First and Second-Order Optimality Conditions Using Approximations for Nonsmooth Vector Optimization in Banach Spaces[1]

P. Q. Khanh[2] and N. D. Tuan[3]

[1]This work was partially supported by the National Basic Research Program in Natural Sciences of Vietnam.

[2]Professor, Department of Mathematics, International University of Hochiminh City, Hochiminh City, Vietnam.

[3]Lecturer, Department of Mathematics, University of Natural Sciences of Hochiminh City, Hochiminh City, Vietnam.

Abstract. We use the first and second-order approximations of mappings to establish both necessary and sufficient optimality conditions for unconstrained and constrained nonsmooth vector optimization problems. Ideal solutions, efficient solutions and weakly solutions are considered. The data of the problems need not even be continuous. Some often imposed compactness assumptions are also relaxed. Examples are provided to compare our results and some recent known ones.

Key Words. First and second-order approximations, vector optimization, relative p-compactness, asymptotical p-compactness, optimality conditions.

1. Introduction

Let X and Y be Banach spaces. Let $S \subseteq X$ be nonempty, C be an ordering cone of Y and $f : X \to Y$ be a mapping. Assume that C is closed and convex, with nonempty interior. The aim of the present paper is to establish both necessary and sufficient optimality conditions, of both first and second-order, for ideal solutions, Pareto efficient solutions and weakly efficient solutions of the unconstrained nonsmooth vector problem

$$\text{(P1)} \qquad \min f(x), \text{ s.t. } x \in X,$$

as well as the constrained nonsmooth vector problem

$$\text{(P2)} \qquad \min f(x), \text{ s.t. } x \in S.$$

Our optimality conditions are in terms of the first-order approximation introduced by Jourani and Thibault in Ref. 1 and of the second-order approximation of Allali and Amahroq in Ref. 2.

In Ref. 3, Penot describes the situation in the study of optimality conditions in nonsmooth optimization as follows: "The number of results and the variety of concepts introduced make the task of giving a general view on the subject an almost Herculean work". To motivate our aim we mention only the recent literature. For problems involving mappings with locally Lipschitzian derivatives, the Clarke generalized Hessian introduced in Ref. 4 are used to develop various second-order optimality conditions in Refs. 5-8. The Dini and Hadamard directional derivatives are exploited to establish first and second-order conditions in Refs. 9-15. The approximate Jacobian introduced in Ref. 16 and approximate Hessian in Ref. 17 are also effective generalized derivatives for constructing optimality conditions. See also Refs. 18-22. In Refs. 23-25 optimality conditions are derived for nonsmooth problems with multi-valued objectives. Observing that the Fréchet Hessian, Clarke generalized Hessian and approximate Hessian are all particular cases of second-order approximations and even discontinuous mappings may have approximations, we utilize first and second-order approximations to develop optimality conditions for the problems (P1) and (P2). Our results contain theorems of Refs. 2, 7, 8 and 18 as special cases. To the best of our knowledge, Refs. 1, 2 and 26 are the only papers in the literature to employ approximations to consider optimality conditions. However, Ref. 1 investigates metric regularity in terms of first-order approximations and uses another generalized derivative, the approximate subdifferential proposed by Mordukhovich (Refs. 27-28) for finite dimensions and Ioffe (Ref. 29) for Banach spaces, to formulate optimality conditions. Ref. 2 establishes only second-order necessary optimality conditions for scalar optimization in terms of compact second-order approximations. Ref. 26 uses second-order approximations of scalar functions and a scalarization by support functions to construct second-order optimality conditions under differentiability and compactness assumptions. In this paper we consider both necessary and sufficient conditions for vector optimization and we avoid continuity and differentiability assumptions and relax the compactness assumptions. In Section 2 we highlight basic facts about first and second-order approximations and some comparisons with other generalized derivatives. Section 3 is devoted to optimality conditions for unconstrained optimization and the final section, Section 4, to set-constrained optimization.

2. First and Second-Order Approximations

Throughout the paper, the following notations will be used. Let X and Y be Banach spaces. Let $L(X,Y)$ ($B(X,X,Y)$) stand for the space of the continuous linear mappings from X into Y (continuous bilinear mappings from $X \times X$ into Y, respectively). For $L(X,\mathbb{R})$ we usually use the notation X^*. $\langle x^*, x\rangle$ denotes the value of $x^* \in X^*$ at $x \in X$. B_X stands for the closed unit ball in X. For $A \subseteq L(X,Y)$ and $x \in X$ ($B \subseteq B(X,X,Y)$ and $x, z \in X$) denote $A(x) := \{M(x) : M \in A\}$ ($B(x,z) := \{N(x,z) : N \in B\}$, respectively). For $S \subseteq X$, cl$S$ means the closure of S and co$S$ means the convex hull of S. Let us recall generalized derivatives.

Definition 2.1 See Ref. 1. Let $x_0 \in X$ and $g : X \to Y$. The set $A_g(x_0) \subseteq L(X,Y)$ is said to be a first-order approximation of g at x_0 if there exists a neighborhood U of x_0 such that, for all $x \in U$,

$$g(x) - g(x_0) \in A_g(x_0)(x - x_0) + o(\|x - x_0\|),$$

where $o(\|x - x_0\|)/\|x - x_0\|$ tends to 0 as $x \to x_0$.

Notice that we do not have uniqueness of approximation and for $A \in L(X,Y)$, $\{A\}$ is a first-order approximation of g at x_0 if and only if A is the Fréchet derivative.

Definition 2.2 See Ref. 2. We say that $(A_g(x_0), B_g(x_0)) \subseteq L(X,Y) \times B(X,X,Y)$ is a second-order approximation of $g : X \to Y$ at $x_0 \in X$ if

(i) $A_g(x_0)$ is a first-order approximation of g at x_0;

(ii) $g(x) - g(x_0) \in A_g(x_0)(x - x_0) + B_g(x_0)(x - x_0, x - x_0) + o(\|x - x_0\|^2)$.

Notice that for $A \in L(X,Y)$ and $B \in B(X,X,Y)$, $\{(A,B)\}$ is a second-order approximation of g at x_0 if and only if $A = g^{'}(x_0)$ and $B = \frac{1}{2}g^{''}(x_0)$, where $g^{''}(x_0)$

is the second Fréchet derivative of g at x_0.

Remark 2.1. The notion of approximations has the advantage over many other generalized derivatives that an approximation may exist even for a discontinuous mapping, for instance, let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} \sqrt{x} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ x^{-1} & \text{if } x < 0. \end{cases}$$

Then g is discontinuous at 0 and we can take $A_g(0) = (\alpha, +\infty)$ for any $\alpha > 0$ and $B_g(0) = \{0\}$, the zero mapping from $\mathbb{R}$ to $\mathbb{R}$.

Definition 2.3

(i) See Ref. 30. Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping of class $C^{0,1}$, i.e. g is locally Lipschitz. The Clarke generalized Jacobian of g at $x_0 \in \mathbb{R}^n$, denoted by $\partial_C g(x_0)$ is defined by

$$\partial_C g(x_0) := \text{clco}\{\lim g^{'}(x_i) : x_i \to x_0, g^{'}(x_i) \text{ exists }\}.$$

(ii) See Ref. 4. Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping of class $C^{1,1}$, i.e. g has Fréchet derivatives which are locally Lipschitz. The Clarke generalized Hessian of g at x_0 is $\partial_C(g^{'})(x_0)$, which is denoted by $\partial^2_C g(x_0)$. Clearly,

$$\partial^2_C g(x_0) := \text{clco}\{\lim g^{''}(x_i) : x_i \to x_0, g^{''}(x_i) \text{ exists }\}.$$

We call $\partial_C g(.)$ and $\partial_C^2 g(.)$ the Clarke generalized Jacobian and Clarke generalized Hessian of g. For other generalized derivatives we adopt a similar distinction between derivative mappings and their values at points.

Definition 2.4

(i) See Ref. 16. Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. A closed subset $\partial g(x_0) \subseteq L(\mathbb{R}^n, \mathbb{R}^m)$ is called an approximate Jacobian of g at $x_0 \in \mathbb{R}^n$ if, for each $v \in \mathbb{R}^m$ and $u \in \mathbb{R}^n$,

$$(vg)^+(x_0, u) \leq \sup_{M \in \partial g(x_0)} \langle v, Mu \rangle,$$

where $(.)^+$ denotes the upper Dini directional derivative of $(.)$, i.e.

$$(vg)^+(x_0, u) := \limsup_{t \downarrow 0} \langle v, g(x_0 + tu) - g(x_0) \rangle / t.$$

(ii) See Ref. 20. An approximate Jacobian $\partial g(x_0)$ is termed a Fréchet approximate Jacobian of g at x_0 if there is a neighborhood U of x_0 such that, for each $x \in U$,

$$g(x) - g(x_0) \in \partial g(x_0)(x - x_0) + o(\|x - x_0\|).$$

(iii) See Refs. 16-17. Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable. A closed subset $\partial^2 g(x_0) \subseteq B(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^m)$ is said to be an approximate Hessian of g at x_0 if $\partial^2 g(x_0)$ is an approximate Jacobian of $g'(.)$ at x_0.

It is obvious that any Fréchet approximate Jacobian is a first-order approximation. If an approximate Jacobian $\partial g(.)$ is upper semicontinuous at x_0, then (Ref. 20) $\text{clco}\partial g(x_0)$ is a Fréchet approximate Jacobian and hence is a first-order approximation. To get an improvement of this result and relations between the notions we need the following lemma.

Lemma 2.1 See Ref. 16, Taylor's formula.

(i) If $g : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and, for $x, y \in \mathbb{R}^n$ and $z \in [x, y]$, $\partial g(z)$ is an approximate Jacobian of g at z, then

$$g(x) - g(y) \in \text{clco}(\partial g[x, y](x - y)).$$

(ii) If $g : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable and, for $x, y \in \mathbb{R}^n$ and $z \in [x, y]$, $\partial^2 g(z)$ is an approximate Hessian of g at z, then

$$g(x) - g(y) \in g'(y)(x - y) + \frac{1}{2}\text{clco}(\partial^2 g[x, y](x - y, x - y)).$$

Proposition 2.1. Assume that $g : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and admits an approximate Jacobian $\partial g(.)$ which is upper semicontinuous at x_0. Then $\text{co}\partial g(x_0)$ is a first-order approximation of g at x_0.

Proof. The assumption means that $\forall \epsilon > 0$, $\exists \delta > 0$, $\forall x \in x_0 + \delta B_{\mathbb{R}^n}$,

$$\partial g(x) \subseteq \partial g(x_0) + \tfrac{\epsilon}{2} B_{L(\mathbb{R}^n,\mathbb{R}^m)}.$$

Hence

$$\mathrm{co}\partial g[x_0, x] \subseteq \mathrm{co}\partial g(x_0) + \tfrac{\epsilon}{2} B_{L(\mathbb{R}^n,\mathbb{R}^m)}.$$

By Lemma 2.1(i) one has

$$\begin{aligned} g(x) - g(x_0) &\in \mathrm{co}\partial g(x_0)(x - x_0) + \tfrac{\epsilon}{2} B_{L(\mathbb{R}^n,\mathbb{R}^m)}(x - x_0) \\ &\quad + \tfrac{\epsilon}{2}\|x - x_0\| B_{\mathbb{R}^m} \\ &= \mathrm{co}\partial g(x_0)(x - x_0) + \epsilon\|x - x_0\| B_{\mathbb{R}^m}. \end{aligned}$$

That is $A_g(x_0) = \mathrm{co}\partial g(x_0)$ is a first-order approximation. □

Proposition 2.2. Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be continuous and $x_0 \in \mathbb{R}^n$. If $A_g(x_0)$ is a first-order approximation of g at x_0 then $\mathrm{cl}A_g(x_0)$ is a Fr*é*chet approximate Jacobian of g at x_0.

Proof. By the definition of $A_g(x_0)$, for $u \in \mathbb{R}^n$ and small positive t, one finds $M_t \in A_g(x_0)$ such that

$$g(x_0 + tu) - g(x_0) = M_t(tu) + o(t).$$

Then, for $v \in \mathbb{R}^m$,

$$\tfrac{1}{t}\langle v, g(x_0 + tu) - g(x_0)\rangle \leq \sup_{M \in clA_{g(x_0)}} \langle v, M(u)\rangle + \langle v, o(t)/t\rangle.$$

Taking $\limsup_{t \downarrow 0}$ one sees that

$$(vg)^{+}(x_0, u) \leq \sup_{M \in clA_{g(x_0)}} \langle v, Mu \rangle. \qquad \square$$

Proposition 2.3. Assume that $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuously differentiable in a neighborhood U of x_0 and $\partial^2 g(.)$ is an approximate Hessian which is upper semicontinuous at x_0. Then, $(g^{'}(x_0), \frac{1}{2}\text{co}\partial^2 g(x_0))$ is a second-order approximation of g at x_0.

Proof. For an arbitrarily fixed $\epsilon > 0$, by the assumed upper semicontinuity, without loss of generality, we can assume that, for $B = B_{L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m))}$ and $x \in U$,

$$\text{co}\partial^2 g(x) \subseteq \text{co}\partial^2 g(x_0) + \tfrac{\epsilon}{2} B.$$

Hence, by Lemma 2.1(ii),

$$\begin{aligned} g(x) - g(x_0) \in\ & g^{'}(x_0)(x - x_0) + \tfrac{1}{2}\text{clco}\partial^2 g[x_0, x](x - x_0, x - x_0) \\ & + \tfrac{\epsilon}{2}\|x - x_0\|^2 B_{\mathbb{R}^m} \\ \subseteq\ & g^{'}(x_0)(x - x_0) + \tfrac{1}{2}\text{co}\partial^2 g(x_0)(x - x_0, x - x_0) \\ & + \tfrac{\epsilon}{2}\|x - x_0\|^2 B_{\mathbb{R}^m} + \tfrac{\epsilon}{2}\|x - x_0\|^2 B_{\mathbb{R}^m}. \end{aligned}$$

Thus, $(g^{'}(x_0), \frac{1}{2}\text{co}\partial^2 g(x_0))$ is a second-order approximation of g at x_0. $\square$

The following result is a direct consequence of Propositions 2.1 and 2.3.

Proposition 2.4 See Refs. 1-2.

(i) If $g : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz at x_0 then $\partial_C g(x_0)$ is a first-order approximation of g at x_0.

(ii) If $g : \mathbb{R}^n \to \mathbb{R}^m$ is of class $C^{1,1}$ at x_0 then $(g^{'}(x_0), \frac{1}{2}\partial_C^2 g(x_0))$ is a second-order approximation of g at x_0.

The examples below show that the above generalized derivatives may be equal to or different from each other.

Example 2.1. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(x,y) = \begin{cases} x^2 \sin(1/x) + |y| & \text{if } x \neq 0, \\ |y| & \text{if } x = 0. \end{cases}$$

Then g is locally Lipschitz at (0, 0) and we have an approximate Jacobian

$$\partial g(0,0) = A_g(0,0) = \{(0,\beta) : \beta \in \{-1,1\}\},$$

which is also a Fréchet approximate Jacobian. However,

$$\partial g_C(0,0) = \{(\alpha,\beta) : \alpha,\beta \in [-1,1]\}.$$

Example 2.2. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be given as

$$g(x,y) = (|x| - |y|, |y| - |x|).$$

Then g is locally Lipschitz at $(0,0)$ and

$$\partial g(0,0) = \left\{ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \right\}$$

is an approximate Jacobian but not a Fréchet approximate Jacobian. Moreover a first-order approximation is

$$A_g(0,0) = \partial g(0,0) \bigcup \left\{ \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \right\}$$

which is also a Fréchet approximate Jacobian. We have also $\partial_C g(0,0) = \mathrm{co} A_g(0,0)$.

Example 2.3. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be

$$g(x,y) = (|x|^{1/2}\mathrm{sign}(x), y^{1/3} + |x|).$$

Then g is continuous but $g \notin C^{0,1}$ at $(0,0)$ and a first-order approximation

$$A_g(0,0) = \left\{ \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} : \alpha > 0, \beta = \pm 1, \gamma > 0 \right\}$$

is different from the Fréchet approximate Jacobian

$$\partial_F g(0,0) = \left\{ \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} : \alpha \geq 0, \beta \in [-1,1], \gamma \in \mathbb{R} \right\}.$$

Second-order generalized derivatives have similar situations as shown by the following two examples.

Example 2.4. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(x,y) = \tfrac{1}{2}x^2\mathrm{sign}(x) + \tfrac{1}{2}y^2\mathrm{sign}(y).$$

Then $g \in C^{1,1}$ at $(0,0)$. We have $f'(x,y) = (|x|,|y|)$ and the three second-order derivatives are different:

