyset.$$

Theorem 5 *With the assumptions of Theorem 4 we have*

(a) $W^1((F,G)_+, x_0, (y_0, z_0)) \bigcap -\text{ int } (C_\epsilon(B) \times D) = \emptyset$;

(b) *if* $(u_1, v_1) \in W^1((F,G)_+, x_0, (y_0, z_0)) \bigcap -\text{bd } (C_\epsilon(B) \times D)$, $(u_2, v_2) \in W^2((F,G)_+, x_0, (y_0, z_0), (u_1, v_1)) \bigcap -\text{bd } (C_\epsilon(B) \times D(v_1))$,..., $(u_{m-1}, v_{m-1}) \in W^{m-1}((F,G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-2}, v_{m-2})) \bigcap -\text{bd } (C_\epsilon(B) \times D(v_1))$, $m \geq 2$, *then*

$$W^m((F,G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-1}, v_{m-1})) \bigcap -\text{int } (C_\epsilon(B) \times D(v_1)) = \emptyset.$$

Proving similarly as for Theorem 3 we get the following sufficient condition.

Theorem 6 *Let* x_0, y_0, z_0 *and the generalized convexity condition be as in Theorem 3. Assume that* C *has a base* $B, \delta = \inf\{\|b\|| \ b \in B\}$ *and* $\epsilon \in (0, \delta)$. *Then* (x_0, y_0) *is a strong Henig-properly efficient pair if one of the following condition holds*

(a) $V^1((F,G)_+, x_0, (y_0, z_0)) \bigcap -\ ((\text{int } C_\epsilon(B)) \times D(z_0)) = \emptyset$;

(b) *if* $(u_1, v_1) \in V^1((F,G)_+, x_0, (y_0, z_0)) \bigcap -\text{bd } (C_\epsilon(B) \times D(z_0))$,..., $(u_{m-1}, v_{m-1}) \in V^{m-1}((F,G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-2}, v_{m-2})) \bigcap -\text{bd } (C_\epsilon(B) \times D(z_0)), m \geq 2$, *then*

$$V^m((F,G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-1}, v_{m-1})) \bigcap -((\text{int } C_\epsilon(B)) \times D(z_0)) = \emptyset.$$

Remark 3 We can prove the modifications of Theorems 4-6, where in all formulae with $(u_k, v_k), k = 2, ..., m-1$, involved, $C_\epsilon(B)$ is replaced by (cl $C_\epsilon(B))(u_1)$, similarly as for the other theorems of Section 3. Then the conclusions are stronger since $C_\epsilon(B) \subseteq (\text{cl } C_\epsilon(B))(u_1)$. The present forms of Theorems 4-6 are more convenient to derive the following Theorem 4.1 of Li and Chen (2006).

Corollary 1 (Li and Chen (2006)) *Suppose that* C *has a base* B *with* $\delta = \inf\{\|b\|| \ b \in B\}, (x_0, y_0) \in \text{gr}F, z_0 \in G(x_0) \cap -D$ *and* $(u_i, v_i - y_0, w_i) \in X \times (-C) \times (-D), i = 1, ..., m-1$. *If* (x_0, y_0) *is a strong Henig-properly efficient pair satisfying* (7). *Then*

$$\Big(ED_g^{bm}(F,G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, ..., u_{m-1} - x_0, v_{m-1} - y_0,$$

$$w_{m-1}-z_0)(x)+C\times D+(0,z_0)\Big)\bigcap -\text{int}\,(C_\epsilon(B)\times D)=\emptyset, \qquad (8)$$

for all $x\in\Omega:=\text{dom}ED_g^{bm}(F,G)(x_0,y_0,z_0,u_1-x_0,v_1-y_0,w_1-z_0,...,u_{m-1}-x_0,v_{m-1}-y_0,w_{m-1}-z_0)$.

Proof By the definition of $ED_g^{bm}, x\in\Omega$ and

$$(y,z)\in ED_g^{bm}(F,G)(x_0,y_0,z_0,u_1-x_0,v_1-y_0,w_1-z_0,\\ ...,u_{m-1}-x_0,v_{m-1}-y_0,w_{m-1}-z_0)(x) \qquad (9)$$

mean that

$$(x,y,z)\in T^{bm}_{epi(F,G)}(x_0,y_0,z_0,u_1-x_0,v_1-y_0,w_1-z_0,\\ ...,u_{m-1}-x_0,v_{m-1}-y_0,w_{m-1}-z_0).$$

Therefore,

$$(u_1-x_0,v_1-y_0,w_1-z_0)\in T^{b1}_{epi(F,G)}(x_0,y_0,z_0),\\ ...\\ (u_{m-1}-x_0,v_{m-1}-y_0,w_{m-1}-z_0)\in T^{b(m-1)}_{epi(F,G)}(x_0,y_0,z_0,\\ w_1-z_0,...,u_{m-2}-x_0,v_{m-2}-y_0,w_{m-2}-z_0).$$

