Uniqueness and Hölder Continuity of the Solution to Multivalued Equilibrium Problems in Metric Spaces*

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Abstract. Multivalued equilibrium problems in general metric spaces are considered. Uniqueness and Hölder continuity of the solution are established under Hölder continuity and relaxed Hölder - related monotonicity assumptions. The assumptions appear to be weaker and the conclusion to be properly stronger than that of the recent results in the literature. Furthermore, our theorems include completely some known results for variational inequalities in Hilbert spaces, which were demonstrated via geometrical techniques based on the orthogonal projection in Hilbert spaces and the linearity of the canonical pair $\langle .,. \rangle$.

Keywords: Metric spaces, Multivalued equilibrium problems, Hölder properties, Variational inequalities, Fixed point and coincidence point problems, Vector optimization.

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1 Introduction

Sensitivity analysis and stability of solutions are of the great importance in optimization theory and applications. The equilibrium problem is a problem setting, which includes many optimization-related problems such as variational inequalities, mathematical programming problems, vector optimization, fixed point and coincidence point problems, Nash equilibrium problems, complementarity problems, traffic network equilibria, and the like. There have been numerous papers devoted to the solution existence of equilibrium problems and their generalizations, see e.g. Blum and Oettli (1994), Bianchi, Hadjisavvas and Schaible (1997), Chadli and Riahi (2000), Ansari, Konnov and Yao (2001), Lin, Ansari and Wu (2003), Hai and Khanh (2006, in press), etc and the references therein. However, we observe rather few works in the literature on the sensitivity analysis for equilibrium problems. Muu (1984), Bianchi and Pini (2003) and Anh and Khanh (2004 and to appear) consider semicontinuity of the solution sets. The Hölder continuity of the unique solution is investigated in Bianchi and Pini (2003) for the scalar single-valued case and Anh and Khanh (2006) for vector multivalued equilibrium problems. These papers extend the results of Yen (1995) and Yen and Lee (1997) for variational inequalities in Hilbert spaces to equilibrium problems in metric spaces. However, when applied to this special case of variational inequalities, these extended results become weaker than the original ones, which were established via employing the orthogonal projection of the Hilbert space and the linearity of the canonical pairing $\langle ., . \rangle$ involved in the variational inequality. Inspired by the general observation on many sensitivity results that, in return for a stability property of the solutions, the same property have to be assumed on the data of the problem under consideration, we try to find out assumptions of the

Hölder continuity kind, which relate the metric-space variable and parameters of the problem (in metric spaces) to replace the Hilbert-space structure and the linearity structure of the variational inequality problem in ensuring a stability result which completely includes Yen's result when applied to variational inequalities. It appears that aiming this goal we finally obtain in this note a remarkable improvement of the results in Bianchi and Pini (2003) and Anh and Khanh (2006) which becomes properly stronger than the Yen's result, when applied to variational inequalities.

The organization of the paper is as follows. The rest of this section is devoted to the problem statement and some preliminaries. In Section 2 we establish the main results and provide corollaries and examples to compare with known recent results. Applications to fixed point and coincidence point problems, vector optimization and especially to variational inequalities are presented in Section 3.

Our notations are almost standard. We use $\|.\|$ and $d(.,.)$ for the norm and distance in any normed space and metric space (the context makes it clear what space is encountered). $d(x, A)$ is the distance from x to subset A in X. For a normed space X, X^* is the topological dual and $\langle ., . \rangle$ is the canonical pair. R_+ is the set of nonnegative real numbers. $B(x, r)$ denotes the closed ball of radius $r \geq 0$ and centered at x in a metric space X. int$C$ stands for the interior of a subset C.

Let, throughout the paper if not otherwise specified, X, Z, Λ, M and N be metric spaces, Y be a metric linear space and $K : \Lambda \to 2^X$ be a multifunction with nonempty values. Let $C \subseteq Y$ have int$C \neq \emptyset$. Let $A : X \times N \to 2^Z$ and $F : X \times X \times Z \times M \to 2^Y$ be multifunctions. For each $\lambda \in \Lambda$, $\mu \in M$ and $\eta \in N$ consider the following two equilibrium problems:

(EP):   Find $\bar{x} \in K(\lambda)$ and $\bar{x}^* \in A(\bar{x}, \eta)$ such that, for each $y \in K(\lambda)$,

$$F(\bar{x}, y, \bar{x}^*, \mu) \cap (Y \setminus -\text{int}C) \neq \emptyset.$$

(SEP):   Find $\bar{x} \in K(\lambda)$ and $\bar{x}^* \in A(\bar{x}, \eta)$ such that, for each $y \in K(\lambda)$,

$$F(\bar{x}, y, \bar{x}^*, \mu) \subseteq Y \setminus -\text{int}C.$$

Observe that if $F(x, y, x^*, \mu) \equiv f(y, x^*, \mu)$ and A is single-valued functions, with $f : X \times Z \times M \to Y$ being single-valued mapping, then two problems (EP) and (SEP) collapse to the implicit vector equilibrium problem considered by many authors. We include $A(x, \eta)$ in this problem setting so that the problems are easier to be explained containing particular cases (see Subsections 3.3 and 3.4).

Recall first some Hölder-related notions.

DEFINITION 1.1

(i) (Classical notion.) A multifunction $G : \Lambda \to 2^X$ is said to be $l.\alpha-$Hölder at λ_0 if there exists a neighborhood U of λ_0 such that, $\forall \lambda_1, \lambda_2 \in U$,

$$G(\lambda_1) \subseteq G(\lambda_2) + lB(0, d^\alpha\big(\lambda_1, \lambda_2)\big).$$

(ii) (See Anh and Khanh (2006).) $G : X \times X \to 2^Y$ is called $h.\beta-$Hölder - strongly pseudomonotone of the first type in $S \subseteq X$ if, $\forall x, y \in S, x \neq y$,

$$G(x, y) \nsubseteq -\text{int}C \Longrightarrow G(y, x) + hB\big(0, d^\beta(x, y)\big) \subseteq -C, \tag{1}$$

where $h \geq 0$ and $\beta > 0$. G is called $h.\beta-$Hölder-strongly pseudomonotone of the second type if (1) is replaced by

$$G(x, y) \subseteq Y \setminus -\text{int}C \Longrightarrow G(y, x) + hB\big(0, d^\beta(x, y)\big) \subseteq -C.$$

In the sequel the following Hölder-related assumptions will be essential for considering problem (EP) and (SEP):

For the reference point $(\lambda_0, \mu_0, \eta_0) \in \Lambda \times M \times N$, there are neighborhoods $U(\lambda_0)$, $V(\mu_0)$ and $W(\eta_0)$ of λ_0, μ_0 and η_0, respectively, such that

(A1) $\forall \lambda \in U(\lambda_0)$, $\forall \mu_1, \mu_2 \in V(\mu_0)$, $\forall x, y \in K(\lambda) : x \neq y$, $\forall x_1^*, x_2^* \in A\big(K(\lambda), W(\eta_0)\big)$,

$$F(x, y, x_1^*, \mu_1) \subseteq F(x, y, x_2^*, \mu_2) + mB\Big(0, d^{\theta}(x, y)\big(d^{\zeta}(x_1^*, x_2^*) + d^{\gamma}(\mu_1, \mu_2)\big)\Big),$$

where $m > 0$, $\theta \geq 0$, $\zeta \geq 0$ and $\gamma > 0$;

(A2a) $\forall \mu \in V(\mu_0)$, $\forall \eta \in W(\eta_0)$, $\forall x, y \in K\big(U(\lambda_0)\big) : x \neq y$,

$$hd^{\beta}(x, y) \leq \inf_{x^* \in A(x,\eta)} \inf_{g \in F(x,y,x^*,\mu)} d(g, Y \setminus -\mathrm{int}C) + \inf_{y^* \in A(y,\eta)} \inf_{f \in F(y,x,y^*,\mu)} d(f, Y \setminus -\mathrm{int}C), \quad (2)$$

where $h > 0$, $\beta > \theta$.

(A2b) is (A2a) with (2) replaced by

$$hd^{\beta}(x, y) \leq \inf_{x^* \in A(x,\eta)} \sup_{g \in F(x,y,x^*,\mu)} d(g, Y \setminus -\mathrm{int}C) + \inf_{y^* \in A(y,\eta)} \sup_{f \in F(y,x,y^*,\mu)} d(f, Y \setminus -\mathrm{int}C). \quad (3)$$

REMARK 1.1 These assumptions look seemingly complicated. But they are not hard to be checked as shown by numerous examples below. We now make their meanings clearer.

(i) Assumption (A1) incorporates Hölder continuity with respect to state variables x, y and to parameter μ (in connection also with parameters λ and η). It appears that to ensure our Hölder continuity results for the solutions to (EP) and (SEP) to include properly the mentioned result for variational inequalities in Hilbert spaces of Yen (1995), this natural condition can replace particular orthogonality and linearity of variational inequalities in Hilbert spaces (see Theorem 2.1). Note also that Hölder-related assumptions in Bianchi and Pini (2003) and Anh and Khanh (2006) are imposed separately in x, y and in parameters.

(ii) To explain Assumption (A2a, b) we consider a single-valued real function (without parameters) $f : X \times X \to R$ for the sake of simplicity. Then both (a) and (b) collapse to an assumption of the form: $\forall x, y \in K \subseteq X : x \neq y$,

$$hd^{\beta}(x, y) \leq d\big(f(x, y), R_+\big) + d\big(f(y, x), R_+\big). \tag{4}$$

We have the following relation.

PROPOSITION 1.1

(i) *if $f : X \times X \to R$ satisfies (4) then f is $h.\beta-$Hölder-strongly pseudomonotone in K (the two types defined in Definition 1.1 coincide in this case). Conversely, if f is $h.\beta-$Hölder-strongly pseudomonotone in K and quasimonotone in K (see Bianchi and Schaible, 1996), i.e. $\forall x, y \in K : x \neq y$, $f(x, y) < 0 \Longrightarrow f(y, x) \geq 0$, then f satisfies (4).*

(ii) *If $f : X \times X \to R$ is $h.\beta-$Hölder-strongly monotone in $K \subseteq X$ (see Anh and Khanh, 2006), i.e. $\forall x, y \in K : x \neq y$,*

$$f(x, y) + f(y, x) + hd^{\beta}(x, y) \leq 0,$$

then f satisfies (4).

Proof. (i) If $f(x, y) \geq 0$, then $d\big(f(x, y), R_+\big) = 0$ and (4) implies that $d\big(f(y, x), R_+\big) = -f(y, x)$ and

$$f(y, x) + hB\big(0, d^{\beta}(x, y)\big) \subseteq -R_+.$$

For the converse, assume first that $f(x,y) \geq 0$. Then the assumed pseudomonotonicity one has

$$hd^{\beta}(x,y) \leq -f(y,x) = d\big(f(x,y), R_+\big) + d\big(f(y,x), R_+\big).$$

Next, if $f(x,y) < 0$ then $f(y,x) \geq 0$ by the quasimonotonicity. Hence

$$hd^{\beta}(x,y) \leq -f(y,x) = d\big(f(x,y), R_+\big) + d\big(f(y,x), R_+\big).$$

(ii) By the assumption one has

$$hd^{\beta}(x,y) \leq -f(x,y) - f(y,x) \leq d\big(f(x,y), R_+\big) + d\big(f(y,x), R_+\big).$$

□

The following examples interpret the lacking implications in Proposition 1.1.

EXAMPLE 1.1 Let $X = R$, $K = [1,2]$ and $f(x,y) = -\frac{x}{y}$. Then f satisfies (4) with $h = \beta = 1$. But $f(x,y) < 0$, $\forall x, y \in K$, and hence f is not quasimonotone in K.

EXAMPLE 1.2 Let $X = R$, $K = [0,1]$ and $f(x,y) = \frac{y-x}{1+x}$. Then it is easy to see that f satisfies (4) with $h = \frac{1}{2}$, $\beta = 1$. (f is also quasimonotone and $\frac{1}{2}.1-$Hölder - strongly pseudomonotone.) But, $\forall x, y \in K$,

$$f(x,y) + f(y,x) = \frac{(x-y)^2}{(1+x)(1+y)} \geq 0,$$

and hence f is not $\frac{1}{2}.1-$Hölder - strongly monotone in K.

2 Main result

THEOREM 2.1 *For problem* (EP) *assume that solutions exist in a neighborhood of the considered point* $(\lambda_0, \mu_0, \eta_0) \in \Lambda \times M \times N$ *and the assumptions* (A1) *and* (A2a) *are satisfied. Assume further that*

(i) $K(.)$ *is* $l.\alpha-$*Hölder at* λ_0;

(ii) $\forall(\lambda,\mu,\eta)\in U(\lambda_0)\times V(\mu_0)\times W(\eta_0)$, $\forall x\in K(\lambda)$, $\forall x^*\in A(x,\eta)$, $F(x,.,x^*,\mu)$ *is* $n.\delta-$*Hölder in* $K\big(U(\lambda_0)\big)$;

(iii) $\forall\lambda\in U(\lambda_0)$, $\forall x\in K(\lambda)$, $A(x,.)$ *is* $p.\xi-$*Hölder at* η_0.

Then, in a neighborhood of (λ_0,μ_0,η_0) *the solution* $x(\lambda,\mu,\eta)$ *is unique and satisfies the Hölder condition*

$$d\big(x(\lambda_1,\mu_1,\eta_1),x(\lambda_2,\mu_2,\eta_2)\big)\leq k_1 d^{\gamma/(\beta-\theta)}(\mu_1,\mu_2)+k_2 d^{\alpha\delta/\beta}(\lambda_1,\lambda_2)$$
$$+k_3 d^{\zeta\xi/(\beta-\theta)}(\eta_1,\eta_2),$$

where k_1,k_2 *and* k_3 *are positive constants depending on* $h,\beta,m,\theta,l,\alpha,...$

Proof. *Step 1* (uniqueness). Fix any $(\lambda,\mu,\eta)\in U(\lambda_0)\times V(\mu_0)\times W(\eta_0)$. If $\bar{x}$ is a solution of (EP), then $\exists\bar{x}^*\in A(\bar{x},\eta)$, $\forall y\in K(\lambda)$, $F(\bar{x},y,\bar{x}^*,\mu)\not\subseteq-\text{int}C$. So, for $y\neq\bar{x}$,

$$\inf_{x^*\in A(\bar{x},\eta)}\ \inf_{f\in F(\bar{x},y,\bar{x}^*,\mu)} d(f,Y\setminus-\text{int}C)=0.$$

Then, assumption (A2a) implies that

$$F(y,\bar{x},A(y,\eta),\mu)\subseteq-\text{int}C,$$

i.e. y is not a solution of (EP).

Step 2. We prove that

$$d_1:=d\big(x(\lambda_1,\mu_1,\eta_1),x(\lambda_1,\mu_2,\eta_1)\big)\leq\left(\frac{m}{h}\right)^{1/(\beta-\theta)} d^{\gamma/(\beta-\theta)}(\mu_1,\mu_2). \qquad (5)$$

Let $x(\lambda_1,\mu_1,\eta_1)\neq x(\lambda_1,\mu_2,\eta_1)$ (if the equality holds then we are done). Since $x_1^*\in A(x(\lambda_1,\mu_1,\eta_1),\eta_1)$ exists such that $F\big(x(\lambda_1,\mu_1,\eta_1),x(\lambda_1,\ \mu_2,\eta_1),\ x_1^*,\mu_1\big)\not\subseteq -\text{int}C$ one has

$$\inf_{x^*\in A(x(\lambda_1,\mu_1,\eta_1),\eta_1)}\ \inf_{f\in F(x(\lambda_1,\mu_1,\eta_1),x(\lambda_1,\mu_2,\eta_1),x^*,\mu_1)} d(f,Y\setminus-\text{int}C)=0.$$

Assumption (A2a) implies that

$$hd_1^{\beta} \leq \inf_{x^* \in A(x(\lambda_1,\mu_2,\eta_1),\eta_1)} \inf_{f \in F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_1,\mu_1,\eta_1),x^*,\mu_1)} d(f, Y \setminus -\mathrm{int}C).$$

Hence, $\forall z \in Y \setminus -\mathrm{int}C, \forall x^* \in A(x(\lambda_1, \mu_2, \eta_1), \eta_1)$,

$$d_1^{\beta} \leq \frac{1}{h} d\Big(F\big(x(\lambda_1, \mu_2, \eta_1), x(\lambda_1, \mu_1, \eta_1), x^*, \mu_1\big), z\Big). \tag{6}$$

Since $x(\lambda_1, \mu_2, \eta_1)$ is a solution of (EP), there are $x_2^* \in A(x(\lambda_1, \mu_2, \eta_1), \eta_1)$ and $z_1 \in F(x(\lambda_1, \mu_2, \eta_1), x(\lambda_1, \mu_1, \eta_1),\ x_2^*, \mu_2) \setminus -\mathrm{int}C$. If $\mu_1 \neq \mu_2$ then inequality (6) and assumption (A1) together yield

$$d_1^{\beta} \leq \frac{m}{h} d_1^{\theta} d^{\gamma}(\mu_1, \mu_2).$$

Of course this inequality holds also when $\mu_1 = \mu_2$. Since $\beta > \theta$, this implies (5).

Step 3. Now we show that

$$d_2 := d\big(x(\lambda_1, \mu_2, \eta_1), x(\lambda_2, \mu_2, \eta_1)\big) \leq \left(\frac{2nl^{\delta}}{h}\right)^{1/\beta} d^{\alpha\delta/\beta}(\lambda_1, \lambda_2). \tag{7}$$

Let $x(\lambda_1, \mu_2, \eta_1) \neq x(\lambda_2, \mu_2, \eta_1)$. Thanks to (i) we have $x_1' \in K(\lambda_1)$ and $x_2' \in K(\lambda_2)$ such that

$$d\big(x_1', x(\lambda_2, \mu_2, \eta_1)\big) \leq l d^{\alpha}(\lambda_1, \lambda_2),$$

$$d\big(x(\lambda_1, \mu_2, \eta_1), x_2'\big) \leq l d^{\alpha}(\lambda_1, \lambda_2).$$

By the definition of (EP), $x_3^* \in A(x(\lambda_1, \mu_2, \eta_1), \eta_1)$ and $x_4^* \in A(x(\lambda_2, \mu_2, \eta_1), \eta_1)$ exist such that one has

$$z_2 \in F(x(\lambda_1, \mu_2, \eta_1), x_1', x_3^*, \mu_2) \setminus -\mathrm{int}C,$$

$$z_3 \in F(x(\lambda_2, \mu_2, \eta_1), x_2', x_4^*, \mu_2) \setminus -\mathrm{int}C.$$

It follows from assumption (A2a), (ii) and (i) that

$$
\begin{aligned}
d_2^\beta &\le \frac{1}{h}\inf\{d(g,z_2) : g \in F\big(x(\lambda_1,\mu_2,\eta_1), x(\lambda_2,\mu_2,\eta_1), x_3^*, \mu_2\big)\}\\
&\quad+\frac{1}{h}\inf\{d(f,z_3) : f \in F\big(x(\lambda_2,\mu_2,\eta_1), x(\lambda_1,\mu_2,\eta_1), x_4^*, \mu_2\big)\}\\
&\le \frac{n}{h}\Big(d^\delta\big(x(\lambda_2,\mu_2,\eta_1), x_1'\big) + d^\delta\big(x(\lambda_1,\mu_2,\eta_1), x_2'\big)\Big)\\
&\le \frac{2nl^\delta}{h}\, d^{\alpha\delta}(\lambda_1,\lambda_2),
\end{aligned}
$$

i.e. one gets (7).

Step 4. We check the inequality

$$
d_3 := d\big(x(\lambda_2,\mu_2,\eta_1), x(\lambda_2,\mu_2,\eta_2)\big) \le \left(\frac{mp^\zeta}{h}\right)^{1/(\beta-\theta)} d^{\zeta\xi/(\beta-\theta)}(\eta_1,\eta_2). \tag{8}
$$

As before we can assume that $x(\lambda_2,\mu_2,\eta_1) \ne x(\lambda_2,\mu_2,\eta_2)$. Since some $x_5^* \in A(x(\lambda_2,\mu_2,\eta_1),\eta_1)$ exists such that one can have

$$
\exists z_4 \in F\big(x(\lambda_2,\mu_2,\eta_1), x(\lambda_2,\mu_2,\eta_2), x_4^*, \mu_2\big) \setminus -\mathrm{int}C,
$$

one arrives at

$$
\inf_{x^*\in A(x(\lambda_2,\mu_2,\eta_1),\eta_1)}\ \inf_{f\in F(x(\lambda_2,\mu_2,\eta_1),x(\lambda_2,\mu_2,\eta_2),x^*,\mu_2))} d(f, Y\setminus -\mathrm{int}C) = 0.
$$

Taking (A2a) into account, one sees that, $\forall z \in Y\setminus -\mathrm{int}C$, $\forall x^{*\prime} \in A\big(x(\lambda_2,\mu_2,\eta_2),\eta_1\big)$,

$$
\begin{aligned}
hd_3^\beta \le& \inf_{x^*\in A(x(\lambda_2,\mu_2,\eta_2),\eta_1)}\ \inf_{f\in F(x(\lambda_2,\mu_2,\eta_2),x(\lambda_2,\mu_2,\eta_1),x^*,\mu_2)} d(f, Y\setminus -\mathrm{int}C)\\
&\le d\big(F(x(\lambda_2,\mu_2,\eta_2), x(\lambda_2,\mu_2,\eta_1), x^{*\prime}, \mu_2), z\big).
\end{aligned} \tag{9}
$$

Since $x(\lambda_2,\mu_2,\eta_2)$ is a solution of (EP), there are $x_6^* \in A(x(\lambda_2,\mu_2,\eta_2),\ \eta_2)$ and

$$
z_4 \in F(x(\lambda_2,\mu_2,\eta_2), x(\lambda_2,\mu_2,\eta_1), x_6^*, \mu_2) \setminus -\mathrm{int}C.
$$

By (iii) there is $x_7^* \in A(x(\lambda_2,\mu_2,\eta_2),\eta_1)$ such that

$$
d(x_6^*, x_7^*) \le pd^\xi(\eta_1,\eta_2). \tag{10}
$$

If $\eta_1 \neq \eta_2$, then inequality (9), assumption (A1) and (10) together show that

$$\begin{aligned} hd_3^{\beta} &\leq H\Big(F\big(x(\lambda_2,\mu_2,\eta_2), x(\lambda_2,\mu_2,\eta_1), x_7^*, \mu_2\big), \\ &\qquad F(x(\lambda_2,\mu_2,\eta_2), x(\lambda_2,\mu_2,\eta_1), x_6^*, \mu_2)\Big) \\ &\leq md^{\theta}\big(x(\lambda_2,\mu_2,\eta_1), x(\lambda_2,\mu_2,\eta_2)\big)d^{\zeta}\big(x_6^*, x_7^*\big) \\ &\leq mp^{\zeta}d_3^{\theta}d^{\zeta\xi}(\eta_1,\eta_2), \end{aligned}$$

where $H(.,.)$ is the Hausdorff distance. This inequality holds also for $\eta_1 = \eta_2$. Since $\beta > \theta$, this implies (8).

Step 5. Finally, since

$$d\big(x(\lambda_1,\mu_1,\eta_1), x(\lambda_2,\mu_2,\eta_2)\big) \leq d_1 + d_2 + d_3$$

we get the conclusion of the theorem with $k_1 = \left(\frac{m}{h}\right)^{1/(\beta-\theta)}$, $k_2 = \left(\frac{2nl^{\delta}}{h}\right)^{1/\beta}$ and $k_3 = \left(\frac{mp^{\zeta}}{h}\right)^{1/(\beta-\theta)}$. □

Observe that all assumptions but (A2a) are about Hölder continuity of the data of problem (EP) and so are natural to ensure a Hölder continuity of the solution. Assumption (A2a) is also Hölder-related and cannot be omitted as shown by the following example, similarly as for the corresponding assumption in Anh and Khanh (2006).

EXAMPLE 2.1 Let $X = Y = Z = R, \Lambda \equiv M \equiv N = [1,2], C = R_+, K(\lambda) = [\lambda - 1, 1]$, $A(x,\lambda) = [\lambda, |x| + \lambda]$ and

$$F(x,y,x^*,\lambda) = (-\infty, |\lambda x|^{1/2}x^*(|x|^{1/4} - y)].$$

Then $K(.)$ is 1.1 - Hölder in Λ; $\forall\lambda \in \Lambda$, $\forall x \in K(\lambda)$, $\forall x^* \in A(x,\lambda)$, $F(x,.,x^*,\lambda)$ is $3\sqrt{2}.1$ - Hölder in $K(\lambda)$; $\forall\lambda \in \Lambda$, $\forall x \in K(\lambda)$, $A(x,.)$ is 1.1−Hölder at λ and assumption (A1) is satisfied with $m = 3, \zeta = 1, \gamma = \frac{1}{2}$ and $\theta = 0$. We indicate that the remaining assumption (A2a) is violated. Taking $\lambda = 1, x_1 = 1, y = 0$, $x^* = 1 \in A(1,1) = [1,2], y^* = 1 \in A(0,1) = \{1\}$ we have $F(1,0,1,1) = (-\infty, 1]$ and $F(0,1,1,1) = (-\infty, 0]$. So the right-hand side of (2) in this case is equal to 0, i.e. (2) is violated. Easy calculations give the following solution set $S(\lambda)$ of (EP):

$$S(\lambda) = \begin{cases} \{1\}, & \text{if } 1 < \lambda \leq 2, \\ \{0,1\} & \text{if } \lambda = 1. \end{cases}$$

So the solutions are not unique at $\lambda = 1$ and $S(.)$ is even not lower semicontinuous at $\lambda = 1$.

THEOREM 2.2 *For problem* (SEP) *assume all the assumptions as for problems* (EP) *with the only change that assumption* (A2a) *is replaced by* (A2b)*. Then the solution of* (SEP) *is unique and satisfies the same Hölder condition as in Theorem 2.1.*

We omit the proof since the technique is similar as that for Theorem 2.1 with suitable modifications. The following example indicates that (A2b) is essential.

EXAMPLE 2.2 Let $X, Y, Z, \Lambda, M, N, C, K$ and A be as in Example 2.1. Let

$$F(x, y, x^*, \lambda) = [\lambda^{1/2} x x^*(x - y^2), +\infty).$$

Then all assumptions but (A2b) are satisfied. Direct computations supply the solution set $S(\lambda)$ of problems (SEP) as

$$S(\lambda) = \begin{cases} \{0,1\} & \text{if } \lambda = 1, \\ \{1\}, & \text{if } 1 < \lambda \leq 2. \end{cases}$$

So the solutions are not unique at $\lambda = 1$ and $S(.)$ is even not lower semicontinuous at this point. Picking $\lambda_0 = 1, x = 1, y = 0, x^* = 1, y^* = 1$ we see that the right-hand side of (3) is 0 and hence (A2b) is not satisfied.

REMARK 2.1 (i) A question arrives that can assumption (i) of Theorems 2.1 and 2.2 be reduced to $K(.)$ being $l.\alpha-$pseudo-Hölder at $\big(\lambda_0, x(\lambda_0, \mu_0, \eta_0)\big)$, i.e. there exist neighborhoods $U(\lambda_0)$ of λ_0 and P of the solution $x(\lambda_0, \mu_0, \eta_0)$ such that, $\forall \lambda_1, \lambda_2 \in U(\lambda_0)$,

$$K(\lambda_1) \cap P \subseteq K(\lambda_2) + lB(0, d^{\alpha}\big(\lambda_1, \lambda_2)\big). \tag{11}$$

Note that if $\alpha = 1$, this property is called pseudo-Lipschitz property or Aubin property, see Aubin and Frankowska (1990), and plays important role in multivalued analysis. The following example gives a negative answer to this question.

EXAMPLE 2.3 Let $X = Y = Z = R, \Lambda \equiv M \equiv N = [-1, 1]$, $C = R_+$, $A(x, \lambda) = [0, \lambda]$, $\lambda_0 = 0$ and

$$K(\lambda) = \begin{cases} [-1, 2] & \text{if } \lambda = 0, \\ [0, \frac{1}{|\lambda|}] & \text{otherwise,} \end{cases}$$

$$F(x, y, x^*, \lambda) = \{(\lambda + 2)(x - y)\}.$$

It is not hard to see that all assumptions (A1), (A2a), (ii) and (iii) are satisfied. Furthermore, $K(.)$ is pseudo-Lipschitz at $\lambda_0 = 0$ and $x(0) = 2$ (taking $P = (0, 4)$, $U(\lambda_0) = (-\frac{1}{2}, \frac{1}{2})$). But some computations give the solution

$$x(\lambda) = \begin{cases} \{2\} & \text{if } \lambda = 0, \\ \{\frac{1}{\lambda}\} & \text{if } 0 < \lambda \leq 1, \end{cases}$$

which is discontinuous at $\lambda_0 = 0$.

(ii) If we impose the additional assumptions that $K(.)$ is continuous and $K(\lambda_0)$ is compact, that $A(., \eta_0)$ is upper semicontinuous (usc for short) and has compact values in $K(\lambda_0) \times \{\eta_0\}$ and that $F(., ., ., \mu_0)$ is usc in $K(\lambda_0) \times K(\lambda_0) \times A\big(K(\lambda_0), \eta_0\big)$, then assumption (i) of Theorems 2.1 and 2.2 can be reduced to the mentioned pseudo-Hölder property. Indeed, by assumption (A2a) and Theorem 3.1 of Anh and Khanh (submitted) we derive that the unique solution $x(., ., .)$ is usc at $(\lambda_0, \mu_0, \eta_0)$. Since $K(.)$ is $l.\alpha-$pseudo-Hölder at $\big(\lambda_0, x(\lambda_0, \mu_0, \eta_0)\big)$ we have (11). By the upper semicontinuity of $x(.,.,.)$ at $(\lambda_0, \mu_0, \eta_0)$ we can assume that $x(\lambda_1, \mu_2, \eta_1)$ and $x(\lambda_2, \mu_2, \eta_1)$ belong to P. Therefore (11) can be used instead of assumption (i) in Step 3 of the proof of Theorem 2.1. (Assumption (i) is needed only in Step 3). It is similar for problem (SEP).

REMARK 2.2 If $A(x, \eta) \equiv \{z_0\}$ with some fixed $z_0 \in Z$, our problems (EP) and (SEP) become the corresponding problems investigated in Anh and Khanh (2006). Setting $F(x, y, z_0, \mu) := F(x, y, \mu)$ we obtain the consequences of Theorems 2.1 and 2.2.

