Systems of Set-Valued Quasivariational Inclusion Problems

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Abstract. We propose four kinds of systems of set - valued quasivariational inclusion problems in product spaces, which include many known systems of equilibrium problems and systems of variational inequalities as well as inclusion problems. Sufficient conditions for the solution existence are established. When applied to special cases these conditions improve many existing results in the literature. To ensure the generality of our problem setting and results, applications in the fixed point theory and quasioptimization problems are included.

Key Words. Systems of quasivariational inclusions, equilibrium problems, variational inequalities, fixed points, quasioptimization, G - quasiconvexity, G - quasiconvexlikeness.

1. Introduction

For the last several decades variational inequality problems have been developed rapidly, see e.g. the papers gathered in books devoted to variational inequalities: Refs. 1 - 3, since this theory proved to be very effective and powerful tools for studying a wide class of problems in diverse fields of pure mathematics and applied sciences like mathematical programming, optimization, engineering, traffic equilibrium problems, mathematical economics, game theory, elasticity theory, etc. An important extension of a variational inequality is an equilibrium problem, see e.g. recent works: Refs. 4 - 9. This problem setting includes not only variational inequalities but also complementarity problems, fixed point and coincidence point problems, optimization problems, game theory, etc. Recently variational inclusions were introduced as extensions of equilibrium problems in Refs. 10 - 13. However, it should be added here that the term "variational inclusions" is understood in various ways in the literature. A variational inclusion means a kind of multivalued variational inequality problems in Refs. 14 - 15. It is used for a problem of finding zeroes of maximal monotone mappings in Refs. 16 - 18. Ref. 19 extends this model.

On the other hand, systems of problems of the above kinds began to attract mathematicans. In Ref. 20 a system of variational inequalites in product spaces was considered. Systems of equilibrium problems in product spaces were investigated in Refs. 21 - 23 as a direct generalization of the system studied in Ref. 20. Another concept of a system of equilibrium problems was proposed in Refs. 24

- 25, where the problem under consideration was to find common solutions to a system of two or infinite number of equilibrium problems in the same spaces.

Inspired by this line of works, we introduce four kinds of systems of set-valued quasivariational inclusion problems in product spaces. Our problem setting, which includes almost all the above problems, is as follows. Let I be any index set. For each $i \in I$, let X_i, Y_i and Z_i be Hausdorff topological vector spaces, $A_i \subseteq X_i$ be nonempty convex subsets. Set $X = \prod_{i\in I} X_i$ and $A = \prod_{i\in I} A_i$, the Tichonov products. Let the following multifunctions be given with nonempty values: $K_i : A \to 2^{X_i}$, $T_i : A \to 2^{Z_i}$, $F_i : T_i(A) \times X_i \times A \to 2^{Y_i}$, $G_i : T_i(A) \times X_i \times A \to 2^{Y_i}$. Let x_i be the canonical projection of $x \in X$ on X_i. The systems of generalized quasivariational inclusion problems under our consideration are the following:

(SQIP1) find $\bar{x}$ such that, $\forall i \in I, \bar{x}_i \in A_i \bigcap \mathrm{cl}K_i(\bar{x})$ and $\forall y_i \in K_i(\bar{x}), \exists \bar{t}_i \in T_i(\bar{x})$,

$$F_i(\bar{t}_i, y_i, \bar{x}) \subseteq G_i(\bar{t}_i, \bar{x}_i, \bar{x}); \tag{1}$$

(SQIP2) find $\bar{x}$ such that, $\forall i \in I, \bar{x}_i \in A_i \bigcap \mathrm{cl}K_i(\bar{x})$ and $\forall y_i \in K_i(\bar{x}), \forall \bar{t}_i \in T_i(\bar{x})$, one has (1);

(SQIP3) find $\bar{x}$ such that, $\forall i \in I, \bar{x}_i \in A_i \bigcap \mathrm{cl}K_i(\bar{x})$ and $\forall y_i \in K_i(\bar{x}), \exists \bar{t}_i \in T_i(\bar{x})$,

$$F_i(\bar{t}_i, y_i, \bar{x}) \bigcap G_i(\bar{t}_i, \bar{x}_i, \bar{x}) \neq \emptyset; \tag{2}$$

(SQIP4) find $\bar{x}$ such that, $\forall i \in I, \bar{x}_i \in A_i \bigcap \mathrm{cl}K_i(\bar{x})$ and $\forall y_i \in K_i(\bar{x}), \forall \bar{t}_i \in T_i(\bar{x})$, one has (2),

where cl(.) means closure of the set (.).

To ensure the generality of the above problem setting we consider some special cases in connection with recent papers in the literature.

(a) If $G_i(t_i, y_i, x) = C_i(x)$ or $Y_i \backslash -\mathrm{int}C_i(x)$, where $C_i : X \to 2^{Y_i}$ has the values $C_i(x)$ being closed cones with nonempty interiors, and $A_i = X_i$, then our four systems collapse to the four systems of generalized vector quasiequilibrium problems investigated in Ref. 23.

