VARIATIONAL SETS OF MULTIVALUED MAPPINGS AND A UNIFIED STUDY OF OPTIMALITY CONDITIONS

PHAN QUOC KHANH† AND NGUYEN DINH TUAN‡

Abstract. We propose two kinds of variational sets of any order for multivalued mappings and show that they are advantageous over many known generalized derivatives in the use for establishing optimality conditions. Applying these sets we prove both necessary and sufficient optimality conditions of any order for efficiency and weak efficiency in a unified way. Many corollaries and examples are provided to show that our results include many recent existing ones. The imposed assumptions are very relaxed and the proofs are rather short in comparison with that of recent results in the literature.

Key words. variational sets, variational analysis, nonsmooth analysis, multivalued mappings, vector optimization, efficiency, weak efficiency, optimality conditions

AMS subject classifications. 49J52, 90C29, 90C30, 90C46

1. Introduction. First-order derivatives (of various types, classical or generalized) of mappings are other mappings with linearity nature, used in approximating the given mappings to simplify a problem for studying it. To have better approximations higher-order derivatives are applied. For optimization-related problems this technique is commonly and effectively employed. In particular, to establish optimality conditions we can see this approach used from the classical Fermat theorem to recent results in nonsmooth optimization involving generalized derivatives of multivalued mappings. For generalized derivatives and their applications in variational analysis see excellent books [2, 15, 18] and long papers [8, 16]. Examining existing optimality conditions we can observe that the key argument is included in a separation of suitable sets. To explain the idea let us take the well-known scheme of Dubovitskii-Milyutin [7] for first-order optimality conditions in single-valued scalar optimization problems: the intersection of the cone of the decrease directions of the multiobjective function and the cone of the feasible directions defined by the constraints must be empty at a local minimizer. Here the cone of the decrease directions is defined by a kind of derivatives. For other theories of optimality conditions, specially of higher-order conditions, in more complicated problems we may have separations of sets, not cones. An important point of a necessary optimality condition of this type is that

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This work was supported in part by the National Basic Research Program in Natural Sciences of Vietnam.

the larger the sets are, the stronger the result and the smaller gap with the corresponding sufficient condition are. This is a motivation for us to propose variational sets replacing derivatives so that they are bigger than the sets defined by the known derivatives and can be used in the mentioned separation. The second point is that the proposed variational sets should be not more difficult to be computed than most of the known derivatives. The third point is that the assumptions imposed for optimality conditions to be established should be as relaxed as possible. Computing our variational sets consists of only limits of mutifunctions in the Painlevé-Kuratowski sense. For our necessary optimality conditions no assumptions are imposed and for the sufficient ones only relaxed convexity is needed. It appears that both first-order and higher-order optimality conditions can be shortly proved in a unified study. Due to these advantages our considerations include many existing results as special cases and in many circumstances the proofs of our main results together with the deriving consequences are still shorter than the original proofs.

The layout of the paper is as follows. In the rest of this section we recall some definitions. We propose two kinds of variational sets of any order in Section 2. Both necessary and sufficient conditions for optimality of any order are established in the next Section 3. The final Section 4 contains many recent existing results as consequences of the theorems in Section 3 and also discussions about comparisons.

In the sequel, if not otherwise stated, let X, Y and Z be real normed spaces and let $C \subseteq Y$ and $D \subseteq Z$ be closed convex cones with nonempty interiors. For $H : X \to 2^Y$, the *domain*, *graph* and *epigraph of* H are defined as

$$\mathrm{dom}H = \{x \in X |\ H(x) \neq \emptyset\},\ \mathrm{gr}H = \{(x, y) \in X \times Y |\ y \in H(x)\},$$
$$\mathrm{epi}H = \{(x, y) \in X \times Y |\ y \in H(x) + C\}.$$

If $\mathrm{dom}H = Q$, we also write $H : Q \to 2^Y$ instead of saying $H : X \to 2^Y$ with $\mathrm{dom}H = Q$.

The only kind of limit of mutifunctions we use is the following *Painlevé-Kuratowski sequential upper limit*

$$\limsup_{x \xrightarrow{\mathrm{H}} x_0} H(x) = \{y \in Y |\ \exists x_n \in \mathrm{dom}H : x_n \to x_0, \exists y_n \in H(x_n) : y_n \to y\},$$

where $x \xrightarrow{\mathrm{H}} x_0$ means that $x \in \mathrm{dom}H$ and $x \to x_0$.

The convexity assumptions for our sufficient optimality conditions will be the following relaxed properties. A subset $Q \subseteq X$ is called *star-shaped at* x_0 if $\forall x \in Q, \forall \alpha \in [0, 1], (1-\alpha)x_0 + \alpha x \in Q$. A set-valued mapping $H : X \to 2^Y$ is said to be C*-convex-along-rays at* $(x_0, y_0) \in \mathrm{gr}H$ on a star-shaped set Q if $\forall x \in Q, \forall \alpha \in [0, 1]$,

$$(1-\alpha)H(x_0)+\alpha H(x)\subseteq H((1-\alpha)x_0+\alpha x)+C.$$

$H : X \to 2^Y$ is called *pseudoconvex at* $(x_0, y_0) \in \mathrm{gr}H$ if

$$\mathrm{epi}H \subseteq (x_0,y_0)+T_{epiH}(x_0,y_0),$$

where, for a subset $Q \subseteq X$, the *contingent cone of* Q *at* $\bar{x} \in X$ is

$$T_Q(\bar{x}) = \{u \in X|\ \exists t_n \to 0^+, \exists u_n \to u, \forall n, \bar{x}+t_n u_n \in Q\}.$$

Our notations are almost standard. For $Q \subseteq X$, int Q, cl Q, bd Q denote its interior, closure and boundary, respectively. Furthermore,

$$\text{cone } Q = \{\lambda q|\ \lambda \geq 0, q \in Q\},$$

$$\text{cone}_+ Q = \{\lambda q|\ \lambda > 0, q \in Q\}.$$

B_X stands for the closed unit ball in X and $B_X(u,\delta)$ for the ball centered at $u \in X$ and of radius δ. For $H : X \to 2^Y$, the so-called *profile mapping of* H is H_+ defined by $H_+(x) = H(x) + C, \forall x \in X$ (then clearly $\mathrm{gr}H_+ = \mathrm{epi}H$). For a cone $C \subseteq Y, C^*$ is the *(positive) polar cone*:

$$C^* = \{y^* \in Y^*|\ \langle y^*, c\rangle \geq 0, \forall c \in C\}$$

and, for $u \in X$,

$$C(u) = \text{cone } (C+u).$$

A nonempty convex subset Q of a convex cone C is called a *base of* C if $C =$ cone Q and $0 \notin$ cl Q. $\mathcal{U}(x_0)$ is used to denote the *set of all neighborhoods of* $x_0 \in X$.

2. Variational sets. To approximate multivalued mapping $F : X \to 2^Y$ at $(x_0, y_0) \in \mathrm{gr}F$ we define the following two types of *higher-order variational sets*, where $v_1, ..., v_{m-1} \in Y$.