COROLLARY 2.3 *For* (EP) *assume that solutions exist in a neighborhood of* (λ_0, μ_0). *Assume (i) of Theorem 2.1 and further that there are neighborhoods* $U(\lambda_0)$ *and* $V(\mu_0)$ *of* λ_0 *and* μ_0 *such that*

(A1) $\forall \lambda \in U(\lambda_0), \forall \mu_1, \mu_2 \in V(\mu_0), \forall x, y \in K(\lambda) : x \neq y,$

$$F(x, y, \mu_1) \subseteq F(x, y, \mu_2) + mB\big(0, d^{\theta}(x, y) d^{\gamma}(\mu_1, \mu_2)\big),$$

where $m > 0$, $\theta \geq 0$ *and* $\gamma > 0$.

(A2a) $\forall \mu \in V(\mu_0), \forall x, y \in K\big(U(\lambda_0)\big) : x \neq y,$

$$hd^{\beta}(x, y) \leq \inf_{g \in F(x,y,\mu)} d(g, Y \setminus -\text{int}C) + \inf_{f \in F(y,x,\mu)} d(f, Y \setminus -\text{int}C),$$

where $h > 0$ *and* $\beta > \theta$;

(ii) $\forall (\lambda, \mu) \in U(\lambda_0) \times V(\mu_0), \forall x \in K(\lambda), F(x, ., \mu)$ *is* $n.\delta$ *- Hölder in* $K(U(\lambda_0))$.

Then, in a neighborhood of (λ_0, μ_0), *the solution* $x(\lambda, \mu)$ *is unique and satisfies the Hölder condition*

$$d\big(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)\big) \leq k_1 d^{\gamma/(\beta-\theta)}(\mu_1, \mu_2) + k_2 d^{\alpha\delta/\beta}(\lambda_1, \lambda_2),$$

for some k_1 *and* $k_2 > 0$ *depending on parameters involved in the assumptions.*

COROLLARY 2.4 *For* (SEP) *assume the solution existence and* (i), (A1) *and* (ii) *as in Corollary 2.3 and replace* (A2a) *by*

(A2b) $\forall \mu \in V(\mu_0)$, $\forall x, y \in K\big(U(\lambda_0)\big) : x \neq y$,

$$hd^{\beta}(x,y) \leq \sup_{g \in F(x,y,\mu)} d(g, Y \setminus -\mathrm{int}C) + \sup_{f \in F(y,x,\mu)} d(f, Y \setminus -\mathrm{int}C),$$

where $h > 0$ *and* $\beta > \theta$.

Then we have the same conclusion as that of Corollary 2.3.

REMARK 2.3 Corollaries 2.3 and 2.4 remarkably sharpen Theorems 2.1 and 2.2 of Anh and Khanh (2006) since

(a) assumptions (A2a) and (A2b) are strictly weaker than the corresponding assumptions (ii) and (ii') of the mentioned results (see Proposition 1.1 and Example 1.1 of this paper);

(b) for $\theta > 0$ the conclusions here are properly stronger as the greater degree is, the stronger Hölder continuity is. Furthermore, assumption (A1) does not impose any condition when $x = y$. This also makes Corollaries 2.3 and 2.4 applicable in some cases where Theorems 2.1 and 2.2 of Anh and Khanh (2006) cannot be employed as shown by the following Examples 2.4 and 2.5, respectively.

EXAMPLE 2.4 Let $X = Y = R, \Lambda \equiv M = [0, \pi]$, $C = R_+$, $K(\lambda) = [\lambda, \pi]$, $\lambda_0 = 0$ and

$$F(x,y,\lambda) = \begin{cases} [-1, 1] & \text{if} \quad x = y, \lambda = 0, \\ (-\infty, 0] & \text{if} \quad x = y, \lambda \neq 0, \\ \left\{(y-x)\tan\left(\frac{x+\lambda+\pi}{12}\right)\right\} & \text{otherwise.} \end{cases}$$

Then it is not hard to see that $K(.)$ is $1.1-$ Hölder at λ, $\forall \lambda \in [0, \pi]$; assumption (A2a) is fulfilled with $h = \frac{4}{\sqrt{3}+2}$ and $\beta = 2$; (ii) is satisfied with $n = \delta = 1$ and (A1) is satisfied with $m = 2$ and $\theta = \gamma = 1$. Thus, Corollary 2.3 is applicable. However $F(x, y, .)$ is not Hölder at 0 (for any degree) and hence the mentioned Theorem 2.1 does not work.

EXAMPLE 2.5 Let $X = Y = R$, $\Lambda \equiv M = [0,1]$, $C = R_+$, $K(\lambda) = [\lambda, \lambda+1]$, $\lambda_0 = 0$

$$F(x,y,\lambda) = \begin{cases} [0,1] & \text{if} \quad x = y, \lambda = 0, \\ [0,+\infty) & \text{if} \quad x = y, \lambda \neq 0, \\ \{(y-x)\ln(e^{x+\lambda}+1)\} & \text{otherwise.} \end{cases}$$

Then (i) is satisfied with $l = \alpha = 1$, (A2b)-with $h = \frac{1}{2}$ and $\beta = 2$, (ii)-with $n = \ln(e^3+1)$ and $\delta = 1$ and (A1)-with $m = \theta = \gamma = 1$.Thus, Corollary 2.4 can be used, but the encountered Theorem 2.2 cannot, by the same reason as above.

The following example gives a case where Corollary 2.3 works but Theorem 2.1 of Anh and Khanh (2006) does not, because of its assumption (ii) being not satisfied.

EXAMPLE 2.6 Let X, Y, Λ, M and C be as in Example 2.5. Let $K(\lambda) = [0, \lambda]$ and $F(x,y,\lambda) = [-x-y-\lambda, -x]$. Then all assumptions of Corollary 2.3 are satisfied, but $F(.,.,\lambda)$ is not Hölder-strongly pseudomonotone as required in the mentioned assumption (ii). Direct computations show that the solution set is $S(\lambda) = \{0\}, \forall \lambda \in [0,1]$ and hence the unique solution is Hölder of any degree.

Note that, the results of Bianchi and Pini (2003) clearly also fail to be applied in Examples 2.4-2.6.

REMARK 2.4 When we almost completed the preparation of this paper we found Mansour and Riahi (2005) with similar results. Namely, this paper considers the special case of Anh and Khanh (2006), where $Y = R$ and $F : X \times X \times M \to R$ is single-valued. Then (EP) and (SEP) coincide. Our Corollary 2.3 (or 2.4) is properly stronger than the result (Theorem 2.2.1) of Mansour and Riahi (2005), since assumption (A2a) is more relaxed than the strongly monotonicity assumed in Theorem 2.2.1; (A1) is also more relaxed than the corresponding assumption there, since the case $x = y$ is not involved; and finally, our all assumptions are imposed on x, y in $K\big(U(\lambda_0)\big)$ not globally. The following examples supply cases where Corollaries 2.3 and 2.4 can be applied but the mentioned Theorem 2.2.1 cannot.

EXAMPLE 2.7 Let $X = Y = \varLambda \equiv M = R$, $C = R_+$, $K(\lambda) = [\lambda, \lambda + 1]$, $\lambda_0 = 1$ and $F(x, y, \lambda) = \{\lambda(x^2 - y^2)\}$. Then (A1) is satisfied with $U(\lambda_0) = (0, 2)$, $\theta = \gamma = 1, m = 6$. (A2a) holds with $h = \beta = 2$ and (ii) with $n = 12$ and $\delta = 1$. So Corollary 2.3 is applied. However, $F(., ., \lambda)$ is not globally strongly monotone. Indeed, for $x = \lambda_0$ and $y = -\lambda_0, F(x, y, \lambda_0) = 0$ and $F(y, x, \lambda_0) = 0 \not\leq -d(\lambda, -\lambda) = -2\lambda$. Thus the result of Mansour and Riahi (2005) fails to be employed.