(b) If $Y_i = Y_0, Z_i = X, T_i = \{x\}$ and $\mathrm{cl}K_i(x) = A_i$ for all $i \in I$ and $x \in X$, where Y_0 is a Hausdorff topological vector space, if $F_i(x, y_i, x)$ is single-valued and if $G_i(x, y_i, x) \equiv Y \backslash -\mathrm{int}C$, where C is a convex cone with $\mathrm{int}C \neq \emptyset$, then (SQIP1) and (SQIP2) coincide with the system of vector equilibrium problems studied in Ref. 21.

(c) If $Z_i = X, T_i = \{x\}$, and $\mathrm{cl}K_i(x) = A_i$ for all $i \in I$ and $x \in X$ and if $G_i(x, y, x) = Y_i \backslash -\mathrm{int}C_i(x)$, where $C_i : X \to 2^{Y_i}$ has the values $C_i(x)$ being convex cones with $\mathrm{int}C_i(x) \neq \emptyset$, then (SQIP3) and (SQIP4) become the systems of generalized vector equilibrium problems considered in Ref. 22.

(d) If $X_i = X_i^*, Y_i = R, K_i(x) = A_i$ and $F_i(t_i, y_i, x) =< T_i(x), y_i - x_i >$ and $G_i(t_i, y_i, x) = R_+$ for all $i \in I, y_i \in Y_i$, $x \in X$, where T_i is single-valued, then (SQIP1) and (SQIP2) are reduced to the system of variational inequalities investigated in Ref. 20.

(e) If I is a singleton, $G_i(t_i, y_i, x) = F_i(t_i, x_i, x) + C_i(x)$, where $C_i : X \to 2^{Y_i}$ has the values $C_i(x)$ being convex cones with nonempty interiors, then (SQIP2) becomes the variational inclusion problems sudied in Ref. 11 and is similar to the variational inclusion problems considerd in Refs. 10, 12 and 13, while (SQIP1) is similar to another variational inclusion problem dealt with in Ref. 13.

(f) If I is a singleton, $G_i(t_i, y_i, x) = Y_i \backslash -\mathrm{int}C_i(x)$, where $C_i : X \to 2^{Y_i}$ has the values $C_i(x)$ being convex cones with $\mathrm{int}C_i(x) \neq \emptyset$, then (SQIP1) and (SQIP2) collapse to the quasiequilibrium problems investigated by many authors, see e.g. Refs. 9 and 26 - 28.

In Section 4 we will mention other special cases of the systems (SQIP1) - (SQIP4).

The paper is organized as follows. Section 2 is devoted to definitions, a fixed point theorem and a maximal element theorem needed in the sequel. In Section 3 the main results are established and in Section 4 applications in fixed point theory and quasioptimization problems are presented to see the generality and effectiveness of the main results.

2. Preliminaries

We recall first some definitions. Let X and Y be topological spaces. A multifunction $F : X \to 2^Y$ is said to be upper semicontinuous (usc, in short) at $\hat{x} \in \mathrm{dom}F := \{x \in X : F(x) \neq \emptyset\}$ if for each open subset U, with $F(\hat{x}) \subseteq U$, there is a neighborhood N of $\hat{x}$ such that $F(N) \subseteq U$. F is called usc in $S \subseteq X$ if F is usc at any $x \in S$. If $S =$dom$F$ we delete the term "in S". In the sequel all properties defined at a point will be extentded to a subset in this way. F is called lower semicontinuous (lsc) at $\hat{x} \in \mathrm{dom}F$ if, for each open subset U satisfying $U \cap F(\hat{x}) \neq \emptyset$ there is a neighborhood N of $\hat{x}$ such that $U \cap F(x) \neq \emptyset$ for all $x \in N$. F is said to be continuous at $\hat{x}$ if F is both usc and lsc at $\hat{x}$. F is termed closed at $x \in \mathrm{dom}F$ if $\forall x_\alpha \to \hat{x},\ \forall y_\alpha \in F(x_\alpha)$: $y_\alpha \to y$, $y \in F(\hat{x})$.

The following facts are well known.

(i) F is lsc at $\hat{x}$ if and only if $\forall y \in F(\hat{x}), \forall x_\alpha \to \hat{x}, \exists y_\alpha \in F(x_\alpha)$, $y_\alpha \to y$.

(ii) If F is usc and has closed values, then F is closed.

(iii) If F is usc, has compact values and $A \subseteq X$ is compact, then $F(A)$ is compact.

Recall that a point $x \in X$ is termed a maximal element of $F : X \to 2^Y$, where X and Y are topological spaces, if $F(x) = \emptyset$. The following existence theorem of maximal elements for a family of multifunctions was established in Ref. 29 in a slightly stronger form.

Theorem 2.1. Let for each $i \in I, X_i$ be a Hausdorff topological vector space, $A_i \subseteq X_i$ be a nonempty convex subset and let $S_i : A = \prod_{i \in I} A_i \to 2^{A_i}$ have nonempty convex values. Assume that the following conditions hold

(i) $S_i^{-1}(x_i)$ is open in A for all $x_i \in A_i$ and $i \in I$;

(ii) $x_i \notin S_i(x)$ for each $x \in A$ and $i \in I$;

(iii) if A is not compact then there exists a nonempty compact subset N of A and, $\forall i \in I$, a nonempty compact convex subset B_i of A_i such that, for each $x \in A \backslash N$, there exists $i \in I$ such that $B_i \cap S_i(x) \neq \emptyset$.

Then, there exists $\bar{x} \in A$ such that $S_i(\bar{x}) = \emptyset$ for all $i \in I$.

We propose the following generalized convexity definitions. Let D, K and H be sets, X be a vector space. Let $F, G : D \times X \to 2^K$ and $T : H \to 2^D$ be multifunctions. For $x \in H, F$ is called G-quasiconvex wih respect to $T(x)$ of type 1 if, $\forall \xi, \eta, z \in X, \forall \lambda \in [0, 1]$, one has the implication

$$[F(t, \xi) \not\subseteq G(t, z) \text{ and } F(t, \eta) \not\subseteq G(t, z), \forall t \in T(x)]$$
$$\Rightarrow [F(t, (1-\lambda)\xi + \lambda\eta) \not\subseteq G(t, z), \forall t \in T(x)]. \tag{3}$$

F is said to be G-quasiconvex with respect to $T(x)$ of type 2 if in (3) we replace $\forall t$ by $\exists t$.

F is said to be G-quasiconvexlike with respect to $T(x)$ of type 1 if $\forall \xi, \eta, z \in X, \forall \lambda \in [0, 1]$, one has the implication

$$[F(t, \xi) \bigcap G(t, z) \neq \emptyset \text{ and } F(t, \eta) \bigcap G(t, z) \neq \emptyset, \forall t \in T(x)]$$

$$\Rightarrow [F(t,(1-\lambda)\xi+\lambda\eta)\bigcap G(t,z)\neq\emptyset, \forall t\in T(x)]. \tag{4}$$

If $\forall t$ in (4) is replaced by $\exists t$, we say that F is G-quasiconvexlike with respect to $T(x)$ of type 2.

3. Main Results

In this section we establish sufficient conditions for the solution existence of four problems (SQIP1) - (SQIP4).

Theorem 3.1. For (SQIP1) assume the following.

(i) $\forall i\in I, \forall x\in A$, considering $F_i(t_i,y_i,x)$ and $G_i(t_i,x_i,x)$, F_i is G_i-quasiconvex, in the first two variables with respect to $T_i(x)$ of type 1; moreover, $\forall t_i\in T_i(x), F_i(t_i,x_i,x)\subseteq G_i(t_i,x_i,x)$;

(ii) $\forall i\in I, \forall y_i\in A_i$, $\{x\in A : \exists t_i\in T_i(x), F_i(t_i,y_i,x)\subseteq G_i(t_i,x_i,x)\}$ is closed;

(iii) $\forall i\in I, \forall x\in A$, $A_i\cap K_i(x)\neq\emptyset$, $K_i(x)$ is convex; $\mathrm{cl}K(.)$ is usc and $K_i^{-1}(y_i)$ is open in A for all $y_i\in A_i$;

(iv) if A is not compact then there exist a nonempty compact subset N of A and, $\forall i\in I$, a nonempty compact convex subset B_i of A_i such that, for each $x\in A\backslash N$, there exist $i\in I$, and $\bar{y}_i\in B_i\cap K_i(x)$ with $F_i(t_i,\bar{y}_i,x)\not\subseteq G_i(t_i,x_i,x)$ for all $t_i\in T_i(x)$.

Then, (SQIP1) has solutions.

Proof. For each $i\in I$ and $x\in A$ set

$$E_i=\{x\in A : x_i\in \mathrm{cl}K_i(x)\},$$

$$P_i(x) = \{z_i \in A_i : F_i(t_i, z_i, x) \not\subseteq G_i(t_i, x_i, x), \forall t_i \in T_i(x)\},$$

$$S_i(x) \;=\; \begin{cases} K_i(x) \cap P_i(x) & \text{if } x \in E_i, \\ A_i \cap K_i(x) & \text{if } x \in A \backslash E_i, \end{cases}$$

For $y_i \in A_i$ one has

$$\begin{aligned} S_i^{-1}(y_i) &= \{x \in E_i \,:\, x \in K_i^{-1}(y_i) \cap P_i^{-1}(y_i)\} \cup \{x \in A \backslash E_i \,:\, x \in K_i^{-1}(y_i)\} \\ &= [E_i \cap K_i^{-1}(y_i) \cap P_i^{-1}(y_i)] \cup [(A \backslash E_i) \cap K_i^{-1}(y_i)] \\ &= [(A \backslash E_i) \cup P_i^{-1}(y_i)] \cap K_i^{-1}(y_i). \end{aligned}$$

Hence

$$A \backslash S_i^{-1}(y_i) \;=\; [E_i \cap (A \backslash P_i^{-1}(y_i))] \cup [A \backslash K_i^{-1}(y_i)].$$

Since cl$K_i(.)$ is usc, E_i is closed. By (iii), $A \backslash K_i^{-1}(y_i)$ is also closed. By (ii) the set

$$A \backslash P_i^{-1}(y_i) = \{x \in A : \exists t_i \in T_i(x), F_i(t_i, y_i, x) \subseteq G_i(t_i, x_i, x)\} \tag{5}$$

is closed too. Thus, (5) shows that $S_i^{-1}(y_i)$ is open in A.

By the G-quasiconvexity of F_i assumed in (i), $P_i(x)$ is convex and hence $S_i(x)$ is convex for all $x \in A$.

Furthermore, since $F_i(t_i, x_i, x) \subseteq G_i(t_i, x_i, x), \forall t_i \in T_i(x)$, one has $x_i \notin P_i(x)$. If $x \in E_i$ then $x_i \notin S_i(x)$. If $x \in A \backslash E_i$, then $x_i \notin$cl$K_i(x)$ and hence $x_i \notin S_i(x)$. By assumption (iv), $\forall x \in A \backslash N, \forall i \in I, \exists B_i \subseteq A_i$(nonempty compact convex) such that $\exists i \in I, \exists \bar{y}_i \in B_i \cap K_i(x)$ with $\bar{y}_i \in P_i(x)$. Therefore, $B_i \cap K_i(x) \neq \emptyset$. Now that all the assumptions of Theorem 2.1 are satisfied, there exists $\bar{x} \in A$ such that $S_i(\bar{x}) \neq \emptyset, \forall i \in I$. Since $A_i \cap K_i(\bar{x}) \neq \emptyset$, $\bar{x}$ must be in E_i. Then $\emptyset = S_i(\bar{x}) = K_i(\bar{x}) \cap P_i(\bar{x})$. Consequently, for any $y_i \in K_i(\bar{x})$ one has $\bar{y}_i \notin P_i(\bar{x})$, i.e. $F_i(\bar{t}_i, y_i, \bar{x}) \subseteq G_i(\bar{t}_i, \bar{x}_i, \bar{x})$ for all $i \in I$ and for some $\bar{t}_i \in T_i(\bar{x})$, which means that $\bar{x}$ is a solution of (SQIP1). $\square$

Remark 3.1. If, $\forall i \in I, \forall y_i \in A_i, T_i(.)$ is a usc multifuntion with compact values, $F_i(., y_i, .)$ is lsc and $G_i(., ., .)$ is a usc multifunction with closed values, then assumption (ii) of Theorem 3.1 is fulfilled.

Proof. Let

$$M_{y_i} = \{x \in A : \exists t_i \in T_i(x), F_i(t_i, y_i, x) \subseteq G_i(t_i, x_i, x)\},$$

$x_\alpha \in M_{y_i}, x_\alpha \to x^*$ and $L = \{x_\alpha\} \cup \{x^*\}$. Then $\exists t_{i\alpha} \in T_i(x_\alpha)$ such that

$$F_i(t_{i\alpha}, y_i, x_\alpha) \subseteq G_i(t_{i\alpha}, x_{\alpha i}, x_\alpha).$$

Since $T_i(L)$ is compact, by extracting a subnet if necessary we assume $t_{i\alpha} \to t^*$ for some $t_i^* \in T_i(L)$. Since $T_i(.)$ is closed, $t_i^* \in T_i(x^*)$.

By the lower semicontinuity of $F_i(., y_i, .), \forall z_i^* \in F_i(t_i^*, y_i, x^*), \exists z_{i\alpha} \in F_i(t_{i\alpha}, y_i, x_\alpha)$ such that $z_{i\alpha} \to z_i^*$. Since $z_{i\alpha} \in G_i(t_{i\alpha}, x_{\alpha i}, x_\alpha)$ and G_i is closed, one has $z_i^* \in G_i(t_i^*, x_i^*, x^*)$. Thus M_{y_i} is closed. $\square$

The following example shows that the converse is not true.

Example 3.1. Let $I = \{1\}, X_1 = Y_1 = Z_1 = R$, $A_1 = [0, 1]$, $K_1(x) \equiv [0, 1]$, $G_1(t_1, y_1, x) \equiv R_+$,

$$T_1(x) = \begin{cases} [1.5, 2] & \text{if } x = 0.5, \\ [0, 1] & \text{otherwise,} \end{cases}$$

$$F_1(t_1, y_1, x) = \begin{cases} [0.5, 1] & \text{if } t_1 = x = 0.5, \\ [1, 2] & \text{otherwise.} \end{cases}$$

Then, $M_{y_1} \equiv [0, 1], \forall y_1 \in A_1$, is closed, but $T_1(.)$ is not usc and $F_1(., y_1, .)$ is not lsc.