DEFINITION 2.1. *The first, second and higher-order variational sets of type 1 are the following:*

$$V^1(F,x_0,y_0) = \limsup_{x \xrightarrow{F} x_0, t\to 0^+} \frac{1}{t}(F(x)-y_0),$$

$$V^2(F,x_0,y_0,v_1) = \limsup_{x \xrightarrow{F} x_0, t\to 0^+} \frac{1}{t^2}(F(x)-y_0-tv_1),$$

$$V^m(F,x_0,y_0,v_1,...,v_{m-1}) = \limsup_{x \xrightarrow{F} x_0, t\to 0^+} \frac{1}{t^m}(F(x)-y_0-tv_1-...-t^{m-1}v_{m-1}).$$

DEFINITION 2.2. *The first, second and higher-order variational sets of* *type 2 are the following:*

$$W^1(F, x_0, y_0) = \limsup_{x \xrightarrow{F} x_0} \text{cone}_+(F(x) - y_0),$$

$$W^2(F, x_0, y_0, v_1) = \limsup_{x \xrightarrow{F} x_0, t \to 0^+} \frac{1}{t}(\text{cone}_+(F(x) - y_0) - v_1),$$

$$W^m(F, x_0, y_0, v_1, ..., v_{m-1})$$
$$= \limsup_{x \xrightarrow{F} x_0, t \to 0^+} \frac{1}{t^{m-1}}(\text{cone}_+(F(x) - y_0) - v_1 - ... - t^{m-2}v_{m-1}).$$

Remark 2.1. To see the nature of these definitions let us compare variational sets of first and second-orders with commonly used sets related to well-known derivatives. Similar and general comparisons for higher-order variational sets will be given in Proposition 4.6.

Recall that the (first-order) contingent derivative of $F : X \to 2^Y$ at $(x_0, y_0) \in \text{gr}F$ is a multivalued mapping $DF(x_0, y_0)$ defined by

$$DF(x_0, y_0)(u) = \limsup_{u' \to u, t \to 0^+} \frac{1}{t}(F(x_0 + tu') - y_0).$$

The second-order contingent derivative of F at (x_0, y_0) with respect to (wrt) $(u_1, v_1) \in X \times Y$ is a multivalued mapping $D^2F(x_0, y_0, u_1, v_1)$ defined by

$$D^2F(x_0, y_0, u_1, v_1)(u) = \limsup_{u' \to u, t \to 0^+} \frac{1}{t^2}(F(x_0 + tu_1 + t^2u') - y_0 - tv_1).$$

Then, clearly we have

$$DF(x_0, y_0)X \subseteq V^1(F, x_0, y_0),$$
$$D^2F(x_0, y_0, u_1, v_1)X \subseteq V^2(F, x_0, y_0, v_1).$$

Moreover, it is obvious that

$$\text{cone}(F(x_0) - y_0) \subseteq W^1(F, x_0, y_0),$$
$$\text{cone}(\text{cone}(F(x_0) - y_0) - v_1) \subseteq W^2(F, x_0, y_0, v_1).$$

Note that $DF(x_0, y_0)X$, $D^2F(x_0, y_0, u_1, v_1)X$, $\text{cone}(F(x_0) - y_0)$ and $\text{cone}(\text{cone}(F(x_0) - y_0) - v_1)$ are among the biggest sets related to derivatives and to approximating cones for subsets. So the above comparisons (together with Proposition 4.6 below) show that the variational sets in Definitions 2.1 and 2.2 are really very big as required and expected (see Section 1).

Remark 2.2. For all $m \geq 1$ we have

(i) $V^m(F, x_0, y_0, v_1, ..., v_{m-1}) \subseteq W^m(F, x_0, y_0, v_1, ..., v_{m-1})$;

(ii) $V^m(F, x_0, y_0, 0, ..., 0) = V^1(F, x_0, y_0)$,

$W^m(F, x_0, y_0, 0, ..., 0) = W^1(F, x_0, y_0)$.

(iii) If $v_1 \notin V^1(F, x_0, y_0)$ then $V^2(F, x_0, y_0, v_1) = \emptyset$. If one of the conditions $v_1 \in V^1(F, x_0, y_0), ..., v_{m-1} \in V^{m-1}(F, x_0, y_0, v_1, ..., v_{m-2})$ is violated, then $V^m(F, x_0, y_0, v_1, ..., v_{m-1}) = \emptyset$. The variational sets of type 2 have the same property.

The inclusion in Remark 2.2(i) may be both a strict inclusion and an equality as shown by the following example.

EXAMPLE 2.1. *(i) Let $X = \mathbb{R}, Y = \mathbb{R}^2$ and, for $n = 1, 2, ...$,*

$$F(x) = \begin{cases} \{(0,0)\} & \text{if } x = 0, \\ \{(-n, n)\} & \text{if } x = \dfrac{1}{n}, \\ \left\{\left(\dfrac{1}{n}, 0\right)\right\} & \text{if } x = \ln\left(1 + \dfrac{1}{n}\right), \\ \left\{\left(1, \dfrac{1}{n^2}\right)\right\} & \text{if } x = \sin\dfrac{1}{n}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then, for $(x_0, y_0) = (0, (0, 0)) \in \mathrm{gr}F$ and $v_1 = (1, 0) \in Y$ one has

$$V^1(F, x_0, y_0) = \{(y^1, 0) \in Y |\ y^1 \geq 0\},$$

$$W^1(F, x_0, y_0) = \{(y^1, 0) \in Y |\ y^1 \geq 0\} \cup \{(-y^1, y^1) \in Y |\ y^1 \geq 0\},$$

$$V^2(F, x_0, y_0, v_1) = \{(y^1, 0) \in Y |\ y^1 \in \mathbb{R}\},$$

$$W^2(F, x_0, y_0, v_1) = \{(y^1, y^2) \in Y |\ y^2 \geq 0\}.$$

(ii) Let $X = \mathbb{R}, Y = \mathbb{R}^2$ and F is defined by

$$F(x) = \begin{cases} \{(0,0)\} & \text{if } x = 0, \\ \{(-n, n)\} & \text{if } x = \dfrac{1}{n}, n = 1, 2, ..., \\ \left\{\left(\dfrac{1}{n}, 0\right)\right\} & \text{if } x = \ln\left(1 + \dfrac{1}{n}\right), n = 1, 2, ..., \\ \emptyset & \text{otherwise.} \end{cases}$$

For $(x_0, y_0) = (0, (0, 0)) \in \mathrm{gr}F$ and $v_1 = (1, 0) \in Y$ one has

$$V^1(F, x_0, y_0) = \{(y^1, 0) \in Y |\ y^1 \geq 0\},$$

$$W^1(F, x_0, y_0) = \{(y^1, 0) \in Y |\ y^1 \geq 0\} \cup \{(-y^1, y^1) \in Y |\ y^1 \geq 0\},$$

$$V^2(F, x_0, y_0, v_1) = W^2(F, x_0, y_0, v_1) = \{(y^1, 0) \in Y |\ y^1 \in \mathbb{R}\}.$$

PROPOSITION 2.3. *Assume that $x_0 \in S \subseteq X$ and $y_0 \in F(x_0)$. Assume further one of the following two conditions*

(a) S is star-shaped at x_0 and F is C-convex-along-rays at (x_0, y_0) on S;

(b) F is pseudoconvex at (x_0, y_0).

