SEMICONTINUITY OF THE APPROXIMATE SOLUTION SETS OF MULTIVALUED QUASIEQUILIBRIUM PROBLEMS *

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We consider two kinds of approximate solutions and approximate solution sets to six variants of multivalued quasiequilibrium problems. Sufficient conditions for the lower semicontinuity, upper semicontinuity, Hausdorff upper semicontinuity and closedness of these approximate solution sets are established. Applications in approximate quasivariational inequalities, approximate fixed points and approximate quasioptimization problems are provided.

Keywords: Lower semicontinuity; Upper semicontinuity; Hausdorff upper semicontinuity and closedness of multifunctions; ε-solutions; Quasiequilibrium problems; Quasivariational inequalities; ε-fixed points; ε-quasioptimization problems

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1. INTRODUCTION AND PRELIMINARIES

Among various meanings of stability and sensitivity of the solutions of a problem the semicontinuity has been increasingly interested recently in the literature. Upper semicontinuity is investigated in [10, 16 − 18]for variational inequalities and in [1, 3, 6, 22] for equilibrium problems. Lower semicontinuity is studied in [23] for minimization problems, in [10, 17, 18] for variational inequalities and in [1, 3] for equilibrium problems. Beside semicontinuity we observe only [2, 6, 21], which deal with stability of equilibrium problems, where Hölder continuity is investigated.

On the other hand, in many practical problems, exact solutions do not exist since the data of the problems are not sufficient "regular". Moreover, the data of the problems have often been obtained approximately by measure devices or by statistical results. Mathematical models of practical problems describe real situations also approximately and hence the existence of exact solutions of mathematical models may become unimportant. That is why approximate solutions are of real interest. Observing that multivalued quasiequilibrium problems are rather general problems, which include quasivariational inequalities, complementarity problems, fixed point and coincidence point problems, optimization, Nash equilibrium problems, etc as special cases, in this note we consider the semicontinuity properties of the approximate solutions of quasiequilibrium problems. In particular, many results of [17] are improved and extended to more general settings.

Throughout the paper, unless otherwise stated, let X, M, N and Λ be Hausdorff topological spaces and Y be a metric vector space with invariant metric $d(.,.)$. Let $K : X \times \Lambda \to 2^X$, $G : X \times N \to 2^X$ and $F : X \times X \times M \to 2^Y$

be multifunctions. Let $C \subseteq Y$ be closed with nonempty interior and $C \neq Y$. We consider the following general parametric multivalued vector quasiequilibrium problems formulated in [3], for each $\lambda \in \Lambda$, $\mu \in M$, $\eta \in N$:

(WQEP$_1$) find $\bar{x} \in \mathrm{cl}K(\bar{x},\lambda)$ such that, for each $x \in K(\bar{x},\lambda)$, there exists $\bar{x}^* \in G(\bar{x},\eta)$,

$$F(\bar{x}^*, x, \mu) \cap (Y \setminus -\mathrm{int}C) \neq \emptyset;$$

(MQEP$_1$) find $\bar{x} \in \mathrm{cl}K(\bar{x},\lambda)$ such that, there exists $\bar{x}^* \in G(\bar{x},\eta)$, for all $x \in K(\bar{x},\lambda)$,

$$F(\bar{x}^*, x, \mu) \cap (Y \setminus -\mathrm{int}C) \neq \emptyset;$$

(SQEP$_1$) find $\bar{x} \in \mathrm{cl}K(\bar{x},\lambda)$ such that, for each $x \in K(\bar{x},\lambda)$ and $\bar{x}^* \in G(\bar{x},\eta)$,

$$F(\bar{x}^*, x, \mu) \cap (Y \setminus -\mathrm{int}C) \neq \emptyset;$$

(WQEP$_2$) find $\bar{x} \in \mathrm{cl}K(\bar{x},\lambda)$ such that, for each $x \in K(\bar{x},\lambda)$, there exists $\bar{x}^* \in G(\bar{x},\eta)$,

$$F(\bar{x}^*, x, \mu) \subseteq Y \setminus -\mathrm{int}C;$$

(MQEP$_2$) find $\bar{x} \in \mathrm{cl}K(\bar{x},\lambda)$ such that, there exists $\bar{x}^* \in G(\bar{x},\eta)$, for all $x \in K(\bar{x},\lambda)$,

$$F(\bar{x}^*, x, \mu) \subseteq Y \setminus -\mathrm{int}C;$$

(SQEP$_2$) find $\bar{x} \in \mathrm{cl}K(\bar{x},\lambda)$ such that, for each $x \in K(\bar{x},\lambda)$ and $\bar{x}^* \in G(\bar{x},\eta)$,

$$F(\bar{x}^*, x, \mu) \subseteq Y \setminus -\mathrm{int}C;$$

where cl(.) and int(.) stand for the closure and the interior, respectively, of the set (.). "W", "M" and "S" would be "Weak", "Middle" and "Strong", respectively.

For $\lambda \in \Lambda$, $\mu \in M$ and $\eta \in N$ we denote the set of the solutions of (WQEP$_1$), (MQEP$_1$), (SQEP$_1$), (WQEP$_2$), (MQEP$_2$) and (SQEP$_2$) by $S_1(\lambda,\mu,\eta)$, $S_2(\lambda,\mu,\eta)$,

$S_3(\lambda,\mu,\eta)$ $S_4(\lambda,\mu,\eta)$, $S_5(\lambda,\mu,\eta)$ and $S_6(\lambda,\mu,\eta)$, respectively.

Recall first some notions. Let X and Y be as above and $G : X \to 2^Y$ be a multifunction. G is said to be lower semicontinuous (lsc) at x_0 if $G(x_0) \cap U \neq \emptyset$ for some open set $U \subseteq Y$ implies the existence of a neighborhood N of x_0 such that, for all $x \in N, G(x) \cap U \neq \emptyset$. An equivalent formulation is that: G is lsc at x_0 if $\forall x_\alpha \to x_0$, $\forall y \in G(x_0), \exists y_\alpha \in G(x_\alpha), y_\alpha \to y$. G is called upper semicontinuous (usc) at x_0 if for each open set $U \supseteq G(x_0)$, there is a neighborhood N of x_0 such that $U \supseteq G(N)$. G is termed Hausdorff upper semicontinuous (H-usc) at x_0 if for each neighborhood B of the origin in Y, there is a neighborhood N of x_0 such that $G(N) \subseteq G(x_0) + B$. G is said to be continuous at x_0 if it is both lsc and usc at x_0 and to be H-continuous at x_0 if it is both lsc and H-usc at x_0. G is called closed at x_0 if for each net $(x_\alpha, y_\alpha) \in \text{graph}G := \{(x,y) \mid y \in G(x)\}, (x_\alpha, y_\alpha) \to (x_0, y_0)$, y_0 must belong to $G(x_0)$. The closedness is closely related to the upper (and Hausdorff upper) semicontinuity (see e.g. [1], Preposition 3.1). We say that G satisfies a certain property in a subset $A \subseteq X$ if G satisfies it at every points of A. If $A = \text{dom}G := \{x \in X : G(x) \neq \emptyset\}$ we omit "in dom$G$" in the statement.

We propose the following two definitions of ε-solutions. Let us use the notations

$$\text{comp}(-\text{int}C)_1^\varepsilon = \big\{ y \in Y : d(y, Y \setminus -\text{int}C) \leq \varepsilon \big\},$$

$$\text{comp}(-\text{int}C)_2^\varepsilon = (Y \setminus -\text{int}C) + B_Y^\varepsilon,$$

where $B_Y^\varepsilon = \big\{ y \in Y \mid d(0,y) \leq \varepsilon \big\}$ and the notation "comp(.)" is related to the word "complement".

Definition 1.1

(a) $\bar{x} \in X$ is called an $\varepsilon-$solution of type 1 of problem (WQEP$_1$) if $\bar{x} \in$

cl$K(\bar{x},\lambda)$ and $\forall y \in K(\bar{x},\lambda)$, $\exists \bar{x}^* \in G(\bar{x},\eta)$,

$$F(\bar{x}^*, y, \mu) \cap \text{comp}(-\text{int}C)_1^\varepsilon \neq \emptyset. \tag{1}$$

(b) If (1) is replaced by

$$F(\bar{x}^*, y, \mu) \cap \text{comp}(-\text{int}C)_2^\varepsilon \neq \emptyset,$$

then $\bar{x}$ is said to be an $\varepsilon-$solution of type 2 of problem (WQEP$_1$).

(c) $\varepsilon-$solutions of the other five problems are defined similarly.