Passing to system (SQIP2) we have

Theorem 3.2. Assume (iii) as in Theorem 3.1. Assume further that

(i$_2$) $\forall i \in I, \forall x \in A$, considering $F_i(t_i, y_i, x)$ and $G_i(t_i, x_i, x)$, F_i is G_i-quasiconvex, in the first two variables with respect to $T_i(x)$ of type 2; moreover, $\exists t_i \in T_i(x), F_i(t_i, x_i, x) \subseteq G_i(t_i, x_i, x)$;

(ii$_2$) $\forall i \in I, \forall y_i \in A_i$, $\{x \in A : \forall t_i \in T_i(x), F_i(t_i, y_i, x) \subseteq G_i(t_i, x_i, x)\}$ is closed;

(iv$_2$) if A is not compact then there exists a nonempty compact subset N of A and, $\forall i \in I$, a nonempty compact convex subset B_i of A_i such that, for each $x \in A \backslash N$, there are $i \in I$, and $\bar{y}_i \in B_i \cap K_i(x)$ with $F_i(t_i, \bar{y}_i, x) \not\subseteq G_i(t_i, x_i, x)$ for some $t_i \in T_i(x)$.

Then, (SQIP2) has a solution.

Proof. The argument is similar to that of the proof of Theorem 3.1, but now the definition of P_i is

$$P_i(x) = \{z_i \in A_i : \exists t_i \in T_i(x), F_i(t_i, z_i, x) \not\subseteq G_i(t_i, x_i, x)\}. \qquad \square$$

Similarly as in Remark 3.1 it is not hard to prove that if $\forall i \in I, \forall y_i \in A_i, F_i(., y_i, .)$ and $T_i(.)$ are lsc and $G_i(., ., .)$ is a usc multifunction with closed values, then condition (ii$_2$) is satisfied.

As a typical example we give a consequence of Theorem 3.2 for one of the special case, which is a system of generalized vector quasiequilibrium problems studied in Ref. 23 and mentioned in (a) of Section 1:

(SQEP2) Find $\bar{x}$ such that, $\forall i \in I, \bar{x}_i \in A_i \bigcap \mathrm{cl} K_i(\bar{x})$ and, $\forall y_i \in K_i(\bar{x}), \forall \bar{t}_i \in T_i(\bar{x})$,

$$F_i(\bar{t}_i, y_i, \bar{x}) \subseteq C_i(\bar{x}).$$

Corollary 3.1. For (SQEP2) assume that

(a) $\forall i \in I, \forall x \in A$, considering $F_i(t_i, y_i, x)$ and $C_i(x)$, F_i is C_i-quasiconvex, in the first two variables, with respect to $T_i(x)$ of type 2; moreover, $\exists t_i \in T_i(x), F_i(t_i, x_i, x) \subseteq C_i(x)$;

(b) $\forall i \in I, \forall y_i \in A_i$, $\{x \in A : \forall t_i \in T_i(x), F_i(t_i, y_i, x) \subseteq C_i(x)\}$ is closed in A;

(c) $\forall i \in I, \forall x \in A, \forall y_i \in A_i$,$A_i \cap K_i(x) \neq \emptyset$, $K_i(x)$ is convex, $\mathrm{cl}K(.)$ is usc and $K_i^{-1}(y_i)$ is open in A;

(d) there exist a nonempty compact subset N of A and, $\forall i \in I$, a nonempty compact convex subset B_i of A_i such that, for each $x \in A \backslash N$, there are $i \in I$, and $\bar{y}_i \in B_i \cap K_i(x)$ with $F_i(t_i, \bar{y}_i, x) \not\subseteq C_i(x)$ for some $t_i \in T_i(x)$.

Then, (SQEP2) has solutions.

Remark 3.2. Corollary 3.1 improves Theorem 3.1 of Ref. 23. Assumptions (a) and (b) are weaker than the corresponding ones in Ref. 23. Namely (b) is weaker than the semicontinuity assumptions as discussed in Remark 3.1. The quasiconvexity assumed in (a) is weaker than the following $C_i(x)$-quasiconvexity assumed in Ref. 23: $\forall \xi, \eta \in X_i, \forall \lambda \in [0,1], \forall t_i \in T_i(x)$, one has either

$$F_i(t_i, \xi, x) \subseteq F_i(t_i, (1-\lambda)\xi + \lambda\eta, x) + C_i(x)$$

or (6)

$$F_i(t_i, \eta, x) \subseteq F_i(t_i, (1-\lambda)\xi + \lambda\eta, x) + C_i(x).$$

Indeed, assume (6). Checking that F_i is C_i-quasiconvex (in the first two variables) with respect to $T_i(x)$ of type 2, we suppose to the contrary that $F_i(t_i, \xi, x) \not\subseteq C_i(x)$ and $F_i(t_i, \eta, x) \not\subseteq C_i(x)$, for some $t_i \in T_i(x)$ but, for all $t_i \in T_i(x)$, $F_i(t_i, (1-\lambda)\xi +$

$\lambda\eta, x) \subseteq C_i(x)$. Then, by (6) either

$$F_i(t_i, \xi, x) \subseteq C_i(x) + C_i(x) = C_i(x)$$

or

$$F_i(t_i, \eta, x) \subseteq C_i(x) + C_i(x) = C_i(x),$$

for all $t_i \in T_i(x)$, a contracdition.

The following example indicates that the converse is not true, i.e. our C_i-quasiconvexity of type 2 is strictly weaker than (6). It gives also a case where Corollary 3.1 can be applied but Theorem 3.1 of Ref. 23 does not work.

Example 3.2. Let $I = \{1\}$ and X_1, Y_1, Z_1, A_1, K_1 be as in Example 3.1. Let $C_1(x) = R_+$,

$$T_1(x) = \begin{cases} [0.5, 2] & \text{if } x = 0.5, \\ [0, 1] & \text{otherwise,} \end{cases}$$

$$F_1(t, y, x) = \begin{cases} [0.5, 1] & \text{if } t = y = x = 0.5, \\ [0, 0.5] & \text{otherwise.} \end{cases}$$

To see that F_1 is not $C_i(x)$-quasiconvex stated in (6) we take $x = 0.5, t = 0.5, \xi = 0, \eta = 1$ and $\lambda = 0.5$. Then,

$$F_1(t, \xi, x) = F_1(0.5, 0, 0.5) = [0, 0.5] \not\subseteq F_1(t, (1-\lambda)\xi + \lambda\eta, x) + C_1(x) = [0.5, 1] + R_+,$$

$$F_1(t, \eta, x) = F_1(0.5, 1, 0.5) = [0, 0.5] \not\subseteq F_1(t, (1-\lambda)\xi + \lambda\eta, x) + C_1(x) = [0.5, 1] + R_+.$$

Moreover, both $T_1(.)$ and $F_1(., y, .)$ are not lsc as required in Theorem 3.1 of Ref. 23. It is not hard to see that all assumptions of Corollary 3.1 are satisfied. So by this corollary the considered problem has solutions. By direct checking one sees that the solution set is [0,1].

Theorem 3.2 can be modified as follows to get a solution existence for (SQIP3).

Theorem 3.3. For (SQIP3) assume (iii) as in Theorem 3.1 and

(i_3) $\forall i \in I, \forall x \in A$, considering $F_i(t_i, y_i, x)$ and $G_i(t_i, x_i, x)$, F_i is G_i-quasiconvexlike, in the first two variables, with respect to $T_i(x)$ of type 1; moreover, $F_i(t_i, x_i, x) \subseteq G_i(t_i, x_i, x), \forall t_i \in T_i(x)$;

(ii_3) $\forall i \in I, \forall y_i \in A_i$, $\{x \in A : \exists t_i \in T_i(x), F_i(t_i, y_i, x) \cap G_i(t_i, x_i, x) \neq \emptyset\}$ is closed in A;

(iv_3) if A is not compact, then there are a nonempty compact subset N of A and, $\forall i \in I$, a nonempty compact convex subset B_i of A_i such that, for each $x \in A \backslash N$, there are $i \in I$ and $\bar{y}_i \in B_i \cap K_i(x)$ with $F_i(t_i, \bar{y}_i, x) \cap G_i(t_i, x_i, x) \neq \emptyset$ for all $t_i \in T_i(x)$.

Then, (SQIP3) has a solution.

Proof. By using another set $P_i(x)$ defined by

$$P_i(x) = \{z_i \in A_i : F_i(t_i, z_i, x) \cap G_i(t_i, x_i, x) \neq \emptyset, \forall t_i \in T_i(x)\}$$

and similar argument as that of the proof of Theorem 3.1 one gets the conclusion.□

Remark 3.3. Assumption (ii_3) is weaker than the following semicontinuity assumption: $\forall i \in I, \forall y_i \in A_i, F_i(., y_i, .)$ and $T_i(.)$ are usc and have nonempty compact values and $G_i(., ., .)$ is usc and has nonempty closed values. Also our G_i-quasiconvexlikeness assumed in (i_3) is weaker than the $C(x)$-quasiconvexlikeness assumed in Theorem 3.2 of Ref. 23. The proof is similar as that in Remark 3.2. So while applied to the special case considerd in Ref. 23, Theorem 3.3 improves Theorem 3.2 of Ref. 23.

To see the generality of our problem setting we will derive below Corollary 3.2 of Theorem 3.3 for the following system of generalized vector equilibrium problems considered in Ref. 22 and mentioned in (c) of Section 1 :

(SGEP) Find $\bar{x}$ such that, $\forall i \in I$ and $\forall y_i \in A_i$,

$$F_i(y_i, \bar{x}) \not\subseteq -\mathrm{int}C_i(\bar{x}).$$

For the sake of comparison, recall the quasiconvexlikeness introduced in Ref. 20. Let X and Y be vector spaces, D be a set, $F : X \times D \to 2^Y$ and $C : D \to 2^Y$ be multifunctions with $C(x)$ being closed convex cone with nonempty interior for each $x \in D$. Then for $x \in D, F(., x)$ is called $C(x)$-quasiconvexlike if $\forall \xi, \eta \in X, \forall \lambda \in [0, 1]$, either

$$F((1-\lambda)\xi + \lambda\eta, x) \subseteq F(\xi, x) - C(x)$$

or

$$F((1-\lambda)\xi + \lambda\eta, x) \subseteq F(\eta, x) - C(x).$$

Corollary 3.2. For (SGEP) assume that

(a) $\forall i \in I, \forall x \in A$, $F_i(., x)$ is $Y_i \backslash -\mathrm{int}C_i(x)$-quasiconvexlike with respect to $T(x) = \{x\}$ in the sense of (4) , i.e. , $\forall \xi, \eta \in X_i, \forall \lambda \in [0, 1]$, one has

$$[F_i(\xi, x) \subseteq -\mathrm{int}C_i(x) \text{ and } F_i(\eta, x) \subseteq -\mathrm{int}C_i(x)]$$
$$\Rightarrow [F_i((1-\lambda)\xi + \lambda\eta, x) \subseteq -\mathrm{int}C_i(x)];$$

moreover, $F_i(x_i, x) \not\subseteq -\mathrm{int}C_i(x), \forall x \in A$;

(b) $\forall y_i \in A_i, \{x \in A : F_i(y_i, x) \not\subseteq -\mathrm{int}C_i(x)\}$ is closed;

(c) if A is not compact, then there are a nonempty compact subset N of A and, $\forall i \in I$, a nonempty compact convex subset B_i of A_i such that, $\forall x \in A \backslash N$, $\exists i \in I$, $\exists \bar{y}_i \in B_i$ with $F_i(\bar{y}_i, x) \subseteq -\mathrm{int}C_i(x)$.

Then, (SGEP) has solutions.

Observe that similarly as in Remark 3.2 for G_i-quasiconvexity, we can see that the above $C_i(x)$-quasiconvexlikeness, defined in Ref. 20, implies the $Y_i \backslash -\text{int}C_i(x)$-quasiconvexlikeness assumed in (a). The following example shows that the converse does not hold and that Corollary 3.2 improves Theorem 3 of Ref. 22 (and also Theorem 3 of Ref. 20).

Example 3.3. Let $I = \{1\}$, $X_1 = Y_1 = Z_1 = R, A_1 = [0, 1]$, $C_1(x) \equiv R_+$ and $F_1(y, x) = 1 - (y - \frac{1}{2})^2$. Then all assumptions of Corollary 3.2 are satisfied and hence (SGEP) in this case has solutions. In fact, it is clear that the solution set is the whole $A_1 = [0, 1]$. However, F_1 is not $C_i(x)$-quasiconvexlike and then Theorem 3 of Ref. 22 cannot be applied.

Passing finally to (SQIP4) we have

Theorem 3.4. For (SQIP4) assume (iii) as in Theorem 3.1 and

(i$_4$) this is (i$_3$) with "type 1" and "$\forall t_i$" replaced by "type 2" and "$\exists t_i$", respectively;

(ii$_4$) $\forall i \in I, \forall y_i \in A_i$, $\{x \in A : \forall t_i \in T_i(x), F_i(t_i, y_i, x) \cap G_i(t_i, x_i, x) \neq \emptyset\}$ is closed in A;

(iv$_4$) if A is not compact, then there exist a nonempty compact subset N of A and, $\forall i \in I$, a nonempty compact convex subset B_i of A_i such that, for each $x \in A \backslash N$, there exist $i \in I$ and $\bar{y}_i \in B_i \cap K_i(x)$ with $F_i(t_i, \bar{y}_i, x) \cap G_i(t_i, x_i, x) \neq \emptyset$, for some $t_i \in T_i(x)$.

Then, (SQIP4) has solutions.

Remark 3.4. Similarly as for the previous three problems (SQIP1) - (SQIP3),

(ii_4) is satisfied if $\forall i \in I, \forall y_i \in A_i$, $T_i(.)$ is lsc, $F_i(., y_i, .)$ and $G_i(., ., .)$ are usc, $F_i(., y_i, .)$ has nonempty compact values and $G_i(., ., .)$ has nonempty closed values. The converse is not true.

4. Applications

To see a variety of applications of the main results in Section 3 let us derive some consequences for the fixed point theory and quasioptimization theory. These topics are seemingly not very close to inclusion and equilibrium problems.

The following fixed point result is Theorem 4.1 of Ref. 23 and is proved by invoking to Theorem 3.4.

Corollary 4.1. Let, $\forall i \in I, X_i$ is a Hilbert space, $A_i \subseteq X_i$ is a closed convex subset. Let $A = \prod_{i\in I} A_i$. Assume that

(i) $\forall i \in I, T_i : A \to 2^{A_i}$ is lsc and has nonempty values;

(ii) if A is not compact, then there are a nonempty compact subset N of A and, $\forall i \in I$, a nonempty compact convex subset B_i of A_i such that, for each $x \in A \backslash N$, $\exists i \in I$, $\exists \bar{y}_i \in B_i$ with $< x_i - t_i, \bar{y}_i - x_i >< 0$, for some $t_i \in T_i(x)$.

Then there is $\bar{x} \in A$ such that $\bar{x} \in \prod_{i\in I} T_i(\bar{x})$.