Then, $\forall x \in S$,

$$F(x) - y_0 \subseteq V^1(F_+, x_0, y_0).$$

Proof. Let $(x, y) \in \mathrm{gr}F$ is arbitrary and fixed.

(a) Choose $t_n \to 0^+$ with $t_n \in (0, 1), \forall n$. Then by the assumed generalized convexity, one has for all n,

$$x_n := x_0 + t_n(x - x_0) \in S,$$

$$y_n := y_0 + t_n(y - y_0) \in F_+(x_0 + t_n(x - x_0)).$$

Since $x \xrightarrow{\mathrm{F}} x_0$ and $\frac{1}{t_n}(y_n - y_0) = y - y_0$ one gets $y - y_0 \in V^1(F_+, x_0, y_0)$.

(b) Now assume that F is pseudoconvex at (x_0, y_0). Then

$$(x - x_0, y - y_0) \in T_{epiF}(x_0, y_0).$$

By definition of the contingent cone, there exist $t_n \to 0^+$ and $(x_n, y_n) \in \mathrm{epi}F$ such that

$$\frac{1}{t_n}\Big((x_n, y_n) - (x_0, y_0)\Big) \to (x - x_0, y - y_0).$$

Consequently, $x_n \to x_0$ and $y_n \in F_+(x_n)$ and

$$\frac{1}{t_n}(y_n - y_0) \to y - y_0.$$

Thus, $y - y_0 \in V^1(F_+, x_0, y_0)$. $\square$

3. Main results. For $F : S \to 2^Y$ and $G : S \to 2^Z$ consider the multivalued vector optimization

$$\text{(P)} \qquad \min F(x), \text{ s.t. } x \in S,\ G(x) \cap -D \neq \emptyset.$$

Let $A := \{x \in S |\ G(x) \cap -D \neq \emptyset\}$ and $F(A) := \bigcup_{x \in A} F(x)$. Recall that, for $x_0 \in A$ and $y_0 \in F(x_0)$, (x_0, y_0) is said to be a *local weakly efficient pair (local efficient pair) of problem* (P) if there exists $U \in \mathcal{U}(x_0)$ such that

$$(F(U \cap A) - y_0) \cap -\mathrm{int}\, C = \emptyset$$
$$\Big((F(U \cap A) - y_0) \cap -C \subseteq (-C) \cap C\Big).$$

If $U = X$ the word "local" is omitted from the terminology. Instead of mentioning problem (P) in the above definition we also say that y_0 is a *weakly efficient point (efficient point*, respectively) *of the set* $F(U \cap A) \subseteq Y$. In general, for a subset $T \subseteq Y$, by $\text{WMin}_C T$ ($\text{Min}_C T$, respectively) we denote the set of all weakly efficient points (efficient points, respectively) of T.

The *interior tangent cone of* S *at* x_0 defined in [7] as

$$IT(S, x_0) = \{u \in X |\ \exists \delta > 0, \forall t \in (0, \delta), \forall u' \in B_X(u, \delta), x_0 + tu' \in S\}.$$

PROPOSITION 3.1 [12]. *If* $S \subseteq X$ *is convex,* $x_0 \in \text{cl}S$ *and* $\text{int } S \neq \emptyset$, *then*

$$IT(\text{int } S, x_0) = \text{int cone } (S - x_0).$$

LEMMA 3.2. *If* $K \subseteq X$ *is a closed convex cone with nonempty interior,* $z_0 \in -K$, $z \in -\text{int cone } (K + z_0)$ *and* $\frac{1}{t_n}(z_n - z_0) \to z$ *as* $t_n \to 0^+$ *then* $z_n \in -\text{int } K$ *for large* n.

Proof. Since $-z \in IT(\text{ int } K, -z_0)$, by the definition of $IT(\text{ int } K, -z_0)$ we have $\exists \delta > 0, \forall t \in (0, \delta), \forall u' \in B_X(-z, \delta), -z_0 + tu' \in \text{int } K$. Hence, for n large enough,

$$-z_0 + t_n\Big(-\frac{1}{t_n}(z_n - z_0)\Big) \in \text{int } K,$$

i.e. $z_n \in -\text{ int } K$. □

THEOREM 3.3. *Let* (x_0, y_0) *be a local weakly efficient pair of* (P) *and* $z_0 \in G(x_0) \cap -D$. *Then*

(i) $V^1((F, G)_+, x_0, (y_0, z_0)) \bigcap -\text{int } (C \times D(z_0)) = \emptyset$;

(ii) *if* $(u_1, v_1) \in V^1((F, G)_+, x_0, (y_0, z_0)) \bigcap -\text{bd } (C \times D(z_0))$ *then*

$$V^2((F, G)_+, x_0, (y_0, z_0), (u_1, v_1)) \bigcap -\text{int } (C(u_1) \times D(z_0)) = \emptyset;$$

(iii) *if* $(u_1, v_1) \in V^1((F, G)_+, x_0, (y_0, z_0)) \bigcap -\text{bd } (C \times D(z_0))$, $(u_2, v_2) \in V^2((F, G)_+, x_0, (y_0, z_0), (u_1, v_1)) \bigcap -\text{bd } (C(u_1) \times D(z_0))$,..., $(u_{m-1}, v_{m-1}) \in V^{m-1}((F, G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-2}, v_{m-2})) \bigcap -\text{bd } (C(u_1) \times D(z_0))$, $m \geq 3$, *then*

$$V^m((F, G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-1}, v_{m-1})) \bigcap -\text{int } (C(u_1) \times D(z_0)) = \emptyset. \tag{3.1}$$

Proof. By Remark 2.2(ii), (ii) with $(u_1, v_1) = (0, 0)$ becomes (i) and (iii) with $(u_2, v_2) = ... = (u_{m-1}, v_{m-1}) = (0, 0)$ becomes (ii). Hence, it suffices to prove (iii). Suppose to the contrary that $(u_i, v_i), i = 1, ..., m-1$, are as in (iii) but there is (y, z) in the intersection in (3.1). By Definition 2.1 there exist sequences $x_n \xrightarrow{(F,\,G)} x_0$, $t_n \to 0^+$ and $(y_n, z_n) \in (F, G)(x_n) + C \times D$ such that

$$\frac{1}{t_n}\Big((y_n, z_n) - (y_0, z_0) - t_n(u_1, v_1) - ... - t_n^{m-1}(u_{m-1}, v_{m-1})\Big) \to (y, z),$$

where $y \in -\text{int } C(u_1)$ and $z \in -\text{int } D(z_0)$. For $i = 2, ..., m-1$, $u_i \in -\text{cone}$ $(C + u_1)$ and hence $\alpha_i \geq 0$ and $c_i \in C$ exist such that $u_i = -\alpha_i(c_i + u_1)$. Therefore,

$$\frac{1}{t_n^m}(y_n - y_0 - t_n u_1 - ... - t_n^{m-1} u_{m-1}) = \frac{1}{t_n^m}\Big(y_n - y_0 - t_n u_1 + \textstyle\sum_{i=2}^{m-1} \alpha_i t_n^i (c_i + u_1)\Big)$$

$$= \left(\frac{y_n - y_0 + \sum_{i=2}^{m-1} \alpha_i t_n^i c_i}{t_n(1 - \sum_{i=2}^{m-1} \alpha_i t_n^{i-1})} - u_1\right) \frac{1 - \sum_{i=2}^{m-1} \alpha_i t_n^{i-1}}{t_n^{m-1}} \to y.$$