By Proposition 2,

$$(v_1-y_0,w_1-z_0)\in V^1((F,G)_+,x_0,(y_0,z_0)),\\ ...\\ (v_{m-1}-y_0,w_{m-1}-z_0)\in V^{m-1}((F,G)_+,x_0,(y_0,z_0),\\ (v_1-y_0,w_1-z_0),...,(v_{m-2}-y_0,w_{m-2}-z_0)).$$

On the other hand, by the assumptions of the corollary,

$$(v_1-y_0,w_1-z_0)\in -C\times(D+z_0)\subseteq -\text{ cl }(C_\epsilon(B)\times D(z_0)).$$

Theorem 4(a) then implies that

$$(v_1-y_0,w_1-z_0)\in -\text{bd }(C_\epsilon(B)\times D(z_0)).$$

Similarly, by Theorem 4, for $k=3,...,m$,

$$(v_{k-1}-y_0,w_{k-1}-z_0)\in -\text{bd }(C_\epsilon(B)\times D(z_0)).$$

Hence, it follows from Proposition 2, (9) and Theorem 4(b) that

$$(y,z)\notin -\text{int }(C_\epsilon(B)\times D),$$

i.e. one gets (8). □

Remark 4 (i) Similarly as for Corollary 1 (i.e. Theorem 4.1 of Li and Chen

(2006)), Theorem 4 implies also Theorem 4.2 of Li and Chen (2006) and Theorem 1 of Liu and Gong (2000).

(ii) By Proposition 2, from Theorem 4 we derive also Theorems 4.1 and 4.2 of Li and Chen (2006) with ED_g^{bm} replaced by ED_g^m, which are new results.

The following example explains that Theorem 4 is more advantageous than Theorems 4.1 and 4.2 of Li and Chen (2006) and Theorem 1 of Liu and Gong (2000) in some cases.

Example 5 Let $X = Y = Z = \mathbb{R}, S = X, C = \mathbb{R}_+, D = \mathbb{R}, (x_0, y_0) = (0, 0), G(x) = \{0\}, \forall x \in \mathbb{R}$ and

$$F(x) = \{y \in \mathbb{R} |\ y \geq -x^{m+1}\},$$

where m is a positive integer. Choose $B = \{1\}$ as a base of C then $\delta = 1$ and $C_\epsilon(B) = \mathbb{R}_+, \forall \epsilon \in (0, \delta)$. Let $z_0 = 0, (u_i, v_i, w_i) \in X \times (-C) \times (-D), i = 1, ..., m-1$. Then,

$$\mathrm{epi}(F, G) = \{(x, y, z) \in \mathbb{R}^3 |\ y \geq -x^{m+1}\},$$

$$T^m_{epi(F,G)}(x_0, y_0, z_0, u_1, v_1, w_1, ..., u_{m-1}, v_{m-1}, w_{m-1}) \subseteq \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R},$$

$$D^m(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, ..., u_{m-1}, v_{m-1}, w_{m-1})(x) \subseteq \mathbb{R}_+ \times \mathbb{R}, \forall x \in \mathbb{R}.$$

Then, $\forall x \in \mathbb{R}, \forall \epsilon \in (0, 1)$,

$$D^m(F, G)_+(x_0, y_0, z_0, u_1, v_1, w_1, ..., u_{m-1}, v_{m-1}, w_{m-1})(x) \bigcap -\mathrm{int}\ (C_\epsilon(B) \times D) = \emptyset.$$

Consequently, the necessary optimality conditions stated in three theorems of Li and Chen (2006) and Liu and Gong (2000) cannot be applied. However, our Theorem 4 rejects (x_0, y_0) from candidates for local Henig-proper efficiency, since

$$V^1((F, G)_+, x_0, (y_0, z_0)) = \mathbb{R}^2$$

intersects $-\mathrm{int}\ (C_\epsilon(B) \times D)$ for all $\epsilon \in (0, 1)$.