Proof. Set

$$K_i(x) = A_i, \forall x \in A,$$

$$F_i(t_i, y_i, x) = \{< x_i - t_i, y_i - x_i >\},$$

$$G_i(t_i, y_i, x) \equiv [0, +\infty).$$

It is not hard to see that all the assumptions of Theorem 3.4 are satisfied. Therefore, there exists $\bar{x} \in A$ such that, $\forall i \in I, \forall y_i \in A_i$, $\forall t_i \in T_i(\bar{x})$,

$$< \bar{x}_i - t_i, y_i - \bar{x}_i >\geq 0.$$

Taking $y_i = t_i = \bar{t}_i$ for any fixed $t_i \in T_i(\bar{x})$ one gets $< \bar{x}_i - \bar{t}_i, \bar{t}_i - \bar{x}_i >\geq 0$, and hence $\| \bar{x}_i - \bar{t}_i \| = 0$. So $\bar{x}_i = \bar{t}_i \in T_i(\bar{x}), \forall i \in I$. □

Applying Theorem 3.1 we can modify Corollary 4.1 to get a new fixed point result as follows.

Corollary 4.2. Assume (ii) of Corollary 4.1 and replace (i) by

(i') $\forall i \in I, T_i : A \to 2^{A_i}$ is usc and has nonempty compact values.

Then there exists $\bar{x} \in A$ such that $\bar{x}_i \in T_i(\bar{x}), \forall i \in I$.

Proof. Setting K_i, F_i and G_i as for Corollary 4.1 and applying Theorem 3.1 and Remark 3.1 one obtains the conclusion. □

Passing to quasi-optimization we first state a solution existence for the following quasivariational inclusion problem, which is special case of (SQIP2). Let X, Y and Z be Hausdorff topological vector spaces, $A \in X$ be a nonempty closed convex subset and $K : A \to 2^X, T : A \to 2^Z$ and $F : T(A) \times X \times A \to 2^Y$ be multifunctions. Let Y be ordered by a closed convex cone C with $\text{int}C \neq \emptyset$. The quasivariational inclusion problem is

(QIP) Find $\bar{x} \in K(\bar{x})$, such that, $\forall y \in K(\bar{x})$ and $\forall \bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \subseteq F(\bar{t}, \bar{x}, \bar{x}) + C.$$

The proof of the following consequence of Theorem 3.2, is easy and omitted.

Corollary 4.3. Assume for (QIP) that

(i) A is compact;

(ii) $\forall x \in A$, considering $F(t, y, x)$ and $G(t, x, x) := F(t, x, x) + C$, F is G-quasiconvex, in the first two variables, with respect to $T(x)$ of type 1;

(iii) $\forall y \in A$, $\{x \in A : \forall t \in T(x), F(t, y, x) \subseteq F(t, x, x) + C\}$ is closed in A;

(iv) $\forall x \in A, K(x)$ is nonempty, closed and convex and $A \cap K(x) \neq \emptyset$; $K^{-1}(y)$ is open in A for all $y \in A$.

Then, problem (QIP) has solutions.

We now investigate the following quasi-optimization problem (studied in Refs. 10 and 30)

(QOP) Find $\bar{x} \in K(\bar{x})$ and $\bar{t} \in T(\bar{x})$ such that

$$F(\bar{t}, \bar{x}, \bar{x}) \bigcap \operatorname{Min}\{F(\bar{t}, K(\bar{x}), \bar{x})/C\} \neq \emptyset,$$

where $\operatorname{Min}\{H/C\}$ denotes the set of Pareto efficient points of set $H \subseteq Y$ (with respect to the ordering cone C).

As a consequence of Corollary 4.3 we obtain the following sufficient condition for the solution existence of (QOP).

Corollary 4.4. For (QOP) assume (ii) - (iv) of Corollary 4.3 and replace (i) by

(i') A is compact; the conjugate cone C^* of C has a weak* compact base; $F(t, x, x)$ is compact for all $(t, x) \in T(A) \times A$.

Then (QOP) has solutions.

Proof. Following Corollary 4.3 one has $(\bar{x}, \bar{t}) \in K(\bar{x}) \times T(\bar{x})$ such that, $\forall y \in K(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \subseteq F(\bar{t}, \bar{x}, \bar{x}) + C. \tag{7}$$

By (i'), $\text{Min}F(\bar{t},\bar{x},\bar{x})/C \neq \emptyset$. Suppose that $\bar{v} \in \text{Min}\{F(\bar{t},\bar{x},\bar{x})/C\}$ but $\bar{v} \notin \text{Min}\{F(\bar{t},K(\bar{x}),\bar{x})/C\}$. Then one has $y \in F(\bar{t},K(\bar{x}),\bar{x})$ such that

$$\bar{v} - y \in C \backslash ((-C) \cap C).$$

By virture of (7) $y \in F(\bar{t},\bar{x},\bar{x}) + C$, i.e. $y = \hat{v} + c$ for some $\hat{v} \in F(\bar{t},\bar{x},\bar{x})$ and $c \in C$. Therefore $\bar{v} - \hat{v} \in c + C\backslash((-C) \cap C) = C\backslash((-C) \cap C)$, contradicting the fact that $\bar{v} \in \text{Min}\{F(\bar{t},\bar{x},\bar{x})/C\}$. $\square$

Corollary 4.4 is new. It is similar to the corresponding results in Ref. 10 and 30 but different.

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