By Lemma 3.2, for sufficiently large n,

$$y_n - y_0 + \textstyle\sum_{i=2}^{m-1} \alpha_i t_n^i c_i \in -\text{int } C,$$

and then

$$y_n - y_0 \in -\text{int } C. \tag{3.2}$$

Similarly, for $i = 1, ..., m-1$, since $v_i \in -\text{cone } (D + z_0)$ there exist $\beta_i \geq 0$ and $d_i \in D$ such that $v_i = -\beta_i(d_i + z_0)$. Hence

$$\frac{1}{t_n^m}(z_n - z_0 - t_n v_1 - ... - t_n^{m-1} v_{m-1})$$

$$= \left(\frac{z_n + \sum_{i=1}^{m-1} \beta_i t_n^i d_i}{1 - \sum_{i=1}^{m-1} \beta_i t_n^i} - z_0\right) \frac{1 - \sum_{i=1}^{m-1} \beta_i t_n^i}{t_n^m} \to z.$$

Again Lemma 3.2 implies, for n large enough, that

$$z_n \in -\text{int } D. \tag{3.3}$$

On the other hand, there are $(\bar{y}_n, \bar{z}_n) \in (F, G)(x_n)$ and $(\bar{c}_n, \bar{d}_n) \in C \times D$ such that

$$(y_n, z_n) = (\bar{y}_n, \bar{z}_n) + (\bar{c}_n, \bar{d}_n).$$

Thus, by (3.2) and (3.3), for large n we have

$$\bar{y}_n - y_0 \in -\text{int } C,\ \bar{z}_n \in -\text{int } D,$$

contradicting the weak efficiency of (x_0, y_0). □

By a similar proof we obtain the following necessary condition using the variational sets of type 2.

THEOREM 3.4. *Let* (x_0, y_0) *be a local weakly efficient pair of* (P) *and* $z_0 \in G(x_0) \cap -D$. *Then*

(i) $W^1((F,G)_+, x_0, (y_0, z_0)) \bigcap -\text{int } (C \times D) = \emptyset$;

(ii) *if* $(u_1, v_1) \in W^1((F,G)_+, x_0, (y_0, z_0)) \bigcap -\text{bd } (C \times D)$, *then*

$$W^2((F,G)_+, x_0, (y_0, z_0), (u_1, v_1)) \bigcap -\text{int } (C(u_1) \times D(v_1)) = \emptyset;$$

(iii) *if* $(u_1, v_1) \in W^1((F,G)_+, x_0, (y_0, z_0)) \bigcap -\text{bd } (C \times D)$, $(u_2, v_2) \in W^2((F,G)_+, x_0, (y_0, z_0), (u_1, v_1)) \bigcap -\text{bd } (C(u_1) \times D(v_1))$,..., $(u_{m-1}, v_{m-1}) \in W^{m-1}((F,G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-2}, v_{m-2})) \bigcap -\text{bd } (C(u_1) \times D(v_1))$, $m \geq 3$, *then*

$$W^m((F,G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-1}, v_{m-1}))$$
$$\bigcap -\text{int } (C(u_1) \times D(v_1)) = \emptyset.$$

Remark 3.1. In the preceding theorems no assumptions are imposed. The proof is simple and requires very few preliminaries. In spite of this the conclusions are strong since the variational sets are rather big as we will see in Proposition 4.6. This strength results in a small gap between the necessary conditions and the sufficient ones as shown by the following two theorems.

THEOREM 3.5. *For problem* (P) *assume that* $x_0 \in A$, $y_0 \in F(x_0)$, $z_0 \in G(x_0) \cap -D$ *and* $S \subseteq \text{dom}F \cap \text{dom}G$. *Assume that either* S *is star-shaped at* x_0, F *is* C*-convex-along-rays at* (x_0, y_0), G *is* D*-convex-along-rays at* (x_0, z_0) *or* (F, G) *is pseudoconvex at* $(x_0, (y_0, z_0))$. *Then* (x_0, y_0) *is a weakly efficient pair if one of the following conditions holds*

(i) $V^1((F,G)_+, x_0, (y_0, z_0)) \bigcap -(\text{int } C \times D(z_0)) = \emptyset$;

(ii) *if* $(u_1, v_1) \in V^1((F,G)_+, x_0, (y_0, z_0)) \bigcap -\text{bd } (C \times D(z_0))$, *then*

$$V^2((F,G)_+, x_0, (y_0, z_0), (u_1, v_1)) \bigcap -(\text{int } C(u_1) \times D(z_0)) = \emptyset;$$

(iii) *if* $(u_1, v_1) \in V^1((F,G)_+, x_0, (y_0, z_0)) \bigcap -\text{bd } (C \times D(z_0))$, $(u_2, v_2) \in V^2((F,G)_+, x_0, (y_0, z_0), (u_1, v_1)) \bigcap -\text{bd } (C(u_1) \times D(z_0))$,..., $(u_{m-1}, v_{m-1}) \in V^{m-1}((F,G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-2}, v_{m-2})) \bigcap -\text{bd } (C(u_1) \times D(z_0))$, $m \geq 3$, *then*

$$V^m((F,G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-1}, v_{m-1}))$$
$$\bigcap -(\text{int } C(u_1) \times D(z_0)) = \emptyset.$$

Proof. Condition (ii) is required to be satisfied also for $(u_1, v_1) = (0, 0)$ and hence (ii) implies (i). Similarly, (iii) also implies (i). So we have to consider only condition (i). By Proposition 2.3, $\forall x \in S$,

$$(F,G)(x) - (y_0, z_0) \subseteq V^1((F,G)_+, x_0, (y_0, z_0)).$$

Then

$$((F,G)(x)-(y_0,z_0))\bigcap -(\text{int } C\times D(z_0))=\emptyset. \tag{3.4}$$

Suppose the existence of $x\in A$ and $y\in F(x)$ such that $y-y_0\in -\text{int } C$. For any $z\in G(x)\cap -D$ one has $z-z_0\in -D(z_0)$ and hence $(y,z)-(y_0,z_0)\in -(\text{int } C\times D(z_0))$, contradicting (3.4). □

By a similar proof we have also the following sufficient condition for efficiency.