EXAMPLE 2.8 Let $X = Y = R$, $\varLambda \equiv M = [0, \pi], C = R_+$, $K(\lambda) = [\lambda, \pi]$, $\lambda_0 = 0$ and

$$F(x, y, \lambda) = \begin{cases} \{0\} & \text{if} \quad x = y, \lambda = 0, \\ \tan\frac{\pi - x}{2} & \text{if} \quad x = y, \lambda \neq 0, \\ (y - x)\tan\frac{x + \lambda + \pi}{12} & \text{otherwise.} \end{cases}$$

Then (A1) holds with $m = 2, \theta = \gamma = 1$, (ii) with $n = \delta = 1$. In this case, $\forall\lambda \in [0, \pi]$, $F(., ., \lambda)$ is even Hölder strongly monotone in $K(\lambda)$ with $h = \frac{4}{\sqrt{3}+2}$ and $\beta = 2$. Then Corollary 2.3 is applicable but Mansour and Riahi (2005) is not, since $F(x, y, .)$ does not satisfy assumption (H3) there about Hölder continuity $\big($corresponding to our (A1)$\big)$.

3 Particular cases

We will now apply the main results in Section 2 to some particular problems of importance.

3.1 *Lower and upper bounded equilibrium problems*

Consider the following problem of Chadli, Chiang and Yao (2002) and Congjun (2006), for $(\lambda, \mu) \in \Lambda \times M$,

(BEP) Find $\bar{x} \in K(\lambda)$, $\forall y \in K(\lambda)$, $\alpha \leq F(\bar{x}, y, \mu) \leq \beta$,

where $K : \Lambda \to 2^X$, $F : X \times X \times M \to R$, $\alpha, \beta \in R : \alpha < \beta$.

Setting $Y = R$, $C = (-\infty, -\beta] \cup [-\alpha, +\infty)$, (EP) $\big($or (SEP)$\big)$ becomes (BEP). Then our Corollary 2.3 (and, the same, Corollary 2.4) becomes a new result for (BEP).

3.2 *Multivalued variational inequalities*

In this Subsection 3.2, if not otherwise stated, let X be a reflexive Banach space, N and Λ be metric linear spaces, $K : \Lambda \to 2^X$ and $A : X \times N \to 2^{X^*}$ be multifunctions with $K(\lambda)$ being closed and convex, $\forall \lambda \in \Lambda$. For each $(\lambda, \eta) \in \Lambda \times N$ consider the variational inequality problem

(VI) Find $\bar{x} \in K(\lambda)$ such that $\exists \bar{t} \in A(\bar{x}, \eta)$, $\forall y \in K(\lambda)$,

$$\langle \bar{t}, y - \bar{x} \rangle \geq 0.$$

We need some preliminaries. Let X and Z be Banach spaces. A multifunction $A : X \to 2^{X^*}$ is said to be monotone if, $\forall x, y \in X$, $\forall u \in Ax$, $\forall v \in Ay$, $\langle u - v, x - y \rangle \geq 0$. A is called strongly monotone if there is $\alpha > 0$ such that $\forall x, y \in X$, $\forall u \in Ax$, $\forall v \in Ay$,

$$\langle u - v, x - y \rangle \geq \alpha \|x - y\|^2.$$

$A : X \to 2^{X^*}$ is called maximal monotone if A is monotone and no enlargement of graph$A$ is possible without destroying monotonicity, i.e. if A is monotone and $\forall (x, u) \in$ graph$A$,

$$\langle u - v, x - y \rangle \geq 0$$

then $(y, v) \in \text{graph}A$.

A mapping $B : X \to 2^Z$ is termed locally bounded at $x \in X$ if there is a neighborhood U of x such that $B(U)$ is bounded. B is locally bounded on the subset D if B is locally bounded at each point of D. As usual any property is extended from a point to a subset in this way.

$A : X \to 2^{X^*}$ is called demicontinuous at $x \in X$ if A is upper semicontinuous at x, considering X with the norm topology and X^* with the star-weak topology. It is known, see Kluge (1979), Lemma 2.13, that if A is monotone, demicontinuous and $A(x)$ is closed and convex, $\forall x \in X$, then A is maximal monotone.

To convert (VI) to a special case of (EP) set $Z = X^*, Y = R, C = R_+$ and $F(x, y, x^*) = \langle x^*, y - x\rangle$.

COROLLARY 3.1 *For* (VI) *assume the existence a neighborhood* $U(\lambda_0) \times W(\eta_0)$ *of* $(\lambda_0, \eta_0) \in \Lambda \times N$ *such that*

(A2a) $\forall \eta \in W(\eta_0)$, $\forall x, y \in K\big(U(\lambda_0)\big) : x \neq y$,

$$h\|x - y\|^\beta \leq \inf_{g \in \langle A(x,\eta), y-x\rangle} d(g, R_+) + \inf_{f \in \langle A(y,\eta), x-y\rangle} d(f, R_+);$$

(i) *there is a neighborhood* P *of the solution* $x(\lambda_0, \eta_0)$ *such that,* $\forall \lambda, \lambda' \in U(\lambda_0)$,

$$K(\lambda) \cap P \subseteq K(\lambda') + lB\big(0, d^\alpha(\lambda, \lambda')\big)$$

(*i.e.* $K(.)$ *is* $l.\alpha-$*pseudo-Hölder at* λ_0);

(iii) $\forall \lambda \in U(\lambda_0)$, $\forall x \in K(\lambda)$, $A(x, .)$ *is* $p.\xi-$ *Hölder at* η_0;

(iv) $A(., .)$ *is locally bounded in* $K\big(U(\lambda_0)\big) \times \{\eta_0\}$;

(v) $\forall \eta \in W(\eta_0)$, $A(., \eta)$ *is maximal monotone.*

Then, in a neighborhood of (λ_0, η_0)*, the solution* $x(\lambda, \eta)$ *of* (VI) *is unique and satisfies the Hölder condition*

$$\|x(\lambda_1, \eta_1) - x(\lambda_2, \eta_2)\| \leq k_1 \|\lambda_1 - \lambda_2\|^{\alpha/\beta} + k_2 \|\eta_1 - \eta_2\|^{\xi/(\beta-1)}.$$

Proof. To apply Theorem 2.1 we need to prove

(a) all assumptions of Theorem 2.1 are satisfied, except (i);

(b) in this case the assumed pseudo-Hölder property in (i) is enough to replace the Hölder property, since there are neighborhoods $\overline{U}$ of $x(\lambda_0, \eta_0)$, $B(\lambda_0, r)$ and $\widehat{U}(\eta_0)$ of η_0 such that, $\forall (\lambda, \eta) \in B(\lambda_0, r) \times \widehat{U}(\eta_0)$, $x(\lambda, \eta) \in \overline{U}$;

(c) solutions of (VI) exist in a neighborhood of (λ_0, η_0).

We prove first (b). Set $x_0 = x(\lambda_0, \eta_0)$ and choose positive r_1 and r such that $B(x_0, r_1) \subseteq P$, $B(\lambda_0, r) \subseteq U(\lambda_0)$ and $lr^\alpha \leq r_1$. By (i), $\forall \lambda \in B(\lambda_0, r), \exists x_\lambda \in K(\lambda)$,

$$\|x_0 - x_\lambda\| \leq l d^\alpha(\lambda_0, \lambda) \leq lr^\alpha \leq r_1.$$

Hence $K(\lambda) \cap B(x_0, r_1)$ is (closed, convex and) nonempty. For $\eta \in W(\eta_0)$ consider the multifunction

$$x \mapsto A(x, \eta) + N_{K(\lambda) \cap B(x_0, r_1)}(x), \tag{12}$$

where $N_S(.)$ is the normal cone to a convex set S at (.). Since $N_S(.)$ is maximal monotone (see e.g. Zeidler, 1990, Example 32.15) and so is $A(., \eta)$, by Rockafellar (1970), the multifunction (12) is maximal monotone with bounded domain $K(\lambda) \cap B(x_0, r_1)$. Therefore, by Zeidler (1990), Corollary 32.35, the multifunction (12) is onto. Consequently, there is $\hat{x}(\lambda, \eta) \in K(\lambda) \cap B(x_0, r_1)$ such that

$$0 \in A\big(\hat{x}(\lambda, \eta), \eta\big) + N_{K(\lambda) \cap B(x_0, r_1)}\big(\hat{x}(\lambda, \eta)\big). \tag{13}$$

We claim that

$$N_{K(\lambda) \cap B(x_0, r_1)}\big(\hat{x}(\lambda, \eta)\big) = N_{K(\lambda)}\big(\hat{x}(\lambda, \eta)\big). \tag{14}$$

To check this we show first that, for (λ,η) close enough to (λ_0,η_0), $\hat{x}(\lambda,\eta)\in \mathrm{int}B(x_0,r_1)$.

Since $\hat{x}(\lambda,\eta)\in B(x_0,r_1)\subseteq P$, (i) implies the existence of $\hat{x}_0\in K(\lambda_0)$ with $\|\hat{x}(\lambda,\eta)-\hat{x}_0\|\leq ld^{\alpha}(\lambda,\lambda_0)$. Similarly, $\exists x_\lambda\in K(\lambda)\cap B(x_0,r_1)$, $d(x_0,x_\lambda)\leq ld^{\alpha}(\lambda,\lambda_0)$. On the other hand, by (iv), for (λ,η) in a neighborhood of (λ_0,η_0), there is $q>0$ such that

$$\sup\big\{\|t\|:t\in A\big(\hat{x}(\lambda,\eta),\eta\big)\big\}\leq q,$$

$$\sup\big\{\|t_0\|:t_0\in A(x_0,\eta_0)\big\}\leq q.$$

Since x_0 is a solution of (VI) and $\hat{x}(\lambda,\eta)$ is a solution of (VI) restricted to $K(\lambda)\cap B(x_0,r_1)$, we have

$$\exists\hat{z}\in A\big(\hat{x}(\lambda,\eta),\eta\big),\langle\hat{z},x_\lambda-\hat{x}(\lambda,\eta)\rangle\geq 0,$$

$$\exists z_0\in A(x_0,\eta_0),\langle z_0,\hat{x}_0-x_0\rangle\geq 0.$$

It follows from assumption (A2a) that

$$\begin{aligned}
h\|\hat{x}(\lambda,\eta)-x_0\|^{\beta}\leq&\inf_{g\in\langle A(\hat{x}(\lambda,\eta),\eta),x_0-\hat{x}(\lambda,\eta)\rangle}d\big(g,\langle\hat{z},x_\lambda-\hat{x}(\lambda,\eta)\rangle\big)\\
&+\inf_{f\in\langle A(x_0,\eta_0),\hat{x}(\lambda,\eta)-x_0\rangle}d\big(f,\langle z_0,\hat{x}_0-x_0\rangle\big)\\
\leq& H\big(\langle A\left(\hat{x}(\lambda,\eta),\eta\right),x_0-\hat{x}(\lambda,\eta)\rangle,\langle A\left(\hat{x}(\lambda,\eta),\eta\right),x_\lambda-\hat{x}(\lambda,\eta)\rangle\big)\\
&+H\big(\langle A(x_0,\eta_0),\hat{x}_0-x_0\rangle,\langle A(x_0,\eta_0),\hat{x}(\lambda,\eta)-x_0\rangle\big)\\
\leq&\sup\left\{\|t\|:t\in A\left(\hat{x}(\lambda,\eta),\eta\right)\right\}\|x_0-x_\lambda\|\\
&+\sup\left\{\|t_0\|:t_0\in A(x_0,\eta_0)\right\}\|\hat{x}_0-\hat{x}(\lambda,\eta)\|\\
\leq& 2qld^{\alpha}(\lambda,\lambda_0).
\end{aligned}$$

Hence for (λ,η) close to (λ_0,η_0), $\hat{x}(\lambda,\eta)\in\mathrm{int}B(x_0,r_1)$. Returning back to (14) we see that the inclusion $\supseteq$ is clear. For the opposite inclusion take arbitrarily v

in the left-hand side and $z \in K(\lambda)$. As $\hat{x}(\lambda,\eta) \in \mathrm{int} B(x_0, r_1)$, there is $\varepsilon > 0$ such that

$$z_\alpha := \hat{x}(\lambda,\eta) + \varepsilon(z - x) \in K(\lambda) \cap B(x_0, r_1)$$

and hence

$$\langle v, z - \hat{x}(\lambda,\eta)\rangle = \langle v, \frac{1}{\varepsilon}(z_\alpha - \hat{x}(\lambda,\eta))\rangle \leq 0,$$

i.e. $v \in N_{K(\lambda)}\big(\hat{x}(\lambda,\eta)\big)$. Thus (14) follows and (13) shows that $\hat{x}(\lambda,\eta)$ is a solution of (VI). By assumption (A2a), the solution $x(\lambda,\eta)$ of (VI) is locally unique. So $\hat{x}(\lambda,\eta) = x(\lambda,\eta)$ and we have proved (b) with $\overline{U} = B(x_0, r_1)$.

(c) The solution existence of (VI) has been demonstrated in (b).

(a) We have seen above that, due to assumption (iv), assumption (ii) of Theorem 2.1 is fulfilled with $n = 2q$ and $\delta = 1$. (iii) of Theorem 2.1 is the same (iii) here. Finally (A1) of Theorem 2.1 is clearly satisfied with $m = \theta = \zeta = 1$. Applying Theorem 2.1 completes the proof of the corollary. □

COROLLARY 3.2 *For* (VI) *assume that* $x_0 := x(\lambda_0, \eta_0)$ *is a solution of* (VI) *at* (λ_0, η_0) *and that there is a neighborhood* $U(\lambda_0) \times W(\eta_0)$ *of* (λ_0, η_0) *such that*

(A2) $A(.,\eta)$ *is strongly monotone for each* $\eta \in W(\eta_0)$;

(i) $K(.)$ *is pseudo-Lipschitz in* $U(\lambda_0)$;

(iii) $A(.,.)$ *is Lipschitz in* $P(x_0) \times W(\eta_0)$.

Then, in a neighborhood of (λ_0, η_0), *the solution of* (VI) *satisfies the Hölder condition*

$$\|x(\lambda_1, \eta_1) - x(\lambda_2, \eta_2)\| \leq k_1 d^{1/2}(\lambda_1, \lambda_2) + k_2 d(\eta_1, \eta_2).$$

Proof. We check the assumptions of Corollary 3.1. (i) holds with $\alpha = 1$. (A2a) is satisfied with $\beta = 2$ by (A2). (iii) is fulfilled with $\gamma = 1$ by (iii) of

Corollary 3.2. (iv) follows from (iii) of this corollary. Finally, since $A(.,.)$ is single-valued and $A(.,\eta)$ is monotone and demicontinuous, $A(.,\eta)$ is maximal monotone by the above-mentioned Lemma 2.13 of Kluge (1979). □

If X is a Hilbert space Corollary 3.2 collapses to Theorem 2.1 of Yen (1995).

3.3 *A fixed point problem*

In Subsections 3.3 and 3.4 let X be a Hilbert space, N be a metric linear space and $A : X \times N \to 2^X$ be a multifunction. For each $\eta \in N$, consider the fixed point problem:

(FP) Find $\bar{x} \in X$, such that

$$\bar{x} \in A(\bar{x}, \eta).$$

Setting $X = Z, Y = R, C = R_+$, $K(\lambda) \equiv X$ and

$$F(x, y, x^*) := \langle x - x^*, y - x \rangle$$

(FP) becomes the following special case of (EP):

(EP_1) Find $\bar{x} \in X$, $\bar{x}^* \in A(\bar{x}, \eta)$ such that, $\forall y \in X$,

$$\langle \bar{x} - \bar{x}^*, y - \bar{x} \rangle \geq 0. \tag{15}$$

Indeed, if $\bar{x}$ is a solution of (FP), i.e. $\bar{x} \in A(\bar{x}, \eta)$. Taking $\bar{x}^* = \bar{x}$ we see that $\bar{x}$ is a solution of (EP_1). Conversely, if $\bar{x}$ is a solution of (EP_1). Putting $y = \bar{x}^*$ in (15) we obtain $\|\bar{x} - \bar{x}^*\| = 0$, i.e. $\bar{x}$ is a solution of (FP).

From Theorem 2.1 we derive

COROLLARY 3.3 *For problem* (FP) *assume the existence of a neighborhood* $W(\eta_0)$ *of* η_0 *such that fixed points of* $A(.,\eta)$ *exists in* $W(\eta_0)$ *and that*

(A2a) $\forall \eta \in W(\eta_0)$, $\forall x, y \in X : x \neq y$,

$$h\|x-y\|^\beta \leq \inf_{g\in\langle x-A(x,\eta),y-x\rangle} d(g,R_+) + \inf_{f\in\langle y-A(y,\eta),x-y\rangle} d(f,R_+),$$

for some $h > 0$ *and* $\beta > 1$;

(iii) $\forall x \in X, A(x,.)$ *is* $p.\xi-$*Hölder at* η_0, *for* $p > 0$ *and* $\xi > 0$;

(iv) *multifunction* $(x,\eta) \mapsto x - A(x,\eta)$ *is locally bounded in* $X \times \{\eta_0\}$.

Then, in a neighborhood of η_0, *the fixed point* $x(\eta)$ *of* $A(.,\eta)$ *is unique and satisfies the Hölder condition*

$$\|x(\eta_1) - x(\eta_2)\| \leq kd^{\xi/(\beta-1)}(\eta_1,\eta_2),$$

for some $k > 0$.

Proof. To check the assumptions of Theorem 2.1 we see that (A1) is clearly satisfied with $m = \theta = \zeta = 1$ and (ii) is satisfied with $\delta = 1$ by assumption (iv) here. □

3.4 *A coincidence point problem*

Let X and N be as in Subsection 3.3 and $f, g : X \times N \to 2^X$ be multifunctions. For each $\eta \in N$, consider the coincidence point problem:

(CP) Find $(x^1, x^2) \in X \times X$ such that $x^1 \in f(x^2,\eta)$, $x^2 \in g(x^1,\eta)$.

To restate (CP) as a particular case of (EP) we set $X_1 = X \times X$, $Z = X_1$, $Y = R$, $C = R_+$, $A : X_1 \times N \to 2^{X_1}$ being defined by

$$A(x,\eta) = f(x^2,\eta) \times g(x^1,\eta)$$

and $F : X_1 \times X_1 \times X_1 \times N \to R$ by

$$F(x,y,x^*) := \langle x^1 - x^{*1}, y^1 - x^1\rangle + \langle x^2 - x^{*2}, y^2 - x^2\rangle. \quad (16)$$

Then it is not hard to see that (CP) is equivalent to the problem:

(EP_2) Find $\bar{x} = (\bar{x}^1, \bar{x}^2) \in X_1$ and $\bar{x}^* = (\bar{x}^{*1}, \bar{x}^{*2}) \in f(\bar{x}^2, \eta) \times g(\bar{x}^1, \eta)$ such that, $\forall y = (y^1, y^2) \in X_1$,

$$\langle \bar{x}^1 - \bar{x}^{*1}, y^1 - \bar{x}^1 \rangle + \langle \bar{x}^2 - \bar{x}^{*2}, y^2 - \bar{x}^2 \rangle \geq 0.$$

COROLLARY 3.4 *Assume that there is a neighborhood $W(\eta_0)$ of η_0 such that there is a coincidence point $(x^1(\eta), x^2(\eta))$, $\forall \eta \in W(\eta_0)$. Assume* (iii) *and* (iv) *as in Corollary 3.3. Assume further*

(A2a) $\forall \eta \in W(\eta_0), \forall x, y \in X : x \neq y$, *$F(x, y, x^*)$ defined by (16) satisfies assumption* (A2a) *in Section 2*;

Then, in a neighborhood of η_0, the coincidence point $(x^1(\eta), x^2(\eta))$ of $f(., \eta)$ and $g(., \eta)$ is unique and satisfies the Hölder condition

$$\|x(\eta_1) - x(\eta_2)\| \leq k d^{\xi/(\beta-1)}(\eta_1, \eta_2),$$

for some $k > 0$.

3.5 *A vector optimization problem*

Let X, Y, Λ, N, C and K be as for problem (EP) in Section 1 and $A : X \times N \to 2^Y$ be a multifunction. For each $(\lambda, \eta) \in \Lambda \times N$, consider the following problem of

(VOP) finding $\bar{x} \in K(\lambda)$ and $\bar{x}^* \in A(\bar{x}, \eta)$ such that, $\forall y \in K(\lambda)$,

$$A(y, \eta) - \bar{x}^* \subseteq Y \setminus -\mathrm{int} C.$$

Recall that such a point $\bar{x}$ is said to be a weak minimizer and $\bar{x}^*$ is a weak minimum of the vector optimization problem

$$\min A(y, \eta), s.t. \quad y \in K(\lambda).$$

To convert (VOP) to a special case of (SEP) we simply set $Z = Y, M \equiv N$ and $F(x, y, x^*, \eta) := A(y, \eta) - x^*$. Then, from Theorem 2.2 we have (cf. also the proof of Theorem 2.1)

COROLLARY 3.5 *For* (VOP) *assume that solutions exist in a neighborhood of* $(\lambda_0, \eta_0) \in \Lambda \times N$. *Assume further that there are neighborhoods* $U(\lambda_0)$ *of* λ_0 *and* $W(\eta_0)$ *of* η_0 *such that*

(A1) $\forall \lambda \in U(\lambda_0), \forall \eta_1, \eta_2 \in W(\eta_0), \forall y \in K(\lambda), \forall x_1^*, x_2^* \in A\big(K(\lambda), W(\eta_0)\big)$,

$$A(y, \eta_1) - x_1^* \subseteq A(y, \eta_2) - x_2^* + m\|y\|^\theta B\big(0, \|x_1^* - x_2^*\| + d^\gamma(\eta_1, \eta_2)\big),$$

where $m > 0, \theta \geq 0$ *and* $\gamma > 0$;

(A2b) $\forall \eta \in W(\eta_0), \forall x, y \in K(U(\lambda_0)) : x \neq y$,

$$hd^\beta(x, y) \leq \inf_{x^* \in A(x,\eta)} \sup_{g \in A(y,\eta) - x^*} d(g, Y \setminus -\mathrm{int}C)$$

$$+ \inf_{y^* \in A(y,\eta)} \sup_{f \in A(x,\eta) - y^*} d(f, Y \setminus -\mathrm{int}C),$$

for $h > 0$ *and* $\beta > \theta$;

(i) $K(.)$ *is* $l.\alpha-$*Hölder at* λ_0 *with* $l > 0$ *and* $\alpha > 0$;

(ii) $\forall \eta \in W(\eta_0)$, $A(., \eta)$ *is* $n.\delta-$ *Hölder in* $K(U(\lambda_0))$;

(iii) $\forall \lambda \in U(\lambda_0), \forall y \in K(\lambda), A(y, .)$ *is* $p.\xi$ *Hölder at* η_0.

Then, in a neighborhood of (λ_0, η_0), *the solution* $x(\lambda, \eta)$ *of (VOP) is unique and satisfies the Hölder condition*

$$d(x(\lambda_1, \eta_1), x(\lambda_2, \eta_2)) \leq k_1 d^{\alpha\delta/\beta}(\lambda_1, \lambda_2) + k_2 d^{\tau/(\beta-\theta)}(\eta_1, \eta_2),$$

where $\tau := \min\{\gamma, \xi\}$, k_1 *and* k_2 *are positive constants depending on* $h, \beta, m, \theta, ...$

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