Lower Semicontinuity and Upper Semicontinuity of the Solution Sets and Approximate Solution Sets of Parametric Multivalued Quasivariational Inequalities[1]

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Communicated by S. Schaible

[1]This research was partially supported by the National Basic Rerearch Program in Natural Sciences of Vietnam. The final part of this work was completed during a stay of the first author at the Department of Mathematics, University of Pau, Pau, France, whose hospitality is acknowledged.

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Abstract. We consider the semicontinuity of the solution set and the approximate solution set of parametric multivalued quasivariational inequalities in topological vector spaces. Three kinds of problems arising from the multivalued situation are investigated. A rather complete picture, which is symmetric for the two kinds of semicontinuity (lower and upper semicontinuity) and for the three kinds of the multivalued quasivariational inequality problems, is supplied. Moreover, we use a simple technique to prove the results. The results obtained improve several known ones in the literature.

Key Words. Multivalued quasivariational inequalities, topological vector spaces, lower semicontinuity, upper semicontinuity, ϵ-solutions.

1. Introduction

Stability of the solution set, to a parametric optimization problem in general and to a parametric variational inequality problem in particular, has been studied intensively in the literature. The meaning of stability varies from one author to another. Stability can be understood as the semicontinuity, continuity, Lipschitz continuity or some kinds of (generalized) differentiability of the optimal value of the objective function. Stability or sensitivity can also be referred to as one of these properties of the solution set.

Observe that for parametric variational inequality problems, many efforts have been made to establish the continuity, Lipschitz continuity or (generalized) differentiability, see e.g. Refs. 1-12. Among them, only Ref. 6 dealt with multivalued problems, while only Ref. 2 investigated quasivariational inequalities.

Much fewer works have been devoted to the semicontinuity of the solution set, see Refs. 13-16, or Ref. 17 and Ref. 18 for related properties.

However, it should be noted that in many applications, a semicontinuity property is sufficient. For example, to prove the existence of an equilibrium for a competitive economy in the Walras-Ward model and Arrow-Deubreu-Mckenzie model, usually based on fixed point arguments, it requires only the upper semicontinuity of the solution set to a variational inequality, see e.g. Ref. 19. Note further that lower semicontinuity seems to be less often required in applications. Moreover, lower semicontinuity is stronger than upper semicontinuity and also more difficult to study, for a discussion on this issue see e.g. Ref. 22. These two factors may be the main reasons for lower semicontinuity to be considered very little even in optimization in general. We observed only Refs. 13, 15, 16, 20-23 on lower semicontinuity of the solution set: Refs. 20-22 for minimization problems, Ref. 23 for generalized inequality systems and Refs.13, 15 and Ref. 16 for variational inequalities.

Since lower semicontinuity and upper semicontinuity constitute continuity, the understanding about the continuity of solution sets will be incomplete without lower semicontinuity results. This motivates our study in this work. So, the aim of the present paper is to provide a rather complete consideration on both lower semicontinuity and upper semicontinuity of the solution sets to parametric multivalued quasivariational inequality problems in Hausdorff topological vector spaces. We try to go insight the symmetric nature of these two semicontinuity properties. It turns out to us that lower semicontinuity is not so difficult to investigate as several authors mentioned. Moreover, we can prove both properties by the same technique. To emphasize the perturbation nature of stability, in all the proofs we use only a contradiction argument on the convergence of nets. It seems to us that this is one of the shortest, clearest and simplest ways to get semicontinuity. To consider all possible kinds of the solution sets the paper is organized as follows . Section 2 contains definitions and preliminaries. Section 3 is devoted to lower semicontinuity of all three possible solution sets. To reveal a symmetric nature we pass to upper semicontinuity of all these solution sets in Section 4. In Section 5 we come back to lower semicontinuity consideration but for the three approximate solution sets, to see that some strict assumptions can be avoided in this case. Another symmetry of the picture of stability is revealed in Section 6, when we prove that, for upper semicontinuity, another definition of approximate solutions is suitable and the assumptions should be the same as for the corresponding exact solutions. Section 7 concludes the paper.

2. Preliminaries

The problems under our consideration are as follows. Let X be a Hausdorff topological vector space, U be a Hausdorff topological space and $A \subseteq X$ be a nonempty compact and convex subset. Let $T : U \times A \longrightarrow 2^{X^*}$, $K : U \times A \longrightarrow 2^X$ be multifunctions and $g : U \times A \longrightarrow A$ be a continuous (single-valued) mapping. For $u \in U$, consider the following three problems of

(WQVI$_u$): find $\bar{x} \in A \bigcap \mathrm{cl}K(u, \bar{x})$ such that, $\forall x \in K(u, \bar{x})$, $\exists \bar{t} \in T(u, \bar{x})$ such that

$$\langle \bar{t}, x - g(u, \bar{x}) \rangle \geq 0;$$

(MQVI$_u$): find $\bar{x} \in A \bigcap \mathrm{cl}K(u, \bar{x})$ such that, $\exists \bar{t} \in T(u, \bar{x})$, $\forall x \in K(u, \bar{x})$ such that

$$\langle \bar{t}, x - g(u, \bar{x}) \rangle \geq 0;$$

(SQVI$_u$): find $\bar{x} \in A \bigcap \mathrm{cl}K(u, \bar{x})$ such that, $\forall x \in K(u, \bar{x})$, $\forall t \in T(u, \bar{x})$,

$$\langle t, x - g(u, \bar{x}) \rangle \geq 0.$$

Let $W(u)$, $M(u)$ and $S(u)$ be the solution sets of (WQVI$_u$), (MQVI$_u$) and (SQVI$_u$), respectively. For the special case, where T is a (single-valued) mapping, we clearly have $W(u) = M(u) = S(u)$ since the three problems coincide. For the general case of T being a multifunction, we have

$$S(u) \subseteq M(u) \subseteq W(u)$$

and the inclusions may be proper (see Example 2.4 of Ref. 14).

Let us recall first some definitions. Consider a general case, where X and Y are topological spaces and $F : X \to 2^Y$ is a multifunction. F is said to be upper semicontinuous (usc) at $x_0 \in \mathrm{dom}F := \{x \in X : \; F(x) \neq \emptyset\}$ if for each neighborhood N of $F(x_0)$, there is a neighborhood M of x_0 such that $F(M) \subseteq N$. F is called usc in a set V if it is usc at every $x \in V$. If $V = \mathrm{dom}F$, we simply say that F is usc. In the sequel, any property defined at a point will be extended for a set in this way. F is termed closed at $x \in \text{dom}F$, if $\forall x_\gamma \in \text{dom}F : x_\gamma \to x,\ \forall y_\gamma \in F(x_\gamma) : y_\gamma \to y,\ y \in F(x)$. Upper semicontinuity and closedness are closely related (see e.g. Proposition 1.1 of Ref. 14). Note also that F is closed if and only if its graph

$$\text{gr}F := \{(x,y) \in X \times Y : y \in F(x)\}$$

is a closed subset of $X \times Y$.

A symmetric property (but stronger than upper semicontinuity) is the following lower semicontinuity. $F : X \longrightarrow 2^Y$ is said to be lower semicontinuous (lsc) at $x \in \text{dom}F$, if $\forall y \in F(x),\ \forall x_\gamma \in \text{domF} :\ x_\gamma \to x,\ \exists y_\gamma \in F(x_\gamma),\ y_\gamma \to y$. A multifunction is called continuous (at a point or in a set) if it is both usc and lsc (at a point or in a set, respectively).

Lemma 2.1 Ref. 24. Let X and Y be topological spaces and $F : X \to 2^Y$ be a multifunction. If F has compact values then F is usc at x if and only if for each net $x_\alpha \in X$ which converges to x and for each net $y_\alpha \in F(x_\alpha)$, there are $y \in F(x)$ and a subnet y_β such that $y_\beta \to y$.

3. Lower Semicontinuity of the Solution Sets

Let $U_0 \subseteq U$ be an open subset and $u_0 \in U_0$. Since the solution existence has been already intensively investigated in the literature, we assume now that for each $u \in U_0$, all of $W(u)$, $M(u)$ and $S(u)$ are nonempty.

We consider first the lower semicontinuity of the solution set $W(.)$. We denote

$$E(u) := \{x \in A : x \in \mathrm{cl}K(u, x)\}.$$

Theorem 3.1. Assume for ($\mathrm{WQVI_u}$) that the following conditions are satisfied:

(i) $K(.,.)$ is usc and has compact values in $(u_0, \boldsymbol{A})$ and $E(.)$ is lsc at u_0;

(ii) $\langle T(.,.),.\rangle$ is lsc in $(u_0, \boldsymbol{A}, \boldsymbol{A})$;

(iii) $\forall x \in W(u_0),\ \forall y \in K(u_0, x),\ \exists t \in T(u_0, x),\ \langle t, y - g(u_0, x)\rangle > 0.$

Then $W(.)$ is lsc at u_0.

Proof. Suppose to the contrary that there exist $x_0 \in W(u_0)$ and a net $u_\alpha \to u_0$ such that each net $x_\alpha \in W(u_\alpha)$ does not converge to x_0. By (iii), $\forall y \in K(u_0, x_0),\ \exists t_0 \in T(u_0, x_0)$,

$$\langle t_0, y - g(u_0, x_0)\rangle > 0. \tag{1}$$

Since $x_0 \in E(u_0)$ and $E(.)$ is lsc at u_0, there is a net $\bar{x}_\alpha \in E(u_\alpha)$ which converges to x_0. By the contradiction assumption, there must be a subnet $\bar{x}_\beta$ such that

$\overline{x}_\beta \notin W(\overline{u}_\beta)$, $\forall\beta$. That means $\exists y_\beta \in K(u_\beta, \overline{x}_\beta)$, $\forall t_\beta \in T(u_\beta, \overline{x}_\beta)$,

$$\langle t_\beta, y_\beta - g(u_\beta, \overline{x}_\beta)\rangle < 0. \tag{2}$$

Since $K(.,.)$ is usc and has compact values in $(u_0, \boldsymbol{A})$, by Lemma 2.1 there are subnets, denoted by the same notations as u_β, $\overline{x}_\beta$ and y_β and a point $y_0 \in K(u_0, x_0)$ such that $y_\beta \to y_0$. The assumption (ii) in turn shows the existence of $\bar{t}_\beta \in T(u_\beta, \overline{x}_\beta)$ such that

$$\langle \bar{t}_\beta, y_\beta - g(u_\beta, \overline{x}_\beta)\rangle \to \langle t_0, y_0 - g(u_0, x_0)\rangle.$$

By (2), we have $\langle t_0, y_0 - g(u_0, x_0)\rangle \leq 0$, which contradicts (1). The proof is complete. $\square$

Note that apart from assumption (iii) all the other ones are some kinds of semicontinuity assumptions and then weaker than the assumptions for all the mentioned results for the continuity and differentiability of the solution set. As we can see in Ref. 14 and also below in this paper, upper semicontinuity results can be established without any assumption like (iii). Since the lower semicontinuity is stronger than the upper one in nature, assumption (iii) is essential for Theorem 3.1 as shown by the following example.

Example 3.1. Let $X = R$, $\mathbf{A} = U = [0,1]$, $K(u,x) \equiv [0,1]$ and $g(x,u) = x$. Let $T(u,x) = \{u\}$, $\forall x \in [0,1]$, $\forall u \in [0,1]$. Then, $W(0) = [0,1]$ and $W(u) = \{0\}$ for all $u \neq 0$. Thus, $W(.)$ is not lsc at $u_0 = 0$. One sees that assumptions (i) and (ii) of Theorem 3.1 are satisfied but (iii) is not satisfied in this case.

By similar arguments based on the same technique of proof we obtain the following two theorems for the remaining two problems.