THEOREM 3.6. *For problem* (P) *assume the assumptions as in Theorem 3.5. Assume further that C is pointed. Then (x_0,y_0) is an efficient pair if one of the following conditions holds*

(i) $V^1((F,G)_+,x_0,(y_0,z_0))\bigcap -(C\times D(z_0))=\{(0,0)\}$;

(ii) *if* $(u_1,v_1)\in V^1((F,G)_+,x_0,(y_0,z_0))\bigcap -\text{bd }(C\times D(z_0))$, *then*

$$V^2((F,G)_+,x_0,(y_0,z_0),(u_1,v_1))\bigcap -(C(u_1)\times D(z_0))\subseteq\{(0,0)\};$$

(iii) *if* $(u_1,v_1)\in V^1((F,G)_+,x_0,(y_0,z_0))\bigcap -\text{bd }(C\times D(z_0))$, $(u_2,v_2)\in V^2((F,G)_+,x_0,(y_0,z_0),(u_1,v_1))\bigcap -\text{bd }(C(u_1)\times D(z_0))$,..., $(u_{m-1},v_{m-1})\in V^{m-1}((F,G)_+,x_0,(y_0,z_0),(u_1,v_1),...,(u_{m-2},v_{m-2}))\bigcap -\text{bd }(C(u_1)\times D(z_0))$, $m\geq 3$, *then*

$$V^m((F,G)_+,x_0,(y_0,z_0),(u_1,v_1),...,(u_{m-1},v_{m-1}))$$
$$\bigcap -(C(u_1)\times D(z_0))\subseteq\{(0,0)\}.$$

For the case where $G(x)\equiv\{0\}$, i.e. problem (P) is without explicit constraint, we have the following sufficient condition for local efficiency without any convexity assumption.

THEOREM 3.7. *Assume that C has a compact base Q, $G(x)\equiv\{0\}$ and $(x_0,y_0)\in \text{gr}F$. Then each of the following conditions is sufficient for (x_0,y_0) to be a local efficient pair of* (P).

(i) $W^1(F,x_0,y_0)\bigcap -C=\{0\}$;

(ii) *if* $u_1\in W^1(F,x_0,y_0)\bigcap -\text{bd } C$, *then*

$$W^2(F,x_0,y_0,u_1)\bigcap -C(u_1)\subseteq\{0\};$$

(iii) *if* $u_1\in W^1(F,x_0,y_0)\bigcap -\text{bd } C$, $u_2\in W^2(F,x_0,y_0,u_1)\bigcap -\text{bd } C(u_1)$,..., $u_{m-1}\in W^{m-1}(F,x_0,y_0,u_1,...,u_{m-2})\bigcap -\text{bd } C(u_1)$, $m\geq 3$, *then*

$$W^m(F,x_0,y_0,u_1,...,u_{m-1})\bigcap -C(u_1)\subseteq\{0\}.$$

Proof. Similarly as for Theorem 3.5 we need to prove only (i). Suppose ad absurdum that there are $x_n\xrightarrow{F}x_0$ and $y_n\in F(x_n)$ such that $y_n-y_0\in -C\setminus\{0\}$. Since Q is a base, $y_n-y_0=-r_nb_n$ for some $r_n>0$ and $b_n\in Q$. By the compactness we can assume that $b_n\to b$ for some $b\neq 0$ in Q. Hence

$$\frac{1}{t_n}(y_n - y_0) = -b_n \to -b \in -C \setminus \{0\},$$

i.e. $-b \in W^1(F, x_0, y_0) \cap -C \setminus \{0\}$, a contradiction. □

Observe that $y \notin -\operatorname{int} C$ is equivalent to the existence of $c^* \in C^*$ with $\langle c^*, y\rangle \geq 0$ and that $y \notin -C$ means the existence of $c^* \in C^*$ with $\langle c^*, y\rangle > 0$. Hence we can formulate dual forms of the above theorems by using Lagrange multipliers depending on the points of variational sets as follows. The proofs are straightforward.

THEOREM 3.8. *Assume that* $x_0 \in A, y_0 \in F(x_0), (x_0, y_0)$ *is a local weakly efficient pair of* (P) *and* $z_0 \in G(x_0) \cap -D$. *Then*

(i) $\forall (y, z) \in V^1((F, G)_+, x_0, (y_0, z_0))$, $\exists (c^*, d^*) \in C^* \times D^* \setminus \{(0, 0)\}$ *such that* $\langle d^*, z_0\rangle = 0$ *and* $\langle c^*, y\rangle + \langle d^*, z\rangle \geq 0$;

(ii) *if* $(u_1, v_1) \in V^1((F, G)_+, x_0, (y_0, z_0)) \bigcap -\mathrm{bd}\ (C \times D(z_0))$ *then* $\forall (y, z) \in V^2((F, G)_+, x_0, (y_0, z_0), (u_1, v_1))$, $\exists (c^*, d^*) \in C^* \times D^* \setminus \{(0, 0)\}$ *such that* $\langle c^*, u_1\rangle = \langle d^*, z_0\rangle = 0$ *and* $\langle c^*, y\rangle + \langle d^*, z\rangle \geq 0$;

(iii) *if* $(u_1, v_1) \in V^1((F, G)_+, x_0, (y_0, z_0)) \bigcap -\mathrm{bd}\ (C \times D(z_0))$, $(u_2, v_2) \in V^2((F, G)_+, x_0, (y_0, z_0), (u_1, v_1)) \bigcap -\mathrm{bd}\ (C(u_1) \times D(z_0))$,..., $(u_{m-1}, v_{m-1}) \in V^{m-1}((F, G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-2}, v_{m-2})) \bigcap -\mathrm{bd}\ (C(u_1) \times D(z_0))$, $m \geq 3$, *then* $\forall (y, z) \in V^m((F, G)_+, x_0, (y_0, z_0), (u_1, v_1), ..., (u_{m-1}, v_{m-1}))$, $\exists (c^*, d^*) \in C^* \times D^* \setminus \{(0, 0)\}$ *such that* $\langle c^*, u_1\rangle = \langle d^*, z_0\rangle = 0$ *and* $\langle c^*, y\rangle + \langle d^*, z\rangle \geq 0$.

4. Corollaries. To see the generality of the results in Section 3 we now derive as consequences a number of recent existing optimality conditions using various kinds of generalized derivatives. To this end we need to recall additional notions.

Consider problem (P) without the explicit constraint, i.e. $G(x) \equiv \{0\}$. A direction $u \in X$ is called a *feasible direction for* S *at* $x_0 \in S$ if $\exists \delta > 0$, $\forall t \in (0, \delta)$, $x_0 + tu \in S$. The *upper Dini derivative of* $F : S \to 2^Y$ *at* $(x_0, y_0) \in \mathrm{gr}F$ *in the feasible direction* u is

$$dF(x_0, y_0, u) = \limsup_{t \to 0^+} \frac{1}{t}(F(x_0 + tu) - y_0).$$

COROLLARY 4.1 [5]. *Consider problem* (P) *with* $G(x) \equiv \{0\}$. *If* (x_0, y_0) *is a weakly efficient pair then, for each feasible direction* u *for* S *at* x_0,

$$dF(x_0, y_0, u) \cap (-\mathrm{int}\ C) = \emptyset.$$

Proof. Since, for each feasible direction u for S,

$$dF(x_0, y_0, u) \subseteq V^1(F, x_0, y_0) \subseteq W^1(F, x_0, y_0),$$

the conclusion follows directly from any of Theorem 3.3(i) and Theorem

3.4(i). □

DEFINITION 4.2 [1]. *Let $S \subseteq X$ and $u_1, ..., u_{m-1} \in X, m \geq 1$.*

(i) The mth-order contingent set of S at $(x, u_1, ..., u_{m-1})$ is

$$T_S^m(x, u_1, ..., u_{m-1}) = \limsup_{t \to 0^+} \frac{1}{t^m}(S - x - tu_1 - ... - t^{m-1}u_{m-1}).$$