3.2 λ-proper efficiency

Theorem 7 *Assume that (x_0, y_0) is a local λ-properly efficient pair of problem* (P) *and $z_0 \in G(x_0) \cap -D$. Then*

(a) $(\lambda, I)\Big(V^1((F, G)_+, x_0, (y_0, z_0))\Big) \bigcap -\mathrm{int}\ (\mathbb{R}_+ \times D(z_0)) = \emptyset$;

(b) *if* $(u_1, v_1) \in V^1((F, G)_+, x_0, (y_0, z_0)) \bigcap -(C \times D(z_0))$, $(u_2, v_2) \in V^2((F, G)_+, x_0, (y_0, z_0), (u_1, v_1)) \bigcap -(C(u_1) \times D(z_0))$,..., $(u_{m-1}, v_{m-1}) \in V^{m-1}((F, G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-2}, v_{m-2})) \bigcap -(C(u_1) \times D(z_0)), m \geq$

2, *then*

$$(\lambda, I)\Big(V^m((F,G)_+, x_0, (y_0,z_0), (u_1,v_1), ..., (u_{m-1}, v_{m-1}))\Big)$$
$$\bigcap -\text{int } (\mathbb{R}_+ \times D(z_0)) = \emptyset.$$

Proof It suffices to prove (b). Suppose to the contrary that we have (2) but with $(u_1,v_1), ..., (u_{m-1}, v_{m-1})$ as in (b) we have

$$(\lambda, I)(y,z) \in (\lambda, I)\Big(V^m((F,G)_+, x_0, (y_0,z_0), (u_1,v_1), ..., (u_{m-1}, v_{m-1}))\Big)$$
$$\bigcap -\text{int } (\mathbb{R}_+ \times D(z_0)).$$

By Definition 1, there are $x_n \xrightarrow{(\mathrm{F,\,G})} x_0$, $t_n \to 0^+$, $(y_n, z_n) \in (F,G)(x_n) + C \times D$ such that

$$\frac{1}{t_n^m}(y_n - y_0 - t_n u_1 - ... - t_n^{m-1} u_{m-1}) \to y \text{ with } \lambda(y) < 0, \quad (10)$$

$$\frac{1}{t_n^m}(z_n - z_0 - t_n v_1 - ... - t_n^{m-1} v_{m-1}) \to z \in -\text{int } D(z_0). \quad (11)$$

$\forall i = 2, ..., m-1$, since $u_i \in -\text{cone } (C + u_1)$, $u_i = -\alpha_i(c_i + u_1)$ for some $\alpha_i \geq 0$ and $c_i \in C$. Hence (10) implies that

$$\frac{1}{t_n^m}\Big(y_n - y_0 + \textstyle\sum_{i=2}^{m-1} \alpha_i t_n^i c_i - t_n(1 - \sum_{i=2}^{m-1} \alpha_i t_n^{i-1}) u_1\Big)$$
$$:= \frac{1}{t_n^m}(y_n - y_0 + c_0) \to y.$$

For large n, since $c_0 \in C$ and $\lambda \in C^*$, we have

$$\frac{1}{t_n^m}\lambda(y_n - y_0 + c_0) \to \lambda(y) < 0.$$

Hence

$$\lambda(y_n - y_0) < 0.$$

Similarly, for large n, from (11) we have $z_n \in -\text{int } D$.

On the other hand, there are $(\bar{y}_n, \bar{z}_n) \in (F,G)(x_n)$ and $(\bar{c}_n, \bar{d}_n) \in C \times D$ such that

$$(y_n, z_n) = (\bar{y}_n, \bar{z}_n) + (\bar{c}_n, \bar{d}_n).$$

Therefore, for large n, $\lambda(\bar{y}_n - y_0) < 0$ and $\bar{z}_n \in -\text{int } D$, a contradiction with the assumed local efficiency. □

By a similar proof we get

Theorem 8 *Assume the same as for Theorem 7. Then*

(a) $(\lambda, I)\Big(W^1((F,G)_+, x_0, (y_0, z_0))\Big)\bigcap -\text{int }(\mathbb{R}_+ \times D) = \emptyset$;

(b) *if* $(u_1, v_1) \in W^1((F,G)_+, x_0, (y_0, z_0))\bigcap -(C \times D)$, $(u_2, v_2) \in W^2((F,G)_+, x_0, (y_0, z_0), (u_1, v_1))\bigcap -(C(u_1) \times D(v_1))$,..., $(u_{m-1}, v_{m-1}) \in W^{m-1}((F,G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-2}, v_{m-2}))\bigcap -(C(u_1)\times D(v_1)), m \geq 2$, *then*

$$(\lambda, I)\Big(W^m((F,G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-1}, v_{m-1}))\Big)$$
$$\bigcap -\text{int }(\mathbb{R}_+ \times D(v_1)) = \emptyset.$$

For a sufficient condition we have

Theorem 9 *Let the generalized convexity assumption of Theorem 3 be satisfied. Then* (x_0, y_0) *is a* λ*-properly efficient pair if one of the following conditions is fulfilled*

(a) $(\lambda, I)\Big(V^1((F,G)_+, x_0, (y_0, z_0))\Big)\bigcap -\text{int }(\mathbb{R}_+ \times D(z_0)) = \emptyset$;