Theorem 3.2. Assume for $(MQVI_u)$ that the condition (i) of Theorem 3.1 is satisfied. Assume further that

(ii) $T(.,.)$ is lsc in (u_0, A);

(iii) $\forall x \in M(u_0),\ \exists t \in T(u_0, x),\ \forall y \in K(u_0, x),\ \langle t, y - g(u_0, x)\rangle > 0.$

Then $M(.)$ is lsc at u_0.

Theorem 3.3. Assume for $(SQVI_u)$ that condition (i) of Theorem 3.1 is satisfied. Assume further that

(ii) $T(.,.)$ is usc and has compact values in (u_0, A);

(iii) $\forall x \in S(u_0)$, $\forall t \in T(u_0, x)$, $\forall y \in K(u_0, x)$, $\langle t, y - g(u_0, x)\rangle > 0$.

Then $S(.)$ is lsc at u_0.

Remark 3.1

(a) The strict assumption (ii) of Theorem 3.3 can be modified as

(ii') $\forall u_\alpha \to u_0$, $\forall x_\alpha \to x_0$ (for some x_0 in A), $\forall y_\alpha \to y$ (for some y in X), $\forall t_\alpha \in T(u_\alpha, x_\alpha)$, $\exists t_\beta$ (subnet), $\exists t_0 \in T(u_0, x_0)$, such that $\langle t_\beta, y_\beta\rangle \to \langle t_0, y\rangle$.

This modification may help to avoid the compactness assumption in (ii) in some situations.

(b) Example 3.1 indicates also that assumptions (iii) of Theorems 3.2 and

3.3 are essential, since in this case $W(u) = M(u) = S(u)$, $\forall u$.

Remark 3.2. To the best of our knowledge, Ref. 13 and Ref. 15 are the only papers in the literature to deal with lower semicontinuity of the solution set of variational inequalities. Ref. 13 considers single-valued vector variational inequalities in finite dimensional spaces and establishes a sufficient condition for the lower semicontinuity with additional assumptions on monotonicity. Ref. 15 studies single-valued scalar variational inequalities in finite dimensional spaces. Under additional assumptions on pseudomonotonicity, a sufficient condition is established for the lower semicontinuity, but of another set of extraordinary solutions defined by

$$I(u) = \{\bar{x} \in A : \langle T(u, \bar{x}), x - \bar{x} \rangle \geq 0, \ \forall x \in K(u)\}$$

(note that here $\bar{x} \in \boldsymbol{A}$, not $\bar{x} \in K(u)$ as usual).

4. Upper Semicontinuity of the Solution Sets

We shall see now that under weaker assumptions, by the same technique of using convergent nets for a contradiction argument as above, we can establish upper semicontinuity of the solution sets. In particular, the strict assumption (iii), which is essential for lower semicontinuity, can be omitted.

First we modify some related results of Ref. 14 in the next two theorems. These results were stated in Ref. 14 in terms of closedness of $W(.)$ and $S(.)$ at u_0 and proved by a direct argument, not by a contradiction one. Here we restate them in terms of upper semicontinuity. Since we can use the same technique for proving in the whole present peper , we omit the proofs here. The reader may adapt the proofs of Theorems 6.1 and 6.3 to check these two theorems.

Theorem 4.1. For problem ($\mathrm{WQVI_u}$) assume that

(i) $K(.,.)$ is lsc in $(u_0, \boldsymbol{A})$ and $\mathrm{cl}K(.,.)$ is usc in $(u_0, \boldsymbol{A})$;

(ii) $T(.,.)$ is usc and has compact values in $(u_0, \boldsymbol{A})$.

Then, $W(.)$ is usc at u_0.

Theorem 4.3. For problem ($\mathrm{SQVI_u}$) assume that

(i) $K(.,.)$ is lsc in $(u_0, \boldsymbol{A})$ and $clK(.,.)$ is usc in $(u_0, \boldsymbol{A})$;

(ii) $\langle T(.,.), .\rangle$ is lsc in $(u_0, \boldsymbol{A}, \boldsymbol{A})$.

Then, $S(.)$ is usc at u_0.

