The solution existence of general variational inclusion problems

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Abstract

We propose general variational inclusion problems which are slightly different from corresponding problems considered in several recent papers in the literature and show that they are advantageous. Sufficient conditions for the solution existence are established. As applications we derive consequences for several special cases of variational inclusion problems, quasioptimization problems, equilibrium problems and implicit variational inequalities and show that they improve the results of some recent existing papers.

Keywords: Variaional inclusion problems; Equilibrium problems; Implicit variational inequalities; Quasioptimization; G-quasiconvexity and G-quasiconvexlikeness with respect to a multifunction; lower and upper C-continuity.

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1. Introduction

Many efforts have been made for the last decade to propose general setting problems related to optimization, beginning with [5] where an equilibrum problem has been considered, see e.g. [2-4,6,10,16].

Very recently inclusion problems were investigated as a generalization of equilibrium problems, in order to include a wide class of problems in diverse fields such as variational inequalities, vector optimization, game theory, fixed point and coincidence point problems, the Nash equilibrium problem, complementarity problems, traffic equilibria, etc. [18-20]. It should be noted here that the term "variational inclusion" is understood in different ways in several recent papers. In [9,13] it means simply multivalued variational inequalities. Variational inclusion problems in [1,7,8] are problems of finding zeroes of maximal monotone mappings. In this note the terminology is similar to [18-20]. Observing that such inclusion problems, although rather general, do not include some general equilibrium problems or are not convenient for studying solution existence (see (c) below), we propose in this note four variants of general inclusion problems to amend existing problem settings and establish sufficient conditions for the solutions existence.

In the sequel, if not otherwise stated, let X, Y and Z be real topological vector spaces, X be Hausdorff and $A \subseteq X$ be a nonempty closed convex subset. Let $C : A \to 2^Y$, $S_1 : A \to 2^X$, $S_2 : A \to 2^X$and $T : A \times X \to 2^Z$ be multifunctions such that $C(x)$ is a closed convex cone with int $C(x) \neq \emptyset$ for each $x \in A$. Let $F : T(A \times X) \times X \times A \to 2^Y$ and $G : T(A \times X) \times X \times A \to 2^Y$ be multivalued mappings. We consider the following four problems.

(IP1) Find $\bar{x} \in S_1(\bar{x})$, such that, $\forall y \in S_2(\bar{x})$, $\forall \bar{t} \in T(\bar{x}, y)$,

$$F(\bar{t}, y, \bar{x}) \subseteq G(\bar{t}, \bar{x}, \bar{x}).$$

(IP2) Find $\bar{x} \in S_1(\bar{x})$, such that, $\forall y \in S_2(\bar{x})$, $\exists \bar{t} \in T(\bar{x}, y)$,

$$F(\bar{t}, y, \bar{x}) \subseteq G(\bar{t}, \bar{x}, \bar{x}).$$

(IP3) Find $\bar{x} \in S_1(\bar{x})$, such that, $\forall y \in S_2(\bar{x})$, $\forall \bar{t} \in T(\bar{x}, y)$,

$$F(\bar{t}, y, \bar{x}) \cap G(\bar{t}, \bar{x}, \bar{x}) \neq \emptyset.$$

(IP4) Find $\bar{x} \in S_1(\bar{x})$, such that, $\forall y \in S_2(\bar{x})$, $\exists \bar{t} \in T(\bar{x}, y)$,

$$F(\bar{t}, y, \bar{x}) \cap G(\bar{t}, \bar{x}, \bar{x}) \neq \emptyset.$$

To motivate the problem setting let us look at several special cases.

(a) If $C(\bar{x}) \equiv C$, $G(t, x, x) = F(t, x, x) + C$, where $C \subseteq Y$ is a closed cone, (IP1) becomes the variational inclusion problem with constraints considered in [18]:

(IP) Find $\bar{x} \in S_1(\bar{x})$ such that, $\forall y \in S_2(\bar{x})$, $\forall \bar{t} \in T(\bar{x}, y)$,

$$F(\bar{t}, y, \bar{x}) \subseteq F(\bar{t}, \bar{x}, \bar{x}) + C.$$

If we replace T by the mapping $(x, y) \mapsto T(x, x)$ we get a variational inclusion problem of Minty type. If T is replaced by the mapping $(x, y) \mapsto T(y, y)$ one has a variational inclusion problem of Stampacchia type.