(ii) The mth-order adjacent set of S at $(x, u_1, ..., u_{m-1})$ is

$$T_S^{bm}(x, u_1, ..., u_{m-1}) = \liminf_{t \to 0^+} \frac{1}{t^m}(S - x - tu_1 - ... - t^{m-1}u_{m-1}).$$

(iii) The mth-order Clarke (or circatangent) set of S at $(x, u_1, ..., u_{m-1})$ is

$$C_S^m(x, u_1, ..., u_{m-1}) = \liminf_{t \to 0^+, z \xrightarrow{S} x} \frac{1}{t^m}(S - z - tu_1 - ... - t^{m-1}u_{m-1}).$$

(iv) [17] The asymptotic second-order tangent cone of S at (x_0, v) is

$$T''(S, x_0, v) = \{w \in X : \exists (t_n, r_n) \to (0^+, 0^+) : \frac{t_n}{r_n} \to 0, \exists w_n \to w,$$
$$\forall n \in \mathbb{N}, x_0 + t_n v + \tfrac{1}{2} t_n r_n w_n \in S\}.$$

DEFINITION 4.3 [1]. *Let $F : S \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$ and $(u_1, v_1), ..., (u_{m-1}, v_{m-1}) \in X \times Y, m \geq 1$.*

(i) The mth-order contingent derivative of F at (x_0, y_0) wrt $(u_1, v_1), ..., (u_{m-1}, v_{m-1})$ is the multivalued mapping $D^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})$ defined by the following graph

$$\mathrm{gr}D^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}) = T^m_{grF}(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}).$$

(ii) The mth-order adjacent derivative of F at (x_0, y_0) wrt $(u_1, v_1), ..., (u_{m-1}, v_{m-1})$ is the multivalued mapping $D^{bm} F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})$ whose graph is

$$\mathrm{gr}D^{bm} F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}) = T^{bm}_{grF}(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}).$$

(iii) The mth-order Clarke (or circatangent) derivative of F at (x_0, y_0) wrt $(u_1, v_1), ..., (u_{m-1}, v_{m-1})$ is the multivalued mapping $C^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})$ whose graph is

$$\mathrm{gr}C^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}) = C^m_{grF}(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1}).$$

DEFINITION 4.4. *Let $F : S \to 2^Y$, $(x_0, y_0) \in \mathrm{gr}F$ and $(u_1, v_1), ..., (u_{m-1}, v_{m-1}) \in X \times Y, m \geq 1$.*

(i) The mth-order contingent epiderivative of F at (x_0, y_0) wrt $(u_1, v_1), ..., (u_{m-1}, v_{m-1})$ is the single-valued mapping $ED^m F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})$

whose epigraph is

$$\mathrm{epi}ED^mF(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1}) = T^m_{epiF}(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1}).$$

(ii) The mth-order adjacent epiderivative $ED^{bm}F(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})$ *and mth-order Clarke epiderivative are defined similarly from the corresponding tangent sets.*

The first-order contingent epiderivative was introduced in [11] and the second-order one in [10]. We define the other and higher-order epiderivatives in a natural way.

DEFINITION 4.5 [14]. *Let* $F : S \to 2^Y$, $(x_0,y_0) \in \mathrm{gr}F$ *and* $(u_1,v_1),...,(u_{m-1},v_{m-1}) \in X \times Y, m \geq 1$.

(i) The mth-order generalized contingent epiderivative of F *at* (x_0,y_0) *wrt* $(u_1,v_1),...,(u_{m-1},v_{m-1})$ *is the multivalued mapping* $ED^m_gF(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})$ *defined by, for* $x \in X$,

$$ED^m_gF(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})(x)$$
$$= \mathrm{Min}_C\{y \in Y : y \in D^mF_+(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})(x)\}.$$

(ii) The mth-order generalized adjacent epiderivative of F *at* (x_0,y_0) *wrt* $(u_1,v_1),...,(u_{m-1},v_{m-1})$ *is the multivalued mapping* $ED^{bm}_gF(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})$ *defined by, for* $x \in X$,

$$ED^{bm}_gF(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})(x)$$
$$= \mathrm{Min}_C\{y \in Y : y \in D^{bm}F_+(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})(x)\}.$$

Of course we can define similarly the *mth-order generalized Clarke epiderivative.*

The following consequence of the encountered definitions is not hard to be checked. It generalizes and extends Remark 2.1.

PROPOSITION 4.6. *Let* $F : S \to 2^Y$, $(x_0,y_0) \in \mathrm{gr}F$, $(u_1,v_1),...,(u_{m-1},v_{m-1}) \in X \times Y, m \geq 1$, *and* $x \in X$.

(i) $ED^mF(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})(x)$

$$\subseteq D^mF_+(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})(x) \subseteq V^m(F_+,x_0,y_0,v_1,...,v_{m-1}).$$

(ii) $ED^{bm}F(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})(x)$

$$\subseteq D^{bm}F_+(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})(x) \subseteq V^m(F_+,x_0,y_0,v_1,...,v_{m-1}).$$

(iii) $ED^m_gF(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})(x)$

$$\subseteq D^mF_+(x_0,y_0,u_1,v_1,...,u_{m-1},v_{m-1})(x) \subseteq V^m(F_+,x_0,y_0,v_1,...,v_{m-1}).$$

(iv) $ED_g^{bm}F(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x)$

$$\subseteq D^{bm}F_+(x_0, y_0, u_1, v_1, ..., u_{m-1}, v_{m-1})(x) \subseteq V^m(F_+, x_0, y_0, v_1, ..., v_{m-1}).$$

The following two examples ensure us that the inclusions in Proposition 4.6 may be strict.

EXAMPLE 4.1. *Let* $X = Y = \mathbb{R}, S = X, C = \mathbb{R}_+, (x_0, y_0) = (0, 0)$ *and* $(u, v) = (1, 0)$*. Let*

$$F(x) = \begin{cases} \{0\} & \text{if } x = 0, \\ \left\{\dfrac{1}{n}\right\} & \text{if } x = \dfrac{1}{n}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $T^1_{epiF}(x_0, y_0) = \mathbb{R}_+ \times \mathbb{R}_+$ *and* $T^2_{epiF}(x_0, y_0, u, v) = \mathbb{R} \times \mathbb{R}_+$.