Remark 1.1

(a) If $\bar{x}$ is an $\varepsilon-$solution of type 2 then $\bar{x}$ is an $\varepsilon-$solution of type 1. Indeed, it suffices to show that

$$\text{comp}(-\text{int}C)_2^\varepsilon \subseteq \text{comp}(-\text{int}C)_1^\varepsilon.$$

We check a more general fact that $A+B_Y^\varepsilon \subseteq A^\varepsilon := \{y \in Y \mid d(y,A) \leq \varepsilon\}$. Let $y \in A + B_Y^\varepsilon$, i.e. $y = a + z$ for some $a \in A$ and $z \in B_Y^\varepsilon$. Then $d(y,a) = d(y-a,0) = d(z,0) \leq \varepsilon$, i.e. $y \in A^\varepsilon$.

(b) If Y is finite dimensional, then the two types of $\varepsilon-$solutions coincide. Indeed, it suffices to check that $A^\varepsilon \subseteq A+B_Y^\varepsilon$, while A is closed. Let $y \in A^\varepsilon$ be arbitrary. Then, for each n, there is $a_n \in A$ with $d(y,a_n) \leq \varepsilon + \frac{1}{n}$. We have some $y_n \in B_Y^\varepsilon$ with $d(y-a_n-y_n,0) = d(y-a_n,y_n) \leq \frac{2}{n}$. By the compactness of B_Y^ε we can assume that $y_n \to y_0 \in B_Y^\varepsilon$. Let $a = y - y_0$, $u_n = y - a_n - y_0 = a - a_n$ and $v_n = y - a_n - y_n$. Then,

$$0 \leq d(a_n,a) = d(u_n,0) \leq d(u_n - v_n, 0) + d(v_n,0) \leq d(y_n - y_0, 0) + \frac{2}{n}.$$

Therefore, $a_n \to a$ and hence $a \in A$. Thus $y = a + y_0 \in A + B_Y^\varepsilon$. ∎

The following two examples show that if Y is infinite dimensional, then in general an $\varepsilon-$solution of type 1 is not guaranteed to be an $\varepsilon-$solution type 2.

Example 1.1 Let $Y = l^\infty, A = \{x^k\}$, where $x_1^k = 1 + \frac{1}{k}, x_{k+1}^k = 1$ and $x_j^k = 0$, $\forall j \neq 1$ and $j \neq k+1$. Then $\|x^k - x^l\| = 1$ if $k \neq l$ and hence A is closed. Taking $y^k \in B_Y^1$ with $y_1^k = -1, y_{k+1}^k = -1$ and $y_j^k = 0$ if $j \neq 1$ and $j \neq k+1$, we have $x^k + y^k = (\frac{1}{k}, 0, 0, ...) \in A + B_Y^1$ and $x^k + y^k \to 0 \notin A + B_Y^1$. Thus, $A + B_Y^1$ is not closed and hence $A + B_Y^1$ is included properly in A^1.

The next example is more complicated but provides the case where A has the form $Y \setminus -\text{int}C$ with $C \neq Y$ as in Definition 1.1.

Example 1.2 Let $Y = l^\infty$. Let

$$U_1 = \left\{ u \in l^\infty \;\middle|\; \frac{1}{2} \leq u_1 \leq 1, 2 \leq u_2 \leq 3 \text{ and } 0 \leq u_k \leq 1 \text{ for } k \geq 3 \right\},$$

$$U_2 = \left\{ u \in l^\infty \;\middle|\; \frac{1}{4} \leq u_1 \leq \frac{1}{2}, 0 \leq u_2 \leq 1, 2 \leq u_3 \leq 3 \text{ and } 0 \leq u_k \leq 1 \text{ for } k \geq 4 \right\},$$

$$\ldots$$

$$U_n = \left\{ u \in l^\infty \;\middle|\; \frac{1}{2^n} \leq u_1 \leq \frac{1}{2^{n-1}}, 2 \leq u_{n+1} \leq 3 \text{ and } 0 \leq u_k \leq 1 \text{ for } k \neq 1, k \neq n+1 \right\}.$$

Then clearly U_n are closed for all n and $d(U_n, U_m) = 1$ if $n \neq m$. Let $A = \bigcup_{n=1}^\infty U_n$. We claim that A is closed. Indeed, assume that $a^i \in A$, $a^i \to a^0$. Since $d(U_n, U_m) = 1$ for $n \neq m$, there exist i_0 and n_0 such that $a^i \in U_{n_0}$ for all $i \geq i_0$. Hence $a^0 \in U_{n_0} \subseteq A$. To see that $A + B_Y^1$ is not closed take sequences $a^k \in A$ and $b^k \in B_Y^1$ with components

$$a_1^k = \frac{1}{2^k}, a_{k+1}^k = 2 \text{ and } a_j^k = 1, \forall j \neq 1, \forall j \neq k+1,$$

$$b_1^k = -1, b_{k+1}^k = -1 \text{ and } b_j^k = 0, \forall j \neq 1, \forall j \neq k+1.$$

Then $a^k + b^k = (-1 + \frac{1}{2^k}, 1, 1, ...) \to (-1, 1, 1, ...) \notin A + B_Y^1$. Thus $A + B_Y^1$ is not closed and contained properly in A^1. Now taking $-C = \mathrm{cl}(l^\infty \setminus A)$ we get $C \neq l^\infty$ and $A = l^\infty \setminus -\mathrm{int}C$ as wanted.

Definition 1.2

(a) The set of $\varepsilon-$solutions of type 1 (of type 2) of problem (WQEP_1) at (λ, μ, η) is denoted by $S_1^{\varepsilon 1}(\lambda, \mu, \eta)$ $\big(S_1^{\varepsilon 2}(\lambda, \mu, \eta), \text{respectively}\big)$. The sets of $\varepsilon-$ solutions of the other five problems are denoted by $S_2^{\varepsilon 1}$ and $S_2^{\varepsilon 2}$, ..., $S_6^{\varepsilon 1}$ and $S_6^{\varepsilon 2}$, respectively in the given order.

(b) Another kind of $\varepsilon-$solution sets is defined, e.g. for type 1, by

$$\widetilde{S_i^{\varepsilon 1}}(\lambda, \mu, \eta) = \begin{cases} S_i(\lambda_0, \mu_0, \eta_0) & \text{if } \quad (\lambda, \mu, \eta) = (\lambda_0, \mu_0, \eta_0), \\ S_i^{\varepsilon 1}(\lambda, \mu, \eta) & \text{otherwise,} \end{cases}$$

where $(\lambda_0, \mu_0, \eta_0)$ is the point under consideration.

We propose the following weak semicontinuity.

Definition 1.3 Let X be a topological space and Y be a topological vector space, and $C \subseteq Y$ with $\mathrm{int}C \neq \emptyset$ and $C \neq Y$.

(a) A multifunction $H : X \longrightarrow 2^Y$ is said to be $C-$quasiupper semicontinuous ($C-$qusc) at x_0 if, for any $x_\alpha \to x_0$, $H(x_0) \subseteq \mathrm{int}C \Rightarrow \exists \bar{\alpha}, H(x_{\bar{\alpha}}) \subseteq \mathrm{int}C$.

(b) H is called $C-$quasilower semicontinuous ($C-$qlsc) at x_0 if, for all $x_\alpha \to x_0$, $H(x_0) \cap \mathrm{int}C \neq \emptyset \Rightarrow \exists \bar{\alpha}, H(x_{\bar{\alpha}}) \cap \mathrm{int}C \neq \emptyset$.

H is said to be $C-$quasicontinuous at x_0, if H is both $C-$qusc and $C-$qlsc at x_0.

(c) H is termed $C-$ Hausdorff quasiupper semicontinuous ($C-$Hqusc) at x_0 if, for any $x_\alpha \to x_0$ and B (open neighborhood of 0 in Y), $H(x_0)+B \subseteq \text{int}C \Rightarrow \exists \bar{\alpha}, H(x_{\bar{\alpha}}) \subseteq \text{int}C$.

Remark 1.2

(a) $C-$quasiupper semicontinuity property is strictly stronger than $C-$Hausdorff quasiupper semicontinuity property;

(b) H is $C-$qusc, $C-$qlsc, $C-$Hqusc at x_0 if and only if H is int$C-$qusc, int$C-$qlsc, int$C-$Hqusc, respectively;

(c) H is usc at x_0, iff H is $C-$qusc at x_0 for all subsets C of Y;

(d) H is lsc at x_0, iff H is $C-$qlsc at x_0 for all subsets C of Y.

Since the solution existence of quasiequilibrium problems has been intensively investigated, see e.g. [4, 5, 7, 9, 12, 14, 15, 19, 20], we do not study this issue, and instead always assume the existence.