(b) If $T(x, y)$ is replaced by $(x, y) \mapsto T(x, x) := T(x)$, $S_1(x) = S_2(x) := S(x)$ and $C(\bar{x}) \equiv C$, a closed cone, then (IP1) coincides with the upper variational inclusion problem investigated in [19]:

(UIP) Find $\bar{x} \in S(\bar{x})$ such that, $\forall y \in S(\bar{x})$, $\forall \bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \subseteq F(\bar{t}, \bar{x}, \bar{x}) + C.$$

(c) Note that (IP) and (UIP) do not include the following general equilibrium problem (without severe assumptions on F):

(EP) Find $\bar{x} \in S_1(\bar{x})$ such that, $\forall y \in S_1(\bar{x})$, $\forall \bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \subseteq C.$$

But our problem (IP1) clearly does.

(d) If $S_1(x) = S_2(x) \equiv A$, T is replaced by $(x, y) \mapsto T(x, x) := T(x)$, $Z = L(X, Y)$ (the space of linear continuous mappings of X into Y), F is single-valued and $G(t, x, x) = Y \backslash -\text{int}C(x)$, then (IP4) collapses to the implicit vector variational inequality studied in [14,15]:

(IVI) Find $\bar{x} \in A$ such that, $\forall y \in A$, $\exists \bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \notin -\text{int } C(\bar{x}).$$

(e) If $S_1(x) = S_2(x) := K(x)$, $Z = L(X, Y)$, $F(t, y, x) = (t, x - y)$ and $G(t, x, x) = Y \backslash -\text{int}C(x)$, where (t, x) denotes the value of a linear mapping t at x, then (IP4) is reduced to the vector quasivariational inequality problem (investigated by many authors):

(QVI) Find $\bar{x} \in K(\bar{x})$ such that, $\forall y \in K(\bar{x})$, $\exists \bar{t} \in T(\bar{x})$,

$$(\bar{t}, y - \bar{x}) \notin -\text{int}C(\bar{x}).$$

2. Preliminaries

We recall first some definitions needed in the sequel. Let X and Y be topological spaces. A multifunction $H : X \to 2^Y$ is said to be upper semicontinuous (usc) at $x_0 \in \mathrm{dom}H := \{x \in X : H(x) \neq \emptyset\}$ if for each neighborhood U of $H(x_0)$, there is a neighborhood N of x_0 such that $H(N) \subseteq U$. H is called usc if H is usc at each point of $\mathrm{dom}H$. In the sequel all properties defined at a point will be extended to domains in this way. H is called lower semicontinuous (lsc) at $x_0 \in \mathrm{dom}H$ if for each open subset U satisfying $U \cap H(x_0) \neq \emptyset$ there exists a neighborhood N of x_0 such that, for all $x \in N$, $U \cap H(x) \neq \emptyset$. An equivelent statement is: H is lsc at $x_0 \in X$ if and only if for any $y_0 \in H(x_0)$ and for any net $\{x_\alpha\}$ in X converging to x_0, there is a net $\{y_\alpha\}$ such that $y_\alpha \in H(x_\alpha)$ for every α and $y_\alpha \to y_0$. H is said to be continuous at $x \in \mathrm{dom}H$ if H is both usc and lsc at x. H is termed closed at $x \in \mathrm{dom}H$ if $\forall x_\alpha \to x$, $\forall y_\alpha \in H(x_\alpha)$ such that $y_\alpha \to y$, then $y \in H(x)$. It is known that if H is usc and has closed values, then H is closed.