Note that in Theorems 4.1 and 4.3 the assumed lower semicontinuity of the fixed point set $E(u)$ of $K(u,.)$, which is difficult to check, is omitted. Note also that the two assumptions (ii) here are symmetric to that of Theorems 3.1 and

3.3. To fulfil the gap of lacking a result for (MQVI$_u$) and to see a total picture of this symmetry let us prove the following results.

Theorem 4.2. For problem (MQVI$_u$) assume (i) and (ii) as in Theorem 4.1. Then, $M(.)$ is usc at u_0.

Proof. Suppose the existence of an open neighborhood V of $M(u_0)$ such that $\forall u_\alpha \to u_0$, $\exists x_\alpha \in M(u_\alpha), x_\alpha \notin V$, $\forall \alpha$. By the compactness of A one can assume that x_α converges to some $x_0 \in A \setminus V$. Because of the upper semicontinuity of $\mathrm{cl}K(.,.)$, $x_0 \in \mathrm{cl}K(u_0, x_0)$. Since $x_0 \notin M(u_0)$, $\forall t \in T(u_0, x_0)$, $\exists y \in K(u_0, x_0)$,

$$\langle t, y - g(u_0, x_0)\rangle < 0. \tag{3}$$

On the contrary, as $x_\alpha \in M(u_\alpha)$, $\exists t_\alpha \in T(u_\alpha, x_\alpha)$, $\forall y_\alpha \in K(u_\alpha, x_\alpha)$,

$$\langle t_\alpha, y_\alpha - g(u_\alpha, x_\alpha)\rangle \geq 0. \tag{4}$$

Assumption (ii) and Lemma 2.1 then imply that a subnet t_β and $t_0 \in T(u_0, x_0)$ exist such that $t_\beta \to t_0$. Let y_0 be corresponding to t_0 according to (3). Since

$K(.,.)$ is lsc there is $y_\beta \in K(u_\beta, x_\beta)$, $y_\beta \to y_0$. Substituting y_β into (4) and passing to limit one gets a contradiction with (3). □

Remark 4.1. Similarly as for Theorem 3.3, assumption (ii) of Theorems 4.1 and 4.2 can be replaced by assumption (ii') stated in Remark 3.1.

Remark 4.2. Let us compare Theorems 4.1 - 4.3 with some known results. Ref. 16 considered the case where X and U were reflexive Banach spaces, $K : U \to 2^X$ and a single-valued bifunction $g(.,.) : X \times X \to R$ replaced $\langle T(.,.),.\rangle$. The solution set $W(.)$ $(= M(.) = S(.))$ was proved to be usc under some semicontinuity and monotonicity assumptions. Ref. 13 considered single-valued vector variational inequalities in finite dimensional spaces and established the upper semicontinuity of the solution set under continuity and pseudomonotonicity assumptions. Theorems 4.1 - 4.3 do not involve any monotonicity assumptions. Ref. 15 considered the set $I(u)$ of extraordinary solutions mentioned in Remark

3.2 above. Ref. 14 considered only $W(.)$ and $S(.)$ and the main results were restated as Theorems 4.1 and 4.3 above.

5. Lower Semicontinuity of the ϵ-Solution Sets

In this section we will establish lower semicontinuity results similar to those in Section 3, but with the strict assumption (iii) omitted, for the approximate solution sets. For a fixed $\epsilon \geq 0$, we define an ϵ-solution to (WQVI$_u$) to be an $\bar{x} \in \boldsymbol{A} \bigcap \mathrm{cl}K(u, \bar{x})$ such that $\forall x \in K(u, \overline{x})$, $\exists \bar{t} \in T(u, \overline{x})$, $\langle \bar{t}, x - g(u, \overline{x}) \rangle \geq -\epsilon$. The set of all ϵ-solutions of (WQVI$_u$) is denoted by $W^\epsilon(u)$. The ϵ-solution sets $M^\epsilon(u)$ and $S^\epsilon(u)$ of (MQVI$_u$) and (SQVI$_u$), respectively, are defined similarly. Stimulated by Ref. 16 we use also the following modified definitions (with u_0 being the considered point)

$$\widetilde{W}^\epsilon(u) = \begin{cases} W(u_0) & \text{if } \ u = u_0, \\ W^\epsilon(u) & \text{if } \ u \neq u_0, \end{cases}$$

$$\widetilde{M}^\epsilon(u) = \begin{cases} M(u_0) & \text{if } \ u = u_0, \\ M^\epsilon(u) & \text{if } \ u \neq u_0, \end{cases}$$

$$\widetilde{S}^{\epsilon}(u) = \begin{cases} S(u_0) & \text{if } u = u_0, \\ S^{\epsilon}(u) & \text{if } u \neq u_0. \end{cases}$$

Theorem 5.1. For problem (WQVI$_u$) assume that the assumptions (i) and (ii) of Theorem 3.1 are satisfied. Then $\widetilde{W}^{\epsilon}(.)$ is lsc at u_0 for each $\epsilon > 0$.

Proof. Suppose to the contrary that there exist $x_0 \in \widetilde{W}^{\epsilon}(u_0)$ and a net $u_\alpha \to u_0$ such that each net $x_\alpha \in \widetilde{W}^{\epsilon}(u_\alpha)$ does not converge to x_0. Since $x_0 \in \widetilde{W}^{\epsilon}(u_0) = W(u_0)$, $\forall y \in K(u_0, x_0)$, $\exists t_0 \in T(u_0, x_0)$ such that

$$\langle t_0, y - g(u_0, x_0)\rangle \geq 0. \tag{5}$$

Since $x_0 \in E(u_0)$ and $E(.)$ is lsc at u_0, there is a net $\bar{x}_\alpha \in E(u_\alpha)$ such that $\overline{x}_\alpha \to x_0$. By the contradiction assumption, one finds a subnet $\bar{x}_\beta$ such that $\overline{x}_\beta \notin \widetilde{W}^{\epsilon}(u_\beta)$, $\forall \beta$, i.e., $\exists y_\beta \in K(u_\beta, \overline{x}_\beta)$, $\forall t_\beta \in T(u_\beta, \overline{x}_\beta)$,

$$\langle t_\beta, y_\beta - g(u_\beta, \overline{x}_\beta)\rangle < -\epsilon. \tag{6}$$

Applying Lemma 2.1 to $K(.,.)$ one has a subnet of y_β, denoted again by y_β, and a point $y_0 \in K(u_0, x_0)$ such that $y_\beta \to y_0$. By the lower semicontinuity of $\langle T(.,.),.\rangle$ at (u_0, x_0, y_0) there is a net $\bar{t}_\beta \in T(u_\beta, \bar{x}_\beta)$ such that

$$\langle \bar{t}_\beta, y_\beta - g(u_\beta, \overline{x}_\beta)\rangle \to \langle t_0, y_0 - g(u_0, x_0)\rangle.$$

By (6) this limit is less than or equal to $-\epsilon$. This contradicts (5) and completes the proof. □

By similar arguments we can establish the corresponding results for $\widetilde{M}^\epsilon(.)$ and $\widetilde{S}^\epsilon(.)$ as follows.

Theorem 5.2. Under the same assumptions (i) and (ii) of Theorem 3.2, $\widetilde{M}^\epsilon(.)$ is lsc at u_0 for each $\epsilon > 0$.

Theorem 5.3. For problem (SQVI_1) assume that conditions (i) and (ii) of Theorem 3.3 are satisfied. Then $\widetilde{S}^\epsilon(.)$ is lsc at u_0 for each $\epsilon > 0$.

Remark 5.1. Assumption (ii) of Theorem 5.3 can be replaced by (ii') in

Remark 3.1(a).