The organization of the paper is as follows. In Section 2 we give sufficient conditions for the approximate solution sets of all the problems to be lower semicontinuous at the point in question. In Section 3 we investigate the sufficient conditions for the approximate solution sets to be upper semicontinuous in each of the three senses. Section 4 is devoted to applications in approximate quasivariational inequalities, approximate fixed points and approximate quasioptimization problems.

2. LOWER SEMICONTINUITY OF THE $\varepsilon-$SOLUTION SETS

Considering the lower semicontinuity of approximate solution sets for our six quasiequilibrium problems we will see below that the sufficient conditions for this lower semicontinuity are the same for the two types of $\varepsilon-$solutions stated in Definition 1.1. The reason is that an $\varepsilon-$solution of type 1, which is not an $\varepsilon-$solution of type 2, much lie on the boundary of comp$(-\text{int})C_1^\varepsilon$, since this set may have more points than comp$(-\text{int})C_2^\varepsilon$ only in the boundary (see Remark 1.1 and Example 1.2). This difference does not affect the lower semicontinuity. But it affects the upper semicontinuity as the next section makes it clear. Moreover, we will see that under usual assumptions lower semicontinuity holds only for the $\varepsilon-$solution sets of the second kind $\widetilde{S_i^{\varepsilon 1}}$ and $\widetilde{S_i^{\varepsilon 2}}$, $i = 1, ..., 6$ (see Definition 1.2) while upper semicontinuity occurs only for the $\varepsilon-$solution sets of the first kind $S_i^{\varepsilon 1}$ and $S_i^{\varepsilon 2}$, $i = 1, ..., 6$ (see Definition 1.2).

In the sequel let, for $\lambda \in \Lambda$,

$$E(\lambda) = \big\{x \in X \mid x \in \text{cl}K(x, \lambda)\big\}.$$

We always assume that $S_i(\lambda, \mu, \eta) \neq \emptyset, i = 1, ..., 6$, for all (λ, μ, η) in a neighborhood of the considered point $(\lambda_0, \mu_0, \eta_0)$.

Theorem 2.1 *Assume that $K(.,.)$ is usc and has compact values in $X \times \{\lambda_0\}$, $E(.)$ is lsc at λ_0 and $F(.,.,.)$ is* comp$(-\text{int}C)_1^\varepsilon-$ *qlsc or* comp$(-\text{int}C)_2^\varepsilon-$ *qlsc in $X \times X \times \{\mu_0\}$. Then the following assertions hold.*

(i) *If $G(.,.)$ is lsc in $X \times \{\eta_0\}$, then $\widetilde{S_1^{\varepsilon 1}}(.,.,.)$, $\widetilde{S_1^{\varepsilon 2}}(.,.,.)$, $\widetilde{S_2^{\varepsilon 1}}(.,.,.)$ and $\widetilde{S_2^{\varepsilon 2}}(.,.,.)$ are lsc at $(\lambda_0, \mu_0, \eta_0)$ for each $\varepsilon > 0$.*

(ii) *If $G(.,.)$ is usc and has compact values in $X \times \{\eta_0\}$, then $\widetilde{S_3^{\varepsilon 1}}(.,.,.)$ and $\widetilde{S_3^{\varepsilon 2}}(.,.,.)$ are lsc at $(\lambda_0, \mu_0, \eta_0)$ for each $\varepsilon > 0$.*

Proof Since the six assertions for the two types of quasilower semicontinuity can be proved by a similar technique we present a proof only for $\widetilde{S_1^{\varepsilon 1}}(.,.,.)$ of problem

(WQEP$_1$). Let $\varepsilon > 0$ be fixed. Suppose that $\widetilde{S_1^{\varepsilon 1}}(.,.,.)$ is not lsc at $(\lambda_0, \mu_0, \eta_0)$, i.e., $\exists x_0 \in \widetilde{S_1^{\varepsilon 1}}(\lambda_0, \mu_0, \eta_0)$, $\exists (\lambda_\alpha, \mu_\alpha, \eta_\alpha) \to (\lambda_0, \mu_0, \eta_0)$, $\forall x_\alpha \in \widetilde{S_1^{\varepsilon 1}}(\lambda_\alpha, \mu_\alpha, \eta_\alpha)$, $x_\alpha \not\to x_0$. Since $x_0 \in \widetilde{S_1^{\varepsilon 1}}(\lambda_0, \mu_0, \eta_0) = S_1(\lambda_0, \mu_0, \eta_0)$, $\forall y_0 \in K(x_0, \lambda_0)$, $\exists x_0^* \in G(x_0, \eta_0)$,

$$F(x_0^*, y_0, \mu_0) \cap (Y \setminus -\text{int}C) \neq \emptyset. \tag{2}$$

By the lower semicontinuity of $E(.)$ at λ_0, there is a net $\bar{x}_\alpha \in E(\lambda_\alpha)$, $\bar{x}_\alpha \to x_0$. By the above contradiction assumption, there must be a subnet $\bar{x}_\beta$ such that, $\forall \beta$, $\bar{x}_\beta \notin \widetilde{S_1^{\varepsilon 1}}(\lambda_\beta, \mu_\beta, \eta_\beta)$, i.e., $\exists y_\beta \in K(\bar{x}_\beta, \lambda_\beta)$, $\forall \bar{x}_\beta^* \in G(\bar{x}_\beta, \eta_\beta)$,

$$F(\bar{x}_\beta^*, y_\beta, \mu_\beta) \cap \text{comp}(-\text{int}C)_1^\varepsilon = \emptyset. \tag{3}$$

Since $K(.,.)$ is usc and $K(x_0, \lambda_0)$ is compact, one can assume that $y_\beta \to y_0$, for some $y_0 \in K(x_0, \lambda_0)$. By the lower semicontinuity of $G(.,.)$ at (x_0, η_0), there exists $\bar{x}_\beta^* \in G(\bar{x}_\beta, \eta_\beta)$, $\bar{x}_\beta^* \to x_0^*$. By the $\text{comp}(-\text{int}C)_1^\varepsilon-$ quasilower semicontinuity of $F(.,.,.)$ at (x_0^*, y_0, μ_0) and $Y \setminus \text{int}C \subset \text{int}\big(\text{comp}(-\text{int}C)_1^\varepsilon\big)$, we see a contradiction between (2) and (3). ∎

The following example shows that Theorem 2.1 is no longer true if we replace the $\varepsilon-$solution sets of the second kind by that of the first kind.

Example 2.1 Let $X = Y = R, \Lambda \equiv M \equiv N = [0,1], C = R_+$, $K(\lambda) = [0,1]$, $G(x, \lambda) = \{x\}$, $\lambda_0 = 0$, and $F(x, y, \lambda) = \{y - x - \varepsilon - \lambda, x - y\}$. Since $G(.,.)$ is single-valued, (WQEP$_1$), (MQEP$_1$) and (SQEP$_1$) are equivalent. From now on if Y is finite dimensional (then the two types of $\varepsilon-$solutions coincide), instead of writing $\varepsilon 1$ or $\varepsilon 2$ in the index of $\varepsilon-$solution sets, we write simply ε, e.g. $S_i^\varepsilon, \widetilde{S_i^\varepsilon}$. It is easy to see that the assumptions of Theorem 2.1 are fulfilled and according to Theorem 2.1, $\widetilde{S_i^\varepsilon}(.)$ are lsc at 0 (in fact $S_i(0) = \{1\}$, $S_i^\varepsilon(\lambda) = [1-\varepsilon, 1]$, $\forall \lambda \in (0,1]$), $\forall i = 1,2,3$. But $S_i^\varepsilon(0) = \{0,1\}$. So $S_i^\varepsilon(.)$ are not lsc at 0, $i = 1,2,3$.

THEOREM 2.2 *Assume that $K(.,.)$ is usc and has compact values in $X \times \{\lambda_0\}$, $E(.)$ is lsc at λ_0 and $F(.,.,.)$ is* $\mathrm{comp}(-\mathrm{int})C_1^\varepsilon-$ *qusc or* $\mathrm{comp}(-\mathrm{int})C_2^\varepsilon-$ *qusc in $X \times X \times \{\mu_0\}$.*

(i) *If $G(.,.)$ is lsc in $X \times \{\eta_0\}$, then $\widetilde{S_4^{\varepsilon 1}}(.,.,.)$, $\widetilde{S_4^{\varepsilon 2}}(.,.,.)$, $\widetilde{S_5^{\varepsilon 1}}(.,.,.)$ and $\widetilde{S_5^{\varepsilon 2}}(.,.,.)$ are lsc at $(\lambda_0, \mu_0, \eta_0)$ for each $\varepsilon > 0$.*

(ii) *If $G(.,.)$ is usc and has compact values in $X \times \{\eta_0\}$, then $\widetilde{S_6^{\varepsilon 1}}(.,.,.)$ and $\widetilde{S_6^{\varepsilon 2}}(.,.,.)$ are lsc at $(\lambda_0, \mu_0, \eta_0)$ for each $\varepsilon > 0$.*

Proof As an example we demonstrate only for problem (SQEP_2). Let $\varepsilon > 0$ be fixed. Suppose to the contrary that $\widetilde{S_6^{\varepsilon 2}}(.,.,.)$ is not lsc at $(\lambda_0, \mu_0, \eta_0)$, i.e., $\exists x_0 \in \widetilde{S_6^{\varepsilon 2}}(\lambda_0, \mu_0, \eta_0)$, $\exists \lambda_\alpha \to \lambda_0, \exists \mu_\alpha \to \mu_0, \exists \eta_\alpha \to \eta_0$, $\forall x_\alpha \in \widetilde{S_6^{\varepsilon 2}}(\lambda_\alpha, \mu_\alpha, \eta_\alpha)$, $x_\alpha \not\to x_0$. Since $x_0 \in \widetilde{S_6^{\varepsilon 2}}(\lambda_0, \mu_0, \eta_0) = S_6(\lambda_0, \mu_0, \eta_0)$, $\forall y_0 \in K(x_0, \lambda_0)$, $\forall x_0^* \in G(x_0, \eta_0)$,

$$F(x_0^*, y_0, \mu_0) \subseteq Y \setminus -\mathrm{int}C. \quad (4)$$

By the lower semicontinuity of $E(.)$ at λ_0, there is a net $\bar{x}_\alpha \in E(\lambda_\alpha)$, $\bar{x}_\alpha \to x_0$. By the above contradiction assumption, there must be a subnet $\bar{x}_\beta$ such that, $\forall \beta$, $\bar{x}_\beta \notin \widetilde{S_6^{\varepsilon 2}}(\lambda_\beta, \mu_\beta, \eta_\beta)$, i.e., $\exists y_\beta \in K(\bar{x}_\beta, \lambda_\beta)$, $\exists \bar{x}_\beta^* \in G(\bar{x}_\beta, \eta_\beta)$,

$$F(\bar{x}_\beta^*, y_\beta, \mu_\beta) \nsubseteq \mathrm{comp}(-\mathrm{int}C)_2^\varepsilon. \quad (5)$$

Since $K(.,.)$ is usc and $K(x_0, \lambda_0)$ is compact, one can assume that $y_\beta \to y_0$, for some $y_0 \in K(x_0, \lambda_0)$. By the upper semicontinuity and the compactness of $G(.,.)$ at (x_0, η_0), one can assume that $\bar{x}_\beta^* \to x_0^*$ for some $x_0^* \in G(x_0, \eta_0)$. By (4) and the $\mathrm{comp}(-\mathrm{int}C)_2^\varepsilon-$ quasiupper semicontinuity of $F(.,.,.)$ at (x_0^*, y_0, μ_0), one has an index $\bar{\beta}$ such that

$$F(\bar{x}_{\bar{\beta}}^*, y_{\bar{\beta}}, \mu_{\bar{\beta}}) \subseteq \mathrm{comp}(-\mathrm{int}C)_2^\varepsilon,$$

which contradicts (5). ■

The following two examples ensure that Theorem 2.2 is no longer true if we replace $\widetilde{S_4^{\varepsilon k}}(.,.,.)$, $\widetilde{S_5^{\varepsilon k}}(.,.,.)$ and $\widetilde{S_6^{\varepsilon k}}(.,.,.)$ by $S_4^{\varepsilon k}(.,.,.)$, $S_5^{\varepsilon k}(.,.,.)$ and $S_6^{\varepsilon k}(.,.,.)$, $k = 1, 2$, respectively.

EXAMPLE 2.2 Let $\varepsilon > 0$ be fixed and small. Let $X = Y = R$, $\Lambda \equiv M \equiv N = [0, 1]$, $C = R_+$, $K(x, \lambda) = [\lambda, \lambda + 1]$, $G(x, \lambda) = \{-x + 1 - \varepsilon, x\}$, $F(x, y, \lambda) = \{x - 1 - \lambda\}$ and $\lambda_0 = 0$. Then, it is clear that the assumptions of Theorem 2.2, (i) hold. Direct calculations yield $S_4(0) = S_5(0) = \{1\}$, $S_4^{\varepsilon}(0) = S_5^{\varepsilon}(0) = \{0\} \cup [1 - \varepsilon, 1]$ and $S_4^{\varepsilon}(\lambda) = S_5^{\varepsilon}(\lambda) = [1 + \lambda - \varepsilon, 1 + \lambda]$, $\forall \lambda \in (0, 1]$. So, $\widetilde{S_4^{\varepsilon}}(.)$ and $\widetilde{S_5^{\varepsilon}}(.)$ are lsc at 0, while $S_4^{\varepsilon}(.)$ and $S_5^{\varepsilon}(.)$ are not lsc at 0.

EXAMPLE 2.3 Let $\varepsilon > 0$ be fixed and small. Let $X = Y = R$, $\Lambda \equiv M \equiv N = [0, 1]$, $C = R_+$, $K(x, \lambda) = \left[0, \frac{1+\varepsilon+\sqrt{(1+\varepsilon)^2+4\lambda}}{2}\right]$, $G(x, \lambda) = \{x\}$, $F(x, y, \lambda) = \{x^2 - (1 - \varepsilon)x - \varepsilon - \lambda\}$ and $\lambda_0 = 0$. Then, it is not hard to see that the assumptions of Theorem 2.2, (ii) hold and $S_6(0) = \{1 + \varepsilon\}$, $S_6^{\varepsilon}(0) = \{0, 1 + \varepsilon\}$ and $S_6^{\varepsilon}(\lambda) = \left\{\frac{1+\varepsilon+\sqrt{(1+\varepsilon)^2+4\lambda}}{2}\right\}$, $\forall \lambda \in (0, 1]$. Hence, $\widetilde{S_6^{\varepsilon}}(.)$ is lsc at 0, while $S_6^{\varepsilon}(.)$ is not lsc at 0.

3. UPPER SEMICONTINUITY OF THE $\varepsilon-$SOLUTION SETS

In this section we consider upper semicontinuity properties in three senses: upper semicontinuity, Hausdorff upper semicontinuity and closedness. We will establish sufficient conditions for the $\varepsilon-$solutions of type 1 of the six problems under our consideration to have these upper semicontinuity properties. We will also see that these results hold only for the $\varepsilon-$solution sets of the first kind.

THEOREM 3.1 *Assume that $K(.,.)$ is lsc in $X \times \{\lambda_0\}$, $E(.)$ is usc, $E(\lambda_0)$ is compact and $F(.,.,.)$ is $Y \backslash \mathrm{comp}(-\mathrm{int}C)_1^{\varepsilon}-$qusc in $X \times X \times \{\mu_0\}$.*

(i) *If $G(.,.)$ is usc and has compact values in $X \times \{\eta_0\}$, then $S_1^{\varepsilon 1}(.,.,.)$ and $S_2^{\varepsilon 1}(.,.,.)$ are both usc and closed at $(\lambda_0, \mu_0, \eta_0)$ for each $\varepsilon \geq 0$.*

(ii) *If $G(.,.)$ is lsc in $X \times \{\eta_0\}$, then $S_3^{\varepsilon 1}(.,.,.)$ is both usc and closed at $(\lambda_0, \mu_0, \eta_0)$ for each $\varepsilon \geq 0$.*

Proof Similar arguments can be applied to prove the three assertions. We present only the proof for problem (SQEP_1). Let $\varepsilon \geq 0$ be fixed. Reasoning "ad absurdum" suppose the existence of an open neighborhood U of $S_3^{\varepsilon 1}(\lambda_0, \mu_0, \eta_0)$, of nets $\lambda_\alpha \to \lambda_0, \mu_\alpha \to \mu_0$, $\eta_\alpha \to \eta_0$, and $x_\alpha \in S_3^{\varepsilon 1}(\lambda_\alpha, \mu_\alpha, \eta_\alpha)$ such that $x_\alpha \notin U, \forall \alpha$. By the upper semicontinuity of $E(.)$ and the compactness of $E(\lambda_0)$, we can assume that $x_\alpha \to x_0$ for some $x_0 \in E(\lambda_0)$. If $x_0 \notin S_3^{\varepsilon 1}(\lambda_0, \mu_0, \eta_0)$, then there are $y_0 \in K(x_0, \lambda_0)$ and $x_0^* \in G(x_0, \eta_0)$,

$$F(x_0^*, y_0, \mu_0) \cap \text{comp}(-\text{int}C)_1^\varepsilon = \emptyset. \tag{6}$$