Now let Y is a topological vector space. A multivalued mapping $H : X \to 2^Y$ is said to be upper C-continuous at $x_0 \in X$ if for any neighborhood V of the origin in Y there is a neighborhood U of x_0 such that $H(x) \subseteq H(x_0) + V + C$, $\forall x \in U$. H is said to be lower C-continuous at $x_0 \in X$ if for any neighborhood V of the origin in Y there is a neighborhood U of x_0 such that $H(x_0) \subseteq H(x) + V - C$ holds for all $x \in U$. H is C-continuous if H is both upper C-continuous and lower C-continuous.

A multifunction H of a subset A of a topological vector space X into X is said to be a KKM mapping in A if for each $\{x_1, ..., x_n\} \subseteq A$, one has $\text{co}\{x_1, ..., x_n\} \subseteq \bigcup_{i=1}^n H(x_i)$, where $\text{co}\{.\}$ stands for the convex hull.

The main tool for proving our results is the following well-known KKM-Fan theorem.

Theorem 2.1. *Assume that X is a topological vector space, $A \subseteq X$ is nonempty convex and $H : A \to 2^X$ is a KKM mapping with closed values. If A is compact, then $\bigcap_{x \in A} H(x) \neq \emptyset$.*

We propose the following generalized convexity definitions. Let D, K be sets, X be a vector space, $A \subseteq X$ be a convex subset. Let $F, G : D \times A \times A \to 2^K$ and $T : X \times A \to 2^D$ be multifunctions. F is called G-quasiconvex with respect to T of type 1 if, for any subset $\{x_1, ..., x_n\} \subseteq A$ and for any $x \in \text{co}\{x_1, ..., x_n\}$ one can find some $i \in \{1, ..., n\}$ such that

$$F(t, x_i, x) \subseteq G(t, x, x), \forall t \in T(x, x_i). \tag{1}$$

F is said to be G-quasiconvex with respect to T of type 2 if in (1) we replace $\forall t$ by $\exists t$.

F is said to be G-quasiconvexlike with respect to T of type 1 if for any subset $\{x_1, ..., x_n\} \subseteq A$ and for any $x \in \text{co}\{x_1, ..., x_n\}$ one can find some $i \in \{1, ..., n\}$ such that

$$F(t, x_i, x) \cap G(t, x, x) \neq \emptyset, \forall t \in T(x, x_i). \tag{2}$$

If $\forall t$ in (2) is replaced by $\exists t$, we say that F is G-quasiconvexlike with respect to $T(x)$ of type 2.

If $T(x,y)=\{x\}$ and $G(t,x,y)=C(y)$, G-quasiconvexity with respect to T of types 1 and 2 collapse to the strong type 1 C-diagonally quasiconvexity in the first argument and G-quasiconvexlikeness with respect to T of types 1 and 2 collapse to the strong type 2 one in [17].

3. Main results

Theorem 3.1. *For problem* (IP1) *assume that the following conditions hold*

(i) *A is nonempty and compact;*

(ii) *$S_1(.)$ is closed, $S_2(x)$ is nonempty with $co(S_2(x)) \subseteq S_1(x)$, $A \cap S_2(x) \neq \emptyset$ and $S_2^{-1}(y)$ is open in A, for all $x, y \in A$;*

(iii) *F is G-quasiconvex with respect to T of type 1;*

(iv) *for each $y \in A$, $\{x \in A : \forall t \in T(x,y), \quad F(t,y,x) \subseteq G(t,x,x)\}$ is closed.*

Then, (IP1) *has a solution.*

Proof. For $x, y \in$ A set

$$E := \{x \in A \,:\, x \in \, S_1(x)\},$$

$$P(x) := \{\, z \in A \,:\exists t \in T(x,z), F(t,z,x) \not\subseteq G(t,x,x)\},$$

$$\Phi(x) \;:=\; \begin{cases} S_2(x) \cap P(x) & \text{if}\, x \in E, \\ A \cap S_2(x) & \text{if}\, x \in A \backslash E, \end{cases}$$

$$Q(y) \;:=\; A \backslash \Phi^{-1}(y).$$

We show first that $Q(.)$ is a KKM mapping in A. Indeed, suppose there is a convex combination $\hat{x} \;:=\; \sum_{j=1}^{n} \alpha_j y_j$ in A such that $\hat{x} \notin \bigcup_{j=1}^{n} Q(y_j)$. Then,

$\hat{x} \notin Q(y_j)$, i.e., $y_j \in \Phi(\hat{x})$ for $j = 1, ..., n$. If $\hat{x} \in E$, one has $y_j \in P(\hat{x})$, i.e., $\exists t \in T(\hat{x}, y_j)$, $F(t, y_j, \hat{x}) \not\subseteq G(t, \hat{x}, \hat{x})$ for $j = 1, ..., n$, contradicting (iii). On the other hand, if $\hat{x} \in A \backslash E$, then $y_j \in \Phi(\hat{x}) = A \cap S_2(\hat{x})$, $j = 1, ..., n$. So $y_j \in \mathrm{co}(S_2(\hat{x}))$ and $\hat{x} := \sum_{j=1}^n \alpha_j y_j \in \mathrm{co}(S_2(\hat{x})) \subseteq S_1(\hat{x})$, another contradiction. Thus, $Q(.)$ must be KKM.

Next we verify the closedness of $Q(y)$, $\forall y \in A$. One has

$$\begin{aligned} \Phi^{-1}(y) &= [E \cap S_2^{-1}(y) \cap P^{-1}(y)] \cup [(A \backslash E) \cap S_2^{-1}(y)] \\ &= [(A \backslash E) \cup P^{-1}(y)] \cap S_2^{-1}(y). \end{aligned}$$

Therefore,

$$\begin{aligned} Q(y) &= A \backslash \{[(A \backslash E) \cup P^{-1}(y)] \cap S_2^{-1}(y)\} \\ &= \{A \backslash [(A \backslash E) \cup P^{-1}(y)]\} \cup (A \backslash S_2^{-1}(y)] \\ &= [E \cap (A \backslash P^{-1}(y))] \cup (A \backslash S_2^{-1}(y)). \end{aligned} \tag{3}$$

As $S_1(.)$ is closed, so is E. We have

$$\begin{aligned} A \backslash P^{-1}(y) &= \{\, x \in A \,:\, y \notin P(x)\} \\ &= \{x \in A : \forall t \in T(x, y), F(t, y, x) \subseteq G(t, x, x)\}, \end{aligned}$$

which is closed by (iv). It follows from (3) that $Q(y)$ is closed.

Applying Theorem 2.1 one obtains a point $\bar{x}$ such that

$$\bar{x} \in \bigcap_{y \in A} Q(y) \;=\; A \backslash \bigcup_{y \in A} \Phi^{-1}(y).$$

So, $\bar{x} \notin \Phi^{-1}(y)$, $\forall y \in A$, i.e., $\Phi(\bar{x}) = \emptyset$. If $\bar{x} \in A \backslash E$, then, $\Phi(\bar{x}) = A \cap S_2(\bar{x})$, contradicting (ii). If $\bar{x} \in E$, one has $\emptyset = \Phi(\bar{x}) = S_2(\bar{x}) \cap P(\bar{x})$. Thus, for all

$y \in S_2(\bar{x})$, $y \notin P(\bar{x})$, i.e., $\forall t \in T(\bar{x}, y)$,$F(t, y, \bar{x}) \subseteq G(t, \bar{x}, \bar{x})$, which means that $\bar{x}$ is a solution of (IP1). □

Theorem 3.2. *For problem* (IP2) *assume* (i) *and* (ii) *of Theorem 3.1. Assume further that*

(iii$_2$) *F is G-quasiconvex with respect to T of type 2;*

(iv$_2$) *for each $y \in A$, $\{x \in A : \exists t \in T(x, y), \quad F(t, y, x) \subseteq G(t, x, x)\}$ is closed.*

Then, (IP2) *is solvable.*

Proof. Using the same argument as in the proof of Theorem 3.1, with

$$P(x) := \{\, z \in A \; : \forall t \in T(x, z), F(t, z, x) \not\subseteq G(t, x, x)\}.$$ □

Theorem 3.3. *For problem* (IP3) *assume* (i) *and* (ii) *of Theorem 3.1 and*

(iii$_3$) *F is G-quasiconvexlike with respect to T of type 1;*

(iv$_3$) *for each $y \in A$, $\{x \in A : \forall t \in T(x, y), \quad F(t, y, x) \cap G(t, x, x) \neq \emptyset\}$ is closed.*

Then, (IP3) *has solutions.*

Proof. By using another set $P(x)$ defined by

$$P(x) := \{\, z \in A \; : \forall t \in T(x, z), F(t, z, x) \cap G(t, x, x) \neq \emptyset\}.$$

and similar reasoning as that of the proof of Theorem 3.1 one gets the conclusion.□

Passing finally to (IP4) we have

Theorem 3.4. *For problem* (IP4) *assume* (i) *and* (ii) *as in Theorem 3.1 and* (iii$_4$) *F is G-quasiconvexlike with respect to T of type 2;*

(iv$_4$) *for each $y \in A$, $\{x \in A : \exists t \in T(x,y), \quad F(t,y,x) \cap G(t,x,x) \neq \emptyset\}$ is closed.*

Then, (IP4) *has solutions.*

4. Special cases and applications

In this section we deal with some particular cases in order to derive direct consequences of our main results and show that these consequences improve several recent results in the literature. So the applications presented here are by no means typical or complete.

First, from Theorem 3.1, we get the following solution existence result for (IP).

Corollary 4.1. *Assume that* (i) *and* (ii) *of Theorem 3.1 hold and*

(iii) *$C(x) \equiv C$ and F is $(F + C)$-quasiconvex with respect to T of type 1;*

(iv) *for each $y \in A$, $\{x \in A : \forall t \in T(x,y), \quad F(t,y,x) \subseteq F(t,x,x) + C\}$ is closed.*

Then, (IP) *has solutions.*

Proof. We simply apply Theorem 3.1 with $G(t,x,x) = F(t,x,x)+C$. □

Remark 4.1. The assumption (iv) in Corollary 4.1 is satisfied provided that

(a) Y is a locally convex space;

(b) $C(x) \equiv C$ is a nonempty closed cone;

(c) for each $y \in A, T(., y)$ is lsc;

(d) for each $y \in A$, F(.,y,.) is lower $(-C)$-continuous; $F(t, x, x)$ is upper C-continuous in (t, x) and has compact values.

Indeed, for every fixed $y \in A$, set

$$M_y := \{x \in A : \forall t \in T(x, y), F(t, y, x) \subseteq F(t, x, x) + C\}.$$

Assume that $\{x_\alpha\} \subseteq M_y$, $x_\alpha \to x^*$. By (c), for every $t^* \in T(x^*, y)$, there exists a net $t_\alpha \in T(x_\alpha, y)$ such that $t_\alpha \to t^*$. We have

$$F(t_\alpha, y, x_\alpha) \subseteq F(t_\alpha, x_\alpha, x_\alpha)+C. \tag{4}$$

As $F(., y, .)$ is lower $(-C)$-continuous, for every neighborhood V_1 of the origin, there is a subnet $\{t_\beta, x_\beta\}$ of $\{t_\alpha, x_\alpha\}$ such that

$$F(t^*, y, x^*) \subseteq F(t_\beta, y, x_\beta)+V_1+C. \tag{5}$$

Since $F(t, x, x)$ is upper C-continuous in (t, x), for every neighborhood V_2 of the origin, we can assume that