Example 5.1. Consider again the problem in Example 3.1. It is obvious that all the assumptions of Theorems 5.1 - 5.3 are satisfied at $u_0 = 0$. Then for each $\epsilon > 0$, $\widetilde{W}^\epsilon(.) \equiv \widetilde{M}^\epsilon(.) \equiv \widetilde{S}^\epsilon(.)$ are lsc at u_0, in spite of the fact that $W(.) \equiv M(.) \equiv S(.)$ are not lsc at u_0 as shown in Example 3.1. Now we verify directly this lower semicontinuity. We claim even more that there exists a neighborhood $U(0)$ of u_0, namely $U(0) = [0, \epsilon]$, such that, $\forall u \in U(0)$, $\widetilde{W}^\epsilon(u) = \widetilde{W}^\epsilon(0) = [0, 1]$. Indeed, we have, $\forall x \in [0, 1]$, $\forall y \in [0, 1]$,

$$u(y - x) \geq -u \geq -\epsilon,$$

i.e., $x \in \widetilde{W}^\epsilon(u)$.

Looking at all the theorems about lower semicontinuity (Theorems 3.1 -

3.3 and 5.1 - 5.3) we see that the assumption about $K(.,.)$ is the same:

(i) $K(.,.)$ is usc and has compact values in $(u_0, \boldsymbol{A})$ and $E(.)$ is lsc at u_0.

The following example gives a case where all assumptions of these six theorems are satisfied, and then all the solution sets $W(.)$, $M(.)$, $S(.)$, $\widetilde{W}^{\epsilon}(.)$, $\widetilde{M}^{\epsilon}(.)$ and $\widetilde{S}^{\epsilon}(.)$ are lsc, but $K(.,.)$ is not lsc.

Example 5.2. Let $X = R$, $\boldsymbol{A} = U = [0,1]$, $g(u,x) = \frac{1}{2}u + \frac{1}{2}x$, $T(u,x) = \{-1\}$ and

$$K(u,x) = \begin{cases} [0, \frac{1}{6}] \bigcup \{1\} \ \ if & u = \frac{1}{2}, x = 0, \\ [0, \frac{1}{3}u + \frac{1}{2}x] & \text{otherwise.} \end{cases}$$

It is easy to see that $E(u) = [0, \frac{2}{3}u]$ and hence even continuous, but $K(.,.)$ is only usc and not lsc (at $(\frac{1}{2}, 0)$). To check assumptions (iii) of all the theorem, for $(u,x) \in U \times \boldsymbol{A}$ we easily obtain by direct calculations that

$$\langle T(u,x), K(u,x) - g(u,x) \rangle := \{ \langle t, y - g(u,x) \rangle : \ t \in T(u,x), \ y \in K(u,x) \}$$

$$= \begin{cases} [\frac{1}{12}, \frac{1}{4}] \bigcup \{-\frac{3}{4}\} \; if & u = \frac{1}{2}, x = 0, \\ [\frac{u}{6}, \frac{u}{2}] & \text{otherwise.} \end{cases}$$

So, 0 does not belong to any image. Hence, all assumptions (iii) are fulfilled. All the other assumptions are also satisfied.

6. Upper Semicontinuity of the ϵ-Solution Sets

In contrast with the case of lower semicontinuity, where we can omit the strict assumption (iii) when passing to the modified approximate solution sets $\widetilde{W}^{\epsilon}(.)$, $\widetilde{M}^{\epsilon}(.)$ and $\widetilde{S}^{\epsilon}(.)$, for upper semicontinuity the assumptions remain the same when passing to approximate solution sets $W^{\epsilon}(.)$, $M^{\epsilon}(.)$ and $S^{\epsilon}(.)$ as follows.

Theorem 6.1. Under the same assumptions (i) and (ii) of Theorem 4.1, $W^{\epsilon}(.)$ is usc at u_0 for any $\epsilon \geq 0$.

Proof. Let $\epsilon \geq 0$ be fixed. Suppose to the contrary that there is an open neighborhood V of $W^{\epsilon}(u_0)$ such that for every net $u_{\alpha} \to u_0$, there is $x_{\alpha} \in W^{\epsilon}(u_{\alpha})$, $x_{\alpha} \notin V$. By the compactness of A one can assume that x_{α} converges to some $x_0 \in A \backslash V$. Since $\mathrm{cl}K(.,.)$ is usc at (u_0, x_0), $x_0 \in \mathrm{cl}K(u_0, x_0)$. Since $x_0 \notin W^\epsilon(u_0)$, there is $y_0 \in K(u_0, x_0)$ such that, $\forall t \in T(u_0, x_0)$,

$$\langle t, y_0 - g(u_0, x_0) \rangle < -\epsilon. \tag{7}$$

By the lower semicontinuity of $K(.,.)$ at (u_0, x_0), one has $y_\alpha \in K(u_\alpha, x_\alpha)$ such that $y_\alpha \to y_0$. Since $x_\alpha \in W^\epsilon(u_\alpha)$, there is $t_\alpha \in T(u_\alpha, x_\alpha)$ such that

$$\langle t_\alpha, y_\alpha - g(u_\alpha, x_\alpha) \rangle \geq -\epsilon. \tag{8}$$

Taking into account Lemma 2.1 one has a subnet t_β and a point $t_0 \in T(u_0, x_0)$ such that $t_\beta \to t_0$. Hence, passing (8) into limit one gets a contradiction with (7). The following two theorems can be proved similarly. $\square$

Theorem 6.2. Under the same assumptions (i) and (ii) of Theorem 4.2, $M^\epsilon(.)$ is usc at u_0 for any $\epsilon \geq 0$.

Remark 6.1. Similarly as Remarks 3.1(a) and 5.1, assumption (ii) of

Theorems 6.1 and 6.2 can be replaced by assumption (ii') in Remark 3.1(a).

Theorem 6.3. Under the same assumptions (i) and (ii) of Theorem 4.3, $S^\epsilon(.)$ is usc at u_0 for any $\epsilon \geq 0$.

The following example shows that Theorem 6.1 is no longer true if we replace $W^\epsilon(.)$ by $\widetilde{W}^\epsilon(.)$.

Example 6.1. Let $X = R$, $U = \boldsymbol{A} = [0,1]$, $g(u,x) = x$, $K(u,x) = [0,1]$ and

$$T(u,x) = \begin{cases} \{-1\} \bigcup \{-2\epsilon\} & if \;\; u = 0,\; x = \frac{1}{2}, \\ \{-1\} & \text{otherwise}, \end{cases}$$

where $\epsilon > 0$ is small. Then, by direct calculation one has $W(0) = \{1\}$, $W^\epsilon(0) = \{\frac{1}{2}\} \bigcup [1-\epsilon, 1]$ and $W^\epsilon(u) = [1-\epsilon, 1]$ if $u \neq 0$. So, $W^\epsilon(.)$ is usc at 0, but $\widetilde{W}^\epsilon(.)$ is not usc at 0. One also easily sees that all assumptions (i) and (ii) of Theorem 6.1 are satisfied.

The example below shows the symmetric fact that Theorem 5.1 is no longer true if $\widetilde{W}^{\epsilon}(.)$ is replaced by $W^{\epsilon}(.)$.

Example 6.2. Let X, A, U, g and K be as in Example 6.1. Let

$$T(u,x) = \begin{cases} \{-\epsilon - u - x\} & if \;\; u + x \le 1 - \epsilon, \\ \{-1\} & \text{otherwise}, \end{cases}$$

where $\epsilon > 0$ is small. Then, it is not hard to check that $W(0) = \{1\}$, $W^{\epsilon}(0) = \{0\} \bigcup [1-\epsilon, 1]$ and $W^{\epsilon}(u) = [1-\epsilon-u, 1]$ if $u \neq 0$. We see that $\widetilde{W}^{\epsilon}(.)$ is lsc at 0 but $W^{\epsilon}(.)$ is not lsc at 0. We see also that all assumptions (i) and (ii) of Theorem 5.1 are satified.

7. Conclusions

In the paper we have obtained sufficient conditions for both lower semicontinuity and upper semicontinuity in a symmetric form for all three possible solution sets. Similar results are also derived for ϵ-solution sets. As an effort to give simple proofs, we have used only one technique based on convergence of nets in a contradiction argument. The obtained results are weaker than the known ones in the literature.

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