Hence, dom $D^1F_+(x_0, y_0) =$ *dom* $ED^1F(x_0, y_0) =$ *dom* $ED^1_gF(x_0, y_0) = \mathbb{R}_+$ *and,* $\forall x \in \mathbb{R}_+$,

$$D^1F_+(x_0, y_0)(x) = \mathbb{R}_+,$$

$$ED^1F(x_0, y_0)(x) = 0, ED^1_gF(x_0, y_0)(x) = \{0\},$$

and, $\forall x \in \mathbb{R}$,

$$D^2F_+(x_0, y_0, u, v)(x) = \mathbb{R}_+,$$

$$ED^2F(x_0, y_0, u, v)(x) = 0, ED^2_gF(x_0, y_0, u, v)(x) = \{0\}.$$

On the other hand

$$V^1(F_+, x_0, y_0) = W^1(F_+, x_0, y_0) = \mathbb{R}_+,$$

$$V^2(F_+, x_0, y_0, v) = W^2(F_+, x_0, y_0, v) = \mathbb{R}_+.$$

EXAMPLE 4.2. *Let* $X, Y, S, C, (x_0, y_0)$ *and* (u, v) *be as in Example 4.1. Let, for* $\alpha \in (1, 2)$, $F(x) = \{y \in \mathbb{R} | \ y \geq |x|^\alpha\}, \forall x \in \mathbb{R}$*. Then*

$$T^2_{epiF}(x_0, y_0, u, v) = \emptyset.$$

Therefore all $D^2F_+(x_0, y_0, u, v), ED^2_gF(x_0, y_0, u, v)$ *and* $ED^2F(x_0, y_0, u, v)$ *do not exist. On the other hand*

$$V^2(F_+, x_0, y_0, v) = W^2(F_+, x_0, y_0, v) = \mathbb{R}_+.$$

From the definitions and above examples we see that the calculation of the upper limits to get our variational sets is not so difficult. The computation of the generalized epiderivatives in Definition 4.5, for instance, is much more complicated.

COROLLARY 4.7 [10]. *Consider problem* (P) *with* $G(x) \equiv \{0\}$. *Let* (x_0, y_0) *be a weakly efficient pair,* $u \in \mathrm{dom} DF_+(x_0, y_0), v \in DF_+(x_0, y_0)(u) \cap -\mathrm{bd}\, C$ *and* $x \in \mathrm{dom} D^2F_+(x_0, y_0, u, v)$. *Then*

$$D^2F_+(x_0, y_0, u, v)(x) \cap -\mathrm{int}\, C(v) = \emptyset.$$

Proof. The conclusion is followed directly Theorem 3.3(ii) and Proposition 4.6(i). It can be derived also from Theorem 3.4(ii) and Proposition 4.6(i). □

As easily as for the preceding two corollaries we can obtain, as direct consequences of any from Theorem 3.3(i) and 3.4(i), Theorem 7 of [11], Theorem 5 of [3], Theorem 4.1 of [4], Proposition 3.1 of [22], Theorem 2.7(a) of [9], Theorem 2 of [5] and Theorem 4.1 of [6]; any from Theorems 3.3(ii) and 3.4(ii), Theorem 3.1 of [10].

The following example shows a case where Theorem 3.3 can be employed but many of the encountered known results cannot.

EXAMPLE 4.3. *Let* $X = Y = \mathbb{R}, S = X, C = \mathbb{R}^+, F(x) = \{-x^3\}$ *and* $(x_0, y_0) = (0, 0)$. *Then* $T^1_{epiF}(x_0, y_0) = \mathbb{R} \times \mathbb{R}_+$. *Hence,* $\forall x \in \mathbb{R}$,

$$D^1F_+(x_0, y_0)(x) = \mathbb{R}_+,$$

$$ED^1F(x_0, y_0)(x) = 0, ED^1_gF(x_0, y_0)(x) = \{0\}.$$

Therefore, Theorem 7 of [11], Theorem 5 of [3], Theorem 4.1 of [4], Proposition 3.1 and Theorem 4.1 of [22] and Theorem 2.7(a) of [9] cannot be applied to reject (x_0, y_0) *as a candidate for a local weakly efficient pair of problem* (P) *with* $G(x) \equiv \{0\}$.

Moreover, $\forall u \in \mathbb{R}, dF(x_0, y_0, u) = \{0\}$ *and then Theorem 2 of [5] (i.e. Corollary 4.1) and Theorem 4.1 of [6] cannot either.*

Furthermore, $\forall u \in \mathbb{R}$,

$$T^2_{epiF}(x_0, y_0, u, 0) = \mathbb{R} \times \mathbb{R}_+,$$

and hence $D^2F_+(x_0, y_0, u, 0)(x) = \mathbb{R}_+, \forall x \in \mathbb{R}$, *which rules out the use of Theorem 3.1 of [10].*

On the other hand, since $V^1(F_+, x_0, y_0) = \mathbb{R}$, *Theorem 3.4 implies that* (x_0, y_0) *is not a local weakly efficient pair.*

In the above corollary, we assume that $\mathrm{dom}F = S$ and u, x are in the appropriate domains. To avoid this assumptions we make use of *tangent sets of* S as follows. For the sake of comparison we consider the special case studied in [13].

COROLLARY 4.8 [13]. *Consider* (P) *with* $G(x) \equiv \{0\}$ *and* $F := f$ *being single-valued. Assume that* f *is twice Fréchet differentiable at the local weakly* *efficient point* $x_0 \in S$. *Then*

(i) $f'(x_0)v \notin -\text{int}\, C, \forall v \in T(S, x_0)$;

(ii) $\forall v \in T(S, x_0)$ *with* $f'(x_0)v \in -\text{bd}\, C$ *one has*

- $f'(x_0)w + f''(x_0)(v, v) \notin -\text{int}\, C(f'(x_0)v), \forall w \in T^2(S, x_0, v)$;
- $f'(x_0)w \notin -\text{int}\, C(f'(x_0)v), \forall w \in T''(S, x_0, v)$.

Proof. (i) By definition, $v \in T(S, x_0)$ means that $\exists t_n \to 0^+, \exists v_n \to v, \forall n, x_0 + t_n v_n \in S$. Then, setting $y_0 = f(x_0)$, one has

$$f'(x_0)v \in V^1(f_+, x_0, y_0) \subseteq W^1(f_+, x_0, y_0),$$

because

$$\lim_{n\to\infty} \frac{1}{t_n}\Big(f(x_0 + t_n v_n) - f(x_0)\Big) = f'(x_0)v.$$

Theorem 3.4(i) then implies that $f'(x_0)v \in -\text{int}\, C$.

(ii) $w \in T^2(S, x_0, v)$ means that $\exists t_n \to 0^+, \exists w_n \to w, \forall n, x_n := x_0 + t_n v + \frac{1}{2}t_n^2 w_n \in S$. Now that $\frac{2}{t_n^2}((x_n - x_0 - t_n v) \to w$, by the Taylor formula one has

$$\lim_{n\to\infty} \frac{2}{t_n^2}\Big(f(x_n) - f(x_0) - t_n f'(x_0)v\Big) = f'(x_0)w + f''(x_0)(v, v).$$

By Definition 2.2, the right-hand side belongs to $W^2(f_+, x_0, y_0, f'(x_0)v)$. So the first assertion follows from Theorem 3.4(ii).

By definition, $w \in T''(S, x_0, v)$ means that $\exists (t_n, r_n) \to (0^+, 0^+) : \dfrac{t_n}{r_n} \to 0, \exists w_n \to w, \forall n, x_n := x_0 + t_n v + \frac{1}{2}t_n r_n w_n \in S$. The Taylor formula gives

$$\frac{2}{t_n r_n}\Big(f(x_n) - f(x_0) - t_n f'(x_0)v\Big) \to f'(x_0)w.$$

By Definition 2.2, $f'(x_0)w \in W^2(f_+, x_0, y_0, f'(x_0)v)$ and the second assertion follows from Theorem 3.4(ii). □

To deduce more consequences let us recall the following.