$$F(t_\beta, x_\beta, x_\beta) \subseteq F(t^*, x^*, x^*)+V_2+C. \tag{6}$$

From (4), (5) and (6) one has, for all neighborhoods V_1 and V_2,

$$F(t^*, y, x^*) \subseteq F(t^*, x^*, x^*)+V_1+V_2+C. \tag{7}$$

We claim that $F(t^*, y, x^*) \subseteq F(t^*, x^*, x^*) + C$. Indeed, suppose there is some $v \in F(t^*, y, x^*)$ and $v \notin F(t^*, x^*, x^*) + C$. Then $F(t^*, x^*, x^*) + C - v := S$ does not meet 0. Because $F(t^*, x^*, x^*)$ is compact, so S is closed. Thus, S^c is open and $0 \in S^c$. Since Y is a locally convex space, there exists a neighborhood V of the origin such that $V = -V$, V is convex and $V \subset S^c$, i.e., $V \cap S = \emptyset$. Then, it's easy to check that $0 \notin (S + \frac{1}{2}V + \frac{1}{2}V)$, i.e., $v \notin F(t^*, x^*, x^*) + C + \frac{1}{2}V + \frac{1}{2}V$, contracdicting (7). Thus, $F(t^*, y, x^*) \subseteq F(t^*, x^*, x^*) + C$, i.e., $x^* \in M_y$ and hence M_y is closed.

Corollary 4.1 improves Thorem 3.3 in [18] since the assumptions are strictly weaker as shown by the following example.

Example 4.1. Let $X = Y = Z = R$, $A = [0,1]$, $S_1(x) = S_2(x) = [0,1]$, $C(x) \equiv R_+$ and

$$T(x,y) = \begin{cases} [-2,-1.5] & \text{if } x = 0.5, \\ [-1,-0.5] & \text{otherwise}, \end{cases}$$

$$F(t,y,x) = \begin{cases} -1 & \text{if } y = x = 0.5, \\ 0 & \text{if } y = x \neq 0.5, \\ 0.5 & \text{if } y = 0.5, x \neq 0.5, \\ 1 & \text{if } y \neq 0.5, y \neq x. \end{cases}$$

It is clear that $T(.,y)$ is not lsc, $F(.,y,.)$ is not lower $(-C)$-continuous and $(t,x) \mapsto F(t,x,x)$ is not upper C-continuous. Hence Theorem 3.3 of [18] does not work. But $M_y \equiv [0,1]$ is closed. So, it is not hard to see that all assumptions of Corollary 4.1 are satisfied. So by this corollary the considered problem has solution. By direct checking one sees that the solution set is [0,1].

We now consider the following quasioptimization problem, see e.g. [11],

(QOP) find $\bar{x} \in S(\bar{x})$ such that, $\forall \bar{t} \in K(\bar{x})$,

$$F(\bar{t},\bar{x},\bar{x}) \bigcap \mathrm{Min}\{F(\bar{t},S(\bar{x}),\bar{x})/C\} \neq \emptyset,$$

where A and F are as in Section 1, $C \subseteq Y$ is a closed cone, $S : A \to 2^X$, $K : A \to 2^Z$

and $\mathrm{Min}\{H/C\}$ denotes the set of the Pareto efficient points of set $H \subseteq Y$ (with respect to the ordering cone C).

As a consequence of Corollary 4.1 we obtain the following result.

Corollary 4.2. *Assume that Y is locally convex, C is pointed, there is a bounded set $M \subseteq Y'$ with $C' = coneM$ and $F(t,x,x)$ is compact for all $(t,x) \in T(A) \times A$. Assume also the following conditions:*

(i) *A is nonempty and compact;*

(ii) *$S(.)$ is closed, $S(x)$ is nonempty and convex, $A \cap S(x) \neq \emptyset$ and $S^{-1}(y)$ is open in A, for all $x, y \in A$;*

(iii) *F is $(F+C)$-quasiconvex with respect to K of type 1;*

(iv) *for each $y \in A$, $\{x \in A : \forall t \in K(x),\quad F(t,y,x) \subseteq F(t,x,x) + C\}$ is closed.*

Then, (QOP) *has a solution.*

Proof. Applying Corollary 4.1 with $S_1(x) = S_2(x) = S(x), T(x,y) = K(x)$ one has $\bar{x} \in S(\bar{x})$ such that, $\forall y \in S(\bar{x}), \forall \bar{t} \in K(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \subseteq F(\bar{t}, \bar{x}, \bar{x}) + C. \qquad (8)$$