Since $K(.,.)$ and $G(.,.)$ are lsc at (x_0, λ_0) and (x_0, η_0), respectively, there are nets $y_\alpha \in K(x_\alpha, \lambda_\alpha)$ and $x_\alpha^* \in G(x_\alpha, \eta_\alpha)$ such that $y_\alpha \to y_0$ and $x_\alpha^* \to x_0^*$. Since $x_\alpha \in S_3^{\varepsilon 1}(\lambda_\alpha, \mu_\alpha, \eta_\alpha)$, one has

$$F(x_\alpha^*, y_\alpha, \mu_\alpha) \cap \text{comp}(-\text{int}C)_1^\varepsilon \neq \emptyset. \tag{7}$$

By the closedness of $\text{comp}(-\text{int}C)_1^\varepsilon$ and the $Y \setminus \text{comp}(-\text{int}C)_1^\varepsilon-$quasiupper semicontinuity of $F(.,.,.)$ at (x_0^*, y_0, μ_0) one has a contradiction between (6) and (7). Thus, $x_0 \in S_3^{\varepsilon 1}(\lambda_0, \mu_0, \eta_0)$, which is again a contradiction, since $x_\beta \notin U, \forall \beta$.

The closedness of $S_3^{\varepsilon 1}(.,.,.)$ can be proved similarly. ■

Theorem 3.1 is no longer true if we replace $S_1^{\varepsilon 1}(.,.,.)$, $S_2^{\varepsilon 1}(.,.,.)$ and $S_3^{\varepsilon 1}(.,.,.)$ by $\widetilde{S_1^{\varepsilon 1}}(.,.,.)$, $\widetilde{S_2^{\varepsilon 1}}(.,.,.)$ and $\widetilde{S_3^{\varepsilon 1}}(.,.,.)$ as shown by the example below.

Example 3.1 Let $\varepsilon > 0$ be fixed and small. Let $X = Y = R$, $\Lambda \equiv M \equiv N =$

$[0,1]$, $C = R_+$, $K(x,\lambda) = [0,1]$, $G(x,\lambda) = \{x\}$, $F(x,y,\lambda) = \{y - x + \lambda\}$ and $\lambda_0 = 0$. Since G is single-valued, (WQEP$_1$), (MQEP$_1$) and (SQEP$_1$) coincide. It is easy to see that the conditions of Theorem 3.1 hold and accordingly, $S_i^{\varepsilon 1}(.)$ are usc at 0 (for all $i = 1,2,3$). $\big($In fact $S_i(0) = \{0\}$, $S_i^{\varepsilon 1}(0) = [0,\varepsilon]$, $S_i^{\varepsilon 1}(\lambda) = [0,\varepsilon+\lambda]$.$\big)$ Thus, $\widetilde{S_i^{\varepsilon 1}}(.)$ is not usc at 0.

Theorem 3.2

(i) *Assume that* $K(.,.)$ *is lsc in* $X \times \{\lambda_0\}$, $E(.)$ *and* $G(.,.)$ *are H-usc and have compact values at* λ_0 *and in* $X \times \{\eta_0\}$, *respectively, and that* $F(.,.,.)$ *is* $Y \setminus \mathrm{comp}(-\mathrm{int}C)_1^{\varepsilon}-$ *Hqusc in* $X \times X \times \{\mu_0\}$.

  (a) *If* $\forall B_X$ (*open neighborhood of 0 in* X), $\forall x \notin S_1^{\varepsilon 1}(\lambda_0,\mu_0,\eta_0) + B_X$, $\exists \rho > 0$, $\exists y \in K(x,\lambda_0)$, $\forall x^* \in G(x,\eta_0)$, $[F(x^*,y,\mu_0) + \mathrm{int}B_Y^{\rho}] \cap \mathrm{comp}(-\mathrm{int}C)_1^{\varepsilon} = \emptyset$, *then* $S_1^{\varepsilon 1}(.,.,.)$ *is H-usc at* (λ_0,μ_0,η_0) *for each* $\varepsilon \geq 0$.

  (b) *If* $\forall B_X$ *as above,* $\forall x \notin S_2^{\varepsilon 1}(\lambda_0,\mu_0,\eta_0) + B_X$, $\exists \rho > 0$, $\forall x^* \in G(x,\eta_0)$, $\exists y \in K(x,\lambda_0)$, $[F(x^*,y,\mu_0) + \mathrm{int}B_Y^{\rho}] \cap \mathrm{comp}(-\mathrm{int}C)_1^{\varepsilon} = \emptyset$, *then* $S_2^{\varepsilon 1}(.,.,.)$ *is H-usc at* (λ_0,μ_0,η_0) *for each* $\varepsilon \geq 0$.

(ii) *Assume that* K *and* E *are as in* (i), $G(.,.)$ *is lsc in* $X \times \{\eta_0\}$ *and* $F(.,.,.)$ *is H-usc in* $X \times X \times \{\mu_0\}$.

  (c) *If* $\forall B_X$ *as above,* $\forall x \notin S_3^{\varepsilon 1}(\lambda_0,\mu_0,\eta_0) + B_X$, $\exists \rho > 0$, $\exists y \in K(x,\lambda_0)$, $\forall x^* \in G(x,\eta_0)$, $[F(x^*,y,\mu_0) + \mathrm{int}B_Y^{\rho}] \cap \mathrm{comp}(-\mathrm{int}C)_1^{\varepsilon} = \emptyset$, *then* $S_3^{\varepsilon 1}(.,.,.)$ *is H-usc at* (λ_0,μ_0,η_0) *for each* $\varepsilon \geq 0$.

Proof We demonstrate only (a). Let $\varepsilon \geq 0$ be fixed. Suppose that $S_1^{\varepsilon 1}(.,.,.)$ is not H-usc at (λ_0,μ_0,η_0), i.e., $\exists B_X$ (open neighborhood of 0 in X), $\exists (\lambda_\alpha,\mu_\alpha,\eta_\alpha) \to (\lambda_0,\mu_0,\eta_0)$, $\exists x_\alpha \in S_1^{\varepsilon 1}(\lambda_\alpha,\mu_\alpha,\eta_\alpha)$ such that $x_\alpha \notin S_1^{\varepsilon 1}(\lambda_0,\mu_0,\eta_0) + B_X, \forall \alpha$. By the Hausdorff upper semicontinuity of $E(.)$ and the compactness of $E(\lambda_0)$, we can assume that $x_\alpha \to x_0$ for some $x_0 \in E(\lambda_0)$. If $x_0 \notin S_1^{\varepsilon 1}(\lambda_0, \mu_0, \eta_0) + B_X$, then the assumption in (a) yields some $\rho > 0$ and some $y_0 \in K(x_0, \lambda_0)$ such that, $\forall x_0^* \in G(x_0, \eta_0)$,

$$[F(x_0^*, y_0, \mu_0) + \text{int} B_Y^\rho] \cap \text{comp}(-\text{int}C)_1^\varepsilon = \emptyset. \tag{8}$$

Since $K(.,.)$ is lsc at (x_0, λ_0), there is a net $y_\alpha \in K(x_\alpha, \lambda_\alpha)$, $y_\alpha \to y_0$. As $x_\alpha \in S_1^{\varepsilon 1}(\lambda_\alpha, \mu_\alpha, \eta_\alpha)$, $\exists x_\alpha^* \in G(x_\alpha, \eta_\alpha)$,

$$F(x_\alpha^*, y_\alpha, \mu_\alpha) \cap \text{comp}(-\text{int}C)_1^\varepsilon \neq \emptyset. \tag{9}$$

As $G(.,.)$ is H-usc and $G(x_0, \eta_0)$ is compact, one has a subnet x_β^* such that $x_\beta^* \to x_0^*$ for some $x_0^* \in G(x_0, \eta_0)$. By the $Y \setminus \text{comp}(-\text{int}C)_1^\varepsilon-$ Hausdorff quasiupper semicontinuity of $F(.,.,.)$ at (x_0^*, y_0, μ_0), we see a contradiction between (8) and (9). Hence, $x_0 \in S_1^{\varepsilon 1}(\lambda_0, \mu_0, \eta_0) + B_X$, which is again a contradiction, since $x_\beta \notin S_1^{\varepsilon 1}(\lambda_0, \mu_0, \eta_0) + B_X, \forall \beta$. ■

We see that to ensure the Hausdorff upper semicontinuity of $S_i^\varepsilon(.,.,.)$, $\forall i = 1, 2, 3$, the upper semicontinuity assumed in Theorem 3.2 is reduced to Hausdorff upper semicontinuity. However, we have to add the assumption in (a). The following example shows that this additional assumption is essential.

Example 3.2 Let $\varepsilon = 0$, $X = Y = R$, $\Lambda \equiv M \equiv N = [0, 1]$, $C = R_+$, $K(x, \lambda) = [0, 1]$, $G(x, \lambda) = \{x\}$, $F(x, y, \lambda) = (-\infty, \lambda x)$ and $\lambda_0 = 0$. As G is single-valued mapping, problems (WQEP$_1$), (MQEP$_1$) and (SQEP$_1$) coincide. It is clear that $S_i^{01}(0) = \{0\}$, $S_i^{01}(\lambda) = [0, 1]$, $\forall i = 1, 2, 3$, $\forall \lambda \in (0, 1]$. So, $S_i^{01}(.)$ is not H-usc at 0. The reason is that assumptions (a) - (c) are violated. Indeed, take $B_X = (-1, 1)$ and $x = 1$. Then, for each $\rho > 0$ and each $y \in [0, 1]$, one has $F(1, y, 0) + \text{int} B_X^\rho = (-\infty, 0)+ (-\rho, \rho) = (-\infty, \rho)$. So, $[F(1, y, 0) + \text{int} B_X^\rho] \cap [-\varepsilon, +\infty) \neq \emptyset$.

THEOREM 3.3 *Assume that* $K(.,.)$ *is lsc in* $X \times \{\lambda_0\}$, $E(.)$ *is usc,* $E(\lambda_0)$ *is compact and* $F(.,.,.)$ *is* $Y\backslash\mathrm{comp}(-\mathrm{int}C)_1^{\varepsilon}-$*qlsc in* $X \times X \times \{\mu_0\}$.

(i) *If* $G(.,.)$ *is usc and has compact values in* $X \times \{\eta_0\}$, *then* $S_4^{\varepsilon 1}(.,.,.)$ *and* $S_5^{\varepsilon 1}(.,.,.)$ *are both usc and closed at* $(\lambda_0, \mu_0, \eta_0)$ *for each* $\varepsilon \geq 0$.

(ii) *If* $G(.,.)$ *is lsc in* $X \times \{\eta_0\}$, *then* $S_6^{\varepsilon 1}(.,.,.)$ *is both usc and closed at* $(\lambda_0, \mu_0, \eta_0)$ *for each* $\varepsilon \geq 0$.

We omit the proof since it is similar to the previous ones.

Example 3.1 ensures also that Theorem 3.3 is no longer true if we replace $S_4^{\varepsilon 1}(.,.,.)$, $S_5^{\varepsilon 1}(.,.,.)$ and $S_6^{\varepsilon 1}(.,.,.)$ by $\widetilde{S_4^{\varepsilon 1}}$, $\widetilde{S_4^{\varepsilon 1}}$ and $\widetilde{S_6^{\varepsilon 1}}$, respectively, since F and G are single-valued functions.

4. APPLICATIONS

Quasiequilibrium problems include as special cases many important problems such as quasivariational inequalities, complementarity problems, fixed point and coincidence point problems, optimization problems, etc. Therefore, applying the results presented in the preceding sections we obviously obtain sufficient conditions for semicontinuity of approximate solution sets of these particular cases. In this section we derive some interesting consequences of the theorems in Section 2 and 3 as examples.

4.1. Quasivariational inequalities

If $Y = R$, $F(x, y, \mu) = \langle T(x, \mu), y - g(x, \mu)\rangle$ and $G(x, \eta) = \{x\}$, where $T : X \times M \to 2^{X^*}$ and $g : X \times M \to X$ is a continuous mapping, then (WQEP$_1$), (MQEP$_1$) and (SQEP$_1$) coincide with (QVI) in [16, 17], and (WQEP$_2$), (MQEP$_2$) and (SQEP$_2$) all become (SQVI) in [16, 17]. The following two corollaries about lower semicontinuity are direct consequences of Theorems 2.1 and 2.2.

Corollary 4.1 *Assume that $K(.,.)$ is usc and has compact values in $X \times \{\lambda_0\}$, $E(.)$ is lsc at λ_0 and $(x, y, \mu) \mapsto \langle T(x, \mu), y - g(x, \mu)\rangle$ is $[-\varepsilon, +\infty)-$qlsc in $X \times X \times \{\mu_0\}$. Then the $\varepsilon-$solution set $\widetilde{S_1^{\varepsilon}}(.,.)$ of* (QVI) *is lsc at (λ_0, μ_0) for each $\varepsilon > 0$.*

Corollary 4.2 *Assume that $K(.,.)$ and $E(.)$ are as in Corollary 4.1 and $(x, y, \mu) \mapsto \langle T(x, \mu), y - g(x, \mu)\rangle$ is $[-\varepsilon, +\infty)-$qusc in $X \times X \times \{\mu_0\}$. Then the $\varepsilon-$solution set $\widetilde{S_2^{\varepsilon}}(.,.)$ of* (SQVI) *is lsc at (λ_0, μ_0) for each $\varepsilon > 0$.*

The next two corollaries about upper semicontinuity are directly derived from Theorems 3.1 and 3.3.

Corollary 4.3 *Let $K(.,.)$ be lsc in $X \times \{\lambda_0\}$ and let $E(.)$ be usc and have compact values at λ_0. Let $(x, y, \mu) \mapsto \langle T(x, \mu), y - g(x, \mu)\rangle$ be $(-\infty, -\varepsilon)-$qusc in $X \times X \times \{\mu_0\}$. Then the $\varepsilon-$solution set $S_1^{\varepsilon}(.,.)$ of* (QVI) *is both usc and closed at (λ_0, μ_0) for each $\varepsilon \geq 0$.*

Corollary 4.4 *Let $K(.,.)$ and $E(.)$ be as in Corollary 4.3. Let $(x, y, \mu) \mapsto \langle T(x, \mu), y - g(x, \mu)\rangle$ be $(-\infty, -\varepsilon)-$qlsc in $X \times X \times \{\mu_0\}$. Then the $\varepsilon-$solution set $S_2^{\varepsilon}(.,.)$ of* (SQVI) *is both usc and closed at (λ_0, μ_0) for each $\varepsilon \geq 0$.*

These four corollaries improve Theorems 5.1, 5.3 and Theorems 6.1, 6.3, respectively, in [17]. The following example ensures that the assumptions of the corollaries are strictly weaker than the corresponding ones imposed in the mentioned theorems.

EXAMPLE 4.1 Let $\varepsilon > 0$ be fixed. Let $X = Y = R$, $\Lambda \equiv M = [0,1]$, $C = R_+$, $K(x,\lambda) = [0,1]$, $\lambda_0 = 0$ and

$$T(x,\lambda) = \begin{cases} \{\varepsilon\} & \text{if } \lambda = 0, \\ [\frac{\varepsilon}{4}, \frac{\varepsilon}{2}] & \text{otherwise.} \end{cases}$$

Then $\langle T(x,0), y-g(x,0)\rangle = [-\varepsilon,\varepsilon]$ and $\langle T(x,\lambda), y-g(x,\lambda)\rangle = [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}], \forall\lambda \in (0,1]$. Hence all the assumptions of Corollaries 4.1 and 4.3 are satisfied. Applying these corollaries we know that $\widetilde{S_1^\varepsilon}(.)$ is lsc at 0 and S_1^ε is both usc and closed at 0. In fact, direct calculation gives $\widetilde{S_1^\varepsilon}(\lambda) = S_1^\varepsilon(\lambda) = [0,1], \forall\lambda \in \Lambda$. However, $\langle T(.,.),.\rangle$ is not lsc and $T(.,.)$ is not usc in $X \times X \times \{0\}$ as required in assumption (ii) of Theorems 5.1 and 6.1 of [17]. So these theorems do not work in this case.