DEFINITION 4.9 [19]. *Let* $F : X \to 2^Y$ *and* $(x_0, y_0) \in \text{gr}F$. *The S-derivative of* F *at* (x_0, y_0) *is the set-valued mapping* $SF(x_0, y_0) : X \to 2^Y$ *defined by*

$$y \in SF(x_0, y_0)(x)$$

$$\Leftrightarrow \exists t_n > 0, \exists x_n \to x, \exists y_n \to y, t_n x_n \to 0 \text{ and } y_0 + t_n y_n \in F(x_0 + t_n x_n), \forall n$$

$$\Leftrightarrow \exists \alpha_n > 0, \exists (x_n, y_n) \in \text{gr}F, x_n \to x_0 \text{ and } \alpha_n(x_n - x_0, y_n - y_0) \to (x, y).$$

Notice that $SF(x_0, y_0)X \subseteq W^1(F, x_0, y_0)$. The inclusion may be strict as shown by

EXAMPLE 4.4. *Let* $X = Y = \mathbb{R}, C = \mathbb{R}_+, (x_0, y_0) = (0, 1)$ *and*

$$F(x) = \begin{cases} \left\{1 + \frac{1}{n}\right\} & \text{if } x = \frac{1}{n}, n = 1, 2, ..., \\ \{1\} & \text{if } x = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then,

$$SF(x_0, y_0)(x) = \begin{cases} \{x\} & \text{if } x \geq 0, \\ \emptyset & \text{if } x < 0, \end{cases}$$

$$W^1(F, x_0, y_0) = \mathbb{R}_+.$$

PROPOSITION 4.10. *Let* $F : X \to 2^Y$, X *be finite dimensional,* $(x_0, y_0) \in \mathrm{gr}F$ *and*

$$SF(x_0, y_0)(x) \cap -C = \begin{cases} \{0\} & \text{if } x = 0, \\ \emptyset & \text{if } x \neq 0. \end{cases}$$

Then $W^1(F, x_0, y_0) \cap -C = \{0\}$.

Proof. Suppose the existence of $y \in -C \setminus \{0\}$, $x_n \xrightarrow{F} x_0, y_n \in F(x_n)$ and $\alpha_n > 0$ such that

$$\alpha_n(y_n - y_0) \to y. \tag{4.1}$$

Without loss of generality it suffices to consider the following three cases.

(a) $\alpha_n(x_n - x_0) \to 0$. Then (4.1) yields $y \in SF(x_0, y_0)(0)$, which is impossible.

(b) $\alpha_n(x_n - x_0) \to x \neq 0$. Then (4.1) yields $y \in SF(x_0, y_0)(x)$, again a contradiction.

(c) $\{\alpha_n(x_n - x_0)\}$ is unbounded and we can assume that $\|\alpha_n(x_n - x_0)\| \to +\infty$. So $\alpha_n \to +\infty$ and $y_n \to y_0$. We can assume further that $\|\alpha_n(x_n - x_0)\|^{-1}\alpha_n(x_n - x_0) \to x_1 \neq 0$. Then, setting $\beta_n = \|x_n - x_0\|^{-1}$ we obtain

$$\beta_n(x_n - x_0, y_n - y_0) = \left(\frac{x_n - x_0}{\|x_n - x_0\|}, \frac{y_n - y_0}{\|y_n - y_0\|}\right) \to (x_1, 0).$$

Thus $0 \in SF(x_0, y_0)(x_1)$, once more a contradiction. □

As a direct consequence of Theorem 3.4(i) we get

COROLLARY 4.11 [21]. *Consider problem* (P) *with* $G(x) \equiv \{0\}$. *If* (x_0, y_0) *is a local weakly efficient pair then,* $\forall x \in X$,

$$SF(x_0, y_0)(x) \cap -\mathrm{int}\, C = \emptyset.$$

In the following example, Theorem 3.4 rejects the suspected point but

Corollary 4.11 (Theorem 3.1 of [21]) cannot do.

EXAMPLE 4.5. *Let* $X = Y = \mathbb{R}, C = \mathbb{R}_+, (x_0, y_0) = (0, 1)$ *and*

$$F(x) = \begin{cases} \left\{1 - \dfrac{1}{n^2}\right\} & \text{if } x = \dfrac{1}{n}, n = 1, 2, ..., \\ \{1\} & \text{if } x = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $SF(x_0, y_0)(x) = \{0\}, \forall x \in \mathbb{R}$ *and Corollary 4.11 says nothing about* (x_0, y_0). *However,* $W^1(F_+, x_0, y_0) = \mathbb{R}$ *and hence this pair is not a local weakly efficient one by Theorem 3.4(i).*

COROLLARY 4.12 [21]. *Consider problem (P) with* $G(x) \equiv \{0\}$. *Let* X *be finite dimensional,* C *have a compact base and* $(x_0, y_0) \in \mathrm{gr}F$. *Assume that*

(i) $SF(x_0, y_0)(0) \cap -C = \{0\}$;

(ii) $SF(x_0, y_0)(x) \cap -C = \emptyset$ *for* $x \in \mathrm{dom}SF(x_0, y_0) \setminus \{0\}$.

Then (x_0, y_0) *is a local efficient pair.*

Proof. The conclusion follows directly from Theorem 3.7(i) and Proposition 4.10. □

The example below gives a case where Theorem 3.7(i) ensures us that (x_0, y_0) is a local efficient pair but Corollary 4.12 (Theorem 4.1 of [21]) cannot be applied.

EXAMPLE 4.6. *Let* $X = Y = \mathbb{R}, C = \mathbb{R}_+, (x_0, y_0) = (0, 1)$ *and*

$$F(x) = \begin{cases} \left\{1 + \dfrac{1}{n^2}\right\} & \text{if } x = \dfrac{1}{n}, n = 1, 2, ..., \\ \{1\} & \text{if } x = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $SF(0, 1)(x) = \{0\}, \forall x \in \mathbb{R}$ *and so Corollary 4.12 says nothing about* (x_0, y_0). *However,* $W^1(F, x_0, y_0) = \mathbb{R}_+$ *and* (x_0, y_0) *is a local efficient pair following Theorem 3.7(i).*

Remark 4.1. Our results do not imply that of [20] since our problem (P) does not involve equality constraints. However, the following example supplies a case where Theorem 3.3 can be used but the corresponding Theorem 11 of [20] cannot.

EXAMPLE 4.7. *Let* $S = X = Y = Z = \mathbb{R}, C = \mathbb{R}_+, , D = \mathbb{R}, (x_0, y_0) = (0, 0), F(x) = \{-\sqrt{|x|}\}$ *and* $G(x) = \{0\}, \forall x \in \mathbb{R}$. *For* $z_0 = 0$ *we have*

$$V^1((F, G)_+, x_0, (y_0, z_0) = \mathbb{R}^2$$

and $\mathbb{R}^2 \cap -\operatorname{int}(C \times D(z_0)) \neq \emptyset$. *By Theorem 3.3,* (x_0, y_0) *is not a local weakly efficient pair of* (P). *However, the fact that* f *is not mth-order Neustadt* *differentiable at* x_0 *(for any* m *and* $x_1, ..., x_{m-1} \in \mathbb{R}$*) rules out the use of Theorem 11 of [20].*

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