By the compactness of $F(\bar{t}, \bar{x}, \bar{x})$, $\mathrm{Min}F(\bar{t}, \bar{x}, \bar{x})/C \neq \emptyset$. Suppose that $\bar{v} \in \mathrm{Min}\{F(\bar{t}, \bar{x}, \bar{x})/C\}$ but $\bar{v} \notin \mathrm{Min}\{F(\bar{t}, K(\bar{x}), \bar{x})/C\}$. One has then $y \in F(\bar{t}, K(\bar{x}), \bar{x})$ such that

$$\bar{v} - y \in C \backslash ((-C) \cap C).$$

By virtue of (8) $y \in F(\bar{t}, \bar{x}, \bar{x}) + C$, i.e. $y = \hat{v} + c$ for some $\hat{v} \in F(\bar{t}, \bar{x}, \bar{x})$ and $c \in C$.

Therefore $\bar{v} - \hat{v} \in c + C \backslash ((-C) \cap C) = C \backslash ((-C) \cap C)$, contradicting the fact that $\bar{v} \in \text{Min}\{F(\bar{t}, \bar{x}, \bar{x})/C\}$. $\square$

Corollary 4.2 has strictly weaker assumptions than Proposition 4.1 in [18].

As the next example of applications consider the following vector equilibrium problem. Let $F : A \times A \to 2^Y, C : A \to 2^Y$.

(EP1) Find $\bar{x} \in A$ such that, $\forall y \in A$,

$$F(y, \bar{x}) \subseteq C(\bar{x}).$$

Corollary 4.3. *For* (EP1) *assume that*

(i) *A is nonempty and compact;*

(ii) *F is C-quasiconvex with respect to T, where $T(x) = \{x\}$;*

(iii) *for each $y \in A$, $\{x \in A : F(y, x) \subseteq C(x)\}$ is closed.*

Then, (EP1) *has a solution.*

Proof. Applying Theorem 3.1 with $S_1(x) = S_2(x) = A, T(x, y) = \{x\}, G(t, x, x) = C(x)$, we have the conclusion. $\square$

Remark 4.2. If A is not compact, then Corollary 4.3 still holds true under the following additional coersivity condition:

(iv) there exists a nonempty compact subset $D \subseteq A$ such that for each finite subset $M \subseteq A$, there is a compact convex subset L_M of A, containing M, such that $\forall x \in L_M \backslash D, \exists y \in L_M,\ F(y, x) \not\subseteq C(x)$.

Therefore Corollary 4.3 has weaker assumptions than Theorem 4.2 of [17].

Finally we apply Theorem 3.4 to problem (IVI) stated in Section 1.

Corollary 4.4. *For* (IVI) *assume that*

(i) *A is nonempty and compact;*

(ii) *F is $(Y\backslash -intC)$-quasiconvexlike with respect to T of type 2;*

(iii) *for each $y \in A$, $\{x \in A : \exists t \in T(x), F(t, y, x) \notin -\ intC(x)\}$ is closed.*

Then, (IVI) *has a solution.*

Proof. Apply Theorem 3.4 with $S_1(x) = S_2(x) = A$, $T(x, y) = \{x\}$ and $G(t, y, x) = Y\backslash -\text{int}C(x)$. □

Remark 4.3. Corollary 4.4 is different from the results in [14,15] since assumption (ii) on generalized convexity is different from the corresponding assumptions in [14,15]. However, the other assumptions of Corollary 4.4 are weaker than the corresponding ones in [14,15]. The following example gives a case where Corollary 4.4 is easily applied, but the theorems in [14,15] do not work.

Example 4.2. Let $X = Y = Z = R$, $A = [0, 1], C(x) = R_+$,

$$T(x) = \begin{cases} [0.5, 1] & \text{if } x = 0.5, \\ [0, 2] & \text{otherwise,} \end{cases}$$

$$F(t, y, x) = \begin{cases} 1 - y^2 & \text{if } y \leq 0.5, \\ xy & \text{if } y > 0.5. \end{cases}$$

It is clear that all assumptions of Corollary 4.4 are fulfilled. But the theorems in [14,15] cannot be applied, since $F(.,y,.)$ is not continuous and $F(t,.,x)$ is not C-convex.

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