(b) *if* $(u_1, v_1) \in W^1((F,G)_+, x_0, (y_0, z_0))\bigcap -(C \times D(z_0))$, $(u_2, v_2) \in W^2((F,G)_+, x_0, (y_0, z_0), (u_1, v_1))\bigcap -(C(u_1) \times D(z_0))$,..., $(u_{m-1}, v_{m-1}) \in W^{m-1}((F,G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-2}, v_{m-2}))\bigcap -(C(u_1)\times D(z_0)), m \geq 2$, *then*

$$(\lambda, I)\Big(W^m((F,G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-1}, v_{m-1}))\Big)$$
$$\bigcap -\text{int }(\mathbb{R}_+ \times D(z_0)) = \emptyset.$$

Proof It is similar to that of Theorem 3. □

The following corollary is an immediate consequence of Theorem 7 (or Theorem 8) and Proposition 2.

Corollary 2 (Liu and Gong (2000)) *If* (x_0, y_0) *is a* λ*-properly efficient pair of* (P), $ED^1F(x_0, y_0)$ *exists and* $S - x_0 \subseteq \text{dom}ED^1F(x_0, y_0)$, *then*, $\forall x \in S$,

$$\langle \lambda, ED^1F(x_0, y_0)(x - x_0)\rangle \geq 0.$$

The example below gives a case where our Theorems 8 and 9 can be used but Corollary 2 (i.e. Theorem 3 of Liu and Gong (2000)) cannot.

Example 6 Let $X = Y = \mathbb{R}, S = \mathbb{R}, C = \mathbb{R}_+, (x_0, y_0) = (0, 0), F(x) = \{y \in \mathbb{R}| y \geq -x^2\}$ and $\lambda = 1$. Then

$$T^1_{epiF}(x_0, y_0) = \mathbb{R} \times \mathbb{R}_+,$$

$$ED^1F(x_0, y_0)(x) = 0, \forall x \in \mathbb{R}.$$

Therefore $\langle \lambda, ED^1F(x_0, y_0)(\mathbb{R})\rangle \geq 0$ and Corollary 2 cannot be used. On the other hand, $V^1(F_+, x_0, y_0) = \mathbb{R}$ and (x_0, y_0) is not a local λ-properly efficient pair, according to Theorem 8 (or Theorem 9).

3.3 Lagrange multiplier forms

We can formulate dual forms of the above optimality conditions by using Lagrange multipliers depending on the points of variational sets as follows. The core of the (simple) proof is the clear observation: $y \notin -\,\mathrm{int}C$ is equivalent to the existence of a Lagrange multiplier $c^* \in C^*$ such that $\langle c^*, y\rangle \geq 0$ and $y \notin -\,\mathrm{cl}C$ is equivalent to $\langle c^*, y\rangle > 0$. Since the proof of the Lagrange multiplier forms of our optimality conditions are straightforward and the formulations of the theorems are similar, we state, without proof, only the theorem corresponding to Theorem 1.

Theorem 10 *Assume that* (x_0, y_0) *is a local Henig-properly efficient pair of problem* (P) *and* $z_0 \in G(x_0) \cap -D$. *Then*

(a) $\forall (y, z) \in V^1((F, G)_+, x_0, (y_0, z_0))$, $\exists (h^*, d^*) \in H^* \times D^* \setminus \{(0, 0)\}$ *such that* $\langle d^*, z_0\rangle = 0$ *and* $\langle h^*, y\rangle + \langle d^*, z\rangle \geq 0$;

(b) *if* $(u_1, v_1) \in V^1((F, G)_+, x_0, (y_0, z_0)) \bigcap -\mathrm{bd}\ ((\mathrm{cl}\ H)\times D(z_0))$, $(u_2, v_2) \in V^2((F, G)_+, x_0, (y_0, z_0), (u_1, v_1)) \bigcap -\mathrm{bd}\ ((\mathrm{cl}\ H)(u_1) \times D(z_0))$,..., $(u_{m-1}, v_{m-1}) \in V^{m-1}((F, G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-2}, v_{m-2})) \bigcap -\mathrm{bd}\ ((\mathrm{cl}\ H)(u_1) \times D(z_0)), m \geq 2$, *then* $\forall (y, z) \in V^m((F, G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-1}, v_{m-1}))$, $\exists (h^*, d^*) \in H^* \times D^* \setminus \{(0, 0)\}$ *such that* $\langle h^*, u_1\rangle = \langle d^*, z_0\rangle = 0$ *and* $\langle h^*, y\rangle + \langle d^*, z\rangle \geq 0$.

From the Lagrange multiplier forms we see that the gaps between our necessary and sufficient conditions are rather "minimal": strict positiveness replaces nonnegativeness. In the primal forms, correspondingly, the gaps are only the boundary of $-C$.

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