Now we pass to quasivariational inequalities with operator solutions introduced in [13]. Let X, M, N, Λ and Y be as above (defined in Section 1). Let $C \subseteq Y$ be closed convex cone with $\text{int}C \neq \emptyset$ and $C \neq Y$. Let $K : L(X,Y)\times\Lambda \to 2^{L(X,Y)}$, $T : L(X,Y) \times L(X,Y) \times M \to 2^X$ be multifunctions. Our quasivariational inequalities with operator solutions are

(OVI) find $\bar{f} \in \text{cl}K(\bar{f},\lambda)$ such that, for each $f \in K(\bar{f},\lambda)$,

$$\big(f - \bar{f}, T(\bar{f},\mu)\big) \cap (Y \setminus -\text{int}C) \neq \emptyset;$$

(SOVI) find $\bar{f} \in \text{cl}K(\bar{f},\lambda)$ such that, for each $f \in K(\bar{f},\lambda)$,

$$\big(f - \bar{f}, T(\bar{f},\mu)\big) \subseteq Y \setminus -\text{int}C.$$

From Theorems 2.1 and 2.2 we derive the following

COROLLARY 4.5 *Assume that $K(.,.)$ is usc and has compact values in $L(X,Y) \times \{\lambda_0\}$, $E(.)$ is lsc at λ_0 and $(f,g,\mu) \mapsto \big(f-g, T(g,\mu)\big)$ is* $\text{comp}(-\text{int}C)_1^\varepsilon$*−qlsc* *or* $\mathrm{comp}(-\mathrm{int}C)_2^\varepsilon-$*qlsc in* $L(X,Y)\times L(X,Y)\times\{\mu_0\}$*. Then the* $\varepsilon-$*solution set* $\widetilde{S_1^\varepsilon}(.,.)$ *of* (OVI) *is lsc at* (λ_0,μ_0) *for each* $\varepsilon>0$*.*

Corollary 4.6 *Assume that* $K(.,.)$ *and* $E(.)$ *are as in Corollary 4.5 and* $(f,g,\mu)\mapsto\big(f-g,T(g,\mu)\big)$ *is* $\mathrm{comp}(-\mathrm{int}C)_1^\varepsilon-$*qusc or* $\mathrm{comp}(-\mathrm{int}C)_2^\varepsilon-$*qusc in* $L(X,Y)\times L(X,Y)\times\{\mu_0\}$*. Then the* $\varepsilon-$*solution set* $\widetilde{S_2^\varepsilon}(.,.)$ *of* (SOVI) *is lsc at* (λ_0,μ_0) *for each* $\varepsilon>0$*.*

Similarly, for upper semicontinuity, but of the other kind of $\varepsilon-$solution sets, Theorems $3.1-3.3$ yield the following consequences.

Corollary 4.7 *Assume that* $K(.,.)$ *is lsc in* $L(X,Y)\times\{\lambda_0\}$*,* $E(.)$ *is usc and has compact values at* λ_0 *and* $(f,g,\mu)\mapsto\big(f-g,T(g,\mu)\big)$ *is* $Y\setminus\mathrm{comp}(-\mathrm{int}C)_1^\varepsilon-$*qusc in* $L(X,Y)\times L(X,Y)\times\{\mu_0\}$*. Then the* $\varepsilon-$*solution set* $S_1^\varepsilon(.,.)$ *of* (OVI) *is usc and closed at* (λ_0,μ_0) *for each* $\varepsilon\geq 0$*.*

Corollary 4.8 *Let* $K(.,.)$ *be lsc in* $L(X,Y)\times\{\lambda_0\}$*,* $E(.)$ *be H-usc and have compact values at* λ_0 *and* $(f,g,\mu)\mapsto\big(f-g,T(g,\mu)\big)$ *be* $Y\setminus\mathrm{comp}(-\mathrm{int}C)_1^\varepsilon-$*Hqusc in* $L(X,Y)\times L(X,Y)\times\{\mu_0\}$*. If* $\forall B$ *(open neighborhood of* 0 *in* $L(X,Y)$*),* $\forall f_0\notin S_1^\varepsilon(\lambda_0,\mu_0)+B$*,* $\exists B_Y$ *(open neighborhood of* 0 *in* Y*),* $\exists f\in K(f_0,\lambda_0)$*,* $\big((f-f_0,T(f_0,\mu_0))+B_Y\big)\cap\mathrm{comp}(-\mathrm{int}C)_1^\varepsilon=\emptyset$*, then* $S_1^\varepsilon(.,.)$ *is H-usc at* (λ_0,μ_0) *for each* $\varepsilon\geq 0$*.*

Corollary 4.9 *Let* $K(.,.)$ *and* $E(.)$ *be as in Corollary 4.7 and* $(f,g,\mu)\mapsto\big(f-g,T(g,\mu)\big)$ *is* $Y\setminus\mathrm{comp}(-\mathrm{int}C)_1^\varepsilon-$*qlsc in* $L(X,Y)\times L(X,Y)\times\{\mu_0\}$*. Then the* $\varepsilon-$*solution set* $S_2^\varepsilon(.,.)$ *of* (SOVI) *is usc at* (λ_0,μ_0) *for each* $\varepsilon\geq 0$*.*

4.2. Approximate fixed points

Let X be a Hilbert space, Λ and M be as in Section 1, $\varepsilon > 0$, $K : \Lambda \to 2^X$ and $T : X \times M \to 2^X$. The set of approximate fixed points of T at (λ, μ) is defined as (see, e.g. a resent paper [8])

(P$^\varepsilon$) $$\mathrm{P}^\varepsilon(\lambda,\mu) = \big\{x \in K(\lambda) \;\big|\; d\big(x, T(x,\mu)\big) \le \varepsilon\big\}.$$

Assume that for each $\lambda \in \Lambda$, $K(\lambda)$ is closed and contains $0 \in X$. The problem of finding $\varepsilon-$fixed points is related to the following problem of finding $\varepsilon-$solutions of a quasiequilibrium problem:

(QEP$^\varepsilon$) Find $\bar{x} \in K(\lambda)$ such that there exists $\bar{t} \in \mathrm{cl}T(\bar{x},\mu)$ with, $\forall y \in K(\lambda)$,

$$\langle \bar{x} - \bar{t}, y + \bar{t} - \bar{x}\rangle \ge -\varepsilon. \tag{10}$$

Proposition 4.1 *If $\bar{x}$ is a solution of* (QEP$^\varepsilon$) *then $\bar{x}$ is a solution of* (P$^{\sqrt{\varepsilon}}$).

Proof By the assumption we have (10). Taking $y = 0$ we get $\|\bar{x} - \bar{t}\|^2 \le \varepsilon$ and hence $d\big(\bar{x}, T(\bar{x},\mu)\big) \le \sqrt{\varepsilon}$. ■

Corollary 4.10 *If $K(.)$ is lsc at λ_0 and $T(.,.)$ is lsc in $X \times \{\mu_0\}$, then the $\varepsilon-$solution set $\widetilde{S^\varepsilon}(.,.)$ of* (QEP$^\varepsilon$) *is lsc at μ_0 for each $\varepsilon > 0$.*

Proof By the lower semicontinuity of $T(.,.)$, $(x,y,\mu) \mapsto \langle x - \mathrm{cl}T(x,\mu), y + \mathrm{cl}T(x,\mu) - x\rangle$ is lsc too. Applying Theorem 2.1 yields the corollary. ■

Corollary 4.11 *If $K(.)$ is usc, $K(\lambda_0)$ is compact and $T(.,.)$ is usc and has compact values in $X \times \{\mu_0\}$, then the $\varepsilon-$solution set $S^\varepsilon(.,.)$ of* (QEP$^\varepsilon$)*is usc and closed at (λ_0,μ_0) for each $\varepsilon \ge 0$.*

Proof $(x,y,\mu) \mapsto \langle x - \mathrm{cl}T(x,\mu), y + \mathrm{cl}T(x,\mu) - x\rangle$ is usc and has compact values in $X \times \{\mu_0\}$. So the conclusion follows directly from Theorem 3.1. ■

By Proposition 4.1, Corollaries 4.10 $-$ 4.11 yield the semicontinuity of a part of the $\sqrt{\varepsilon}-$fixed points of $T(x,\mu)$ with respect to (λ,μ). To deal with the whole set of approximate fixed points we modify problem (QEP$^\varepsilon$) as follows.

(QEP$_1^\varepsilon$) Find $\bar{x} \in K(\lambda)$ such that there exists $\bar{t} \in T^\varepsilon(\bar{x},\mu)$ with, $\forall y \in K(\lambda)$,

$$\langle \bar{x} - \bar{t}, y + \bar{t} - \bar{x} \rangle \geq 0, \tag{11}$$

where $T^\varepsilon(x,\mu) = \{ y \in X \mid d(y, T(x,\mu)) \leq \varepsilon \}$.

Proposition 4.2 *$\bar{x}$ is a solution of* (P$^\varepsilon$) *if and only if $\bar{x}$ is a solution of* (QEP$_1^\varepsilon$).

Proof Being a solution of (QEP$_1^\varepsilon$), $\bar{x}$ yields $\bar{t} \in T^\varepsilon(\bar{x},\mu)$ satisfying (11). Taking $y = 0$ we see that $\|\bar{x} - \bar{t}\| = 0$, and hence $\bar{x} \in T^\varepsilon(\bar{x},\mu)$. Conversely, if $\bar{x}$ is an $\varepsilon-$fixed point of $T(.,\mu)$, i.e. $\bar{x} \in T^\varepsilon(\bar{x},\mu)$. Taking $\bar{t} = \bar{x}$ we see that $\bar{x}$ satisfies (11). ■

Let us denote the fixed point set and the $\varepsilon-$fixed point set of T at (λ,μ) by $P(\lambda,\mu)$ and $P^\varepsilon(\lambda,\mu)$, respectively. Similarly as for quasiequilibrium problems we consider also the following second kind of $\varepsilon-$fixed point set

$$\widetilde{P^\varepsilon}(\lambda,\mu) = \begin{cases} P(\lambda_0,\mu_0) & \text{if } (\lambda,\mu) = (\lambda_0,\mu_0), \\ P^\varepsilon(\lambda,\mu) & \text{otherwise.} \end{cases}$$

Proposition 4.3 *If $K(.)$ is lsc at λ_0 and $T(.,.)$is lsc in $X \times \{\mu_0\}$, then $\widetilde{P^\varepsilon}(.,.)$is lsc at (λ_0,μ_0) for each $\varepsilon > 0$.*

Proof Suppose to the contrary that there are $\lambda_\alpha \to \lambda_0$, $\mu_\alpha \to \mu_0$ and $x_0 \in \widetilde{P^\varepsilon}(\lambda_0,\mu_0)$ such that, $\forall x_\alpha \in \widetilde{P^\varepsilon}(\lambda_\alpha,\mu_\alpha)$, $x_\alpha \not\to x_0$. As $K(.)$ is lsc at λ_0, there exists $\bar{x}_\alpha \in K(\lambda_\alpha)$ such that $\bar{x}_\alpha \to x_0$. Then there must be a subnet $\bar{x}_\beta$ with $\bar{x}_\beta \notin \widetilde{P^\varepsilon}(\lambda_\beta,\mu_\beta)$ for all β. Since $x_0 \in \widetilde{P^\varepsilon}(\lambda_0,\mu_0)$, $x_0 \in T(x_0,\mu_0)$. By the lower semicontinuity of $T(.,.)$ at (x_0,μ_0), there is $t_\beta \in T(\bar{x}_\beta,\mu_\beta)$ such that $t_\beta \to x_0$.

Since, for all $\mu_\beta \neq \mu_0$, $\bar{x}_\beta \notin \widetilde{P^\varepsilon}(\lambda_\beta, \mu_\beta)$, $\bar{x}_\beta \notin T^\varepsilon(\bar{x}_\beta, \mu)$, i.e. $\|\bar{x}_\beta - t_\beta\| \geq \varepsilon$. This is impossible since $\bar{x}_\beta \to x_0$ and $t_\beta \to x_0$. ∎

PROPOSITION 4.4 *If $K(.)$ is usc, $K(\lambda_0)$ is compact and $T(.,.)$is usc and has compact values in $X \times \{\mu_0\}$, then $P^\varepsilon(.,.)$ is usc and closed at (λ_0, μ_0) for each $\varepsilon \geq 0$.*

Proof Arguing by contradiction suppose the existence of a neighborhood U of $P^\varepsilon(\lambda_0, \mu_0)$, of nets $(\lambda_\alpha, \mu_\alpha) \to (\lambda_0, \mu_0)$ and $x_\alpha \in P^\varepsilon(\lambda_\alpha, \mu_\alpha)$, $x_\alpha \notin U, \forall \lambda$. By the assumption on $K(.)$ we have a point $x_0 \in K(\lambda_0)$ and subnet $x_\beta \to x_0$. Since $x_\beta \in P^\varepsilon(\lambda_\beta, \mu_\beta)$, there is $t_\beta \in \text{cl}T(x_\beta, \mu_\beta)$, $\|t_\beta - x_\beta\| \leq \varepsilon$. By the assumption on $T(.,.)$ we can assume that $t_\beta \to t_0$, for some $t_0 \in \text{cl}T(x_0, \mu_0) = T(x_0, \mu_0)$. Hence $\|x_0 - t_0\| \leq \varepsilon$ and then $x_0 \in P^\varepsilon(\lambda_0, \mu_0)$, a contradiction, since $x_\beta \notin U, \forall \beta$.

The closedness of $P^\varepsilon(.,.)$ at (λ_0, μ_0) is similarly verified. ∎

4.3. Approximate quasioptimization problems

Let X, Y, M, Λ, K and C be as in Section 1. Let $T : X \times M \to 2^Y$ be a multifunction. We consider the following quasioptimization problem, for $(\lambda, \mu) \in \Lambda \times M$,

(QOP) find $\bar{x} \in \text{cl}K(\bar{x}, \lambda)$ such that

$$T(\bar{x}, \mu) \cap \text{wMin}\{T\big(K(\bar{x}, \lambda), \mu\big) \,\big|\, C\} \neq \emptyset,$$

where $\text{wMin}\{H \,|\, C\}$ denotes the set of all weakly efficient points y^* of the set $H \subseteq Y$, with respect to the ordering set C, i.e. points $y^* \in H$ such that, $\forall y \in H$, $y - y^* \in Y \setminus -\text{int}C$. Note that here C needs not be a cone as in a usual weak efficiency.

We will show now that (QOP) can be expressed as a case of problem (MQEP_2). Set $X_1 = X \times Y$ and define $K_1 : X_1 \times \Lambda \to 2^{X_1}$, $G_1 : X_1 \times M \to 2^{X_1}$

and $F_1 : X_1 \times X_1 \times M \to 2^Y$ by, for $x_1 = (x^1, y^1)$, $x_2 = (x^2, y^2)$ in X_1,

$$K_1(x_1, \lambda) = K_1\big((x^1, y^1), \lambda\big) = K(x, \lambda) \times \{0_Y\},$$

$$G_1(x_2, \mu) = G_1\big((x^2, y^2), \mu\big) = \{0_X\} \times T(x^2, \mu),$$

$$F_1(x_1, x_2, \mu) = F\big((x^1, y^1), (x^2, y^2), \mu\big) = T(x^2, \mu) - y^2.$$

We consider the following problem (MQEP_2):

(MQEP) find $\bar{x} \in \mathrm{cl}K(\bar{x}, \lambda)$ such that $\exists \bar{y} \in T(\bar{x}, \mu)$, $\forall x \in K(\bar{x}, \lambda)$,

$$F_1\big((0, \bar{y}), (x, 0), \mu\big) \equiv T(x, \mu) - \bar{y} \subseteq Y \setminus -\mathrm{int}C.$$

Proposition 4.5 *$\bar{x}$ is a solution of* (QOP) *if and only if $\bar{x}$ is a solution of* (MQEP).

The proof is direct and so is omitted. The following two approximate problems of (QOP) and (MQEP) are also equivalent

(QOP^ε) find $\bar{x} \in \mathrm{cl}K(\bar{x}, \lambda)$ such that

$$T(\bar{x}, \mu) \cap \mathrm{wMin}\big\{T\big(K(\bar{x}, \lambda), \mu\big)\big|\mathrm{comp}(-\mathrm{int}C)_1^\varepsilon\big\} \neq \emptyset;$$

($\mathrm{MQEP}^\varepsilon$) find $\bar{x} \in \mathrm{cl}K(\bar{x}, \lambda)$ such that $\exists \bar{y} \in T(\bar{x}, \mu)$, $\forall x \in K(\bar{x}, \lambda)$,

$$T(x, \mu) - \bar{y} \subseteq \mathrm{comp}(-\mathrm{int}C)_1^\varepsilon.$$

The following corollaries are direct consequences of Theorems 2.2 and 3.3, respectively.

Corollery 4.12 *Assume that $K(.,.)$ is usc and has compact values in $X \times \{\lambda_0\}$, $E(.)$ is lsc at λ_0 and $T(.,.)$ is lsc and* $\mathrm{comp}(-\mathrm{int}C)_1^\varepsilon$*−qusc in $X \times \{\mu_0\}$. Then the approximate solution set $\widetilde{S^\varepsilon}(.,.)$ of* (QOP) *is lsc at (λ_0, μ_0) for each $\varepsilon > 0$.*

Corollery 4.13 *Assume that $K(.,.)$ is lsc in $X \times \{\lambda_0\}$, $E(.)$ is usc at λ_0 and*

$E(\lambda_0)$ is compact, $T(.,.)$ is usc, $Y \setminus \mathrm{comp}(-\mathrm{int}C)_1^\varepsilon-$qlsc and has compact values in $X \times \{\mu_0\}$. Then the approximate solution set $S^\varepsilon(.,.)$ of (QOP) *is usc and closed at (λ_0, μ_0) for each $\varepsilon \geq 0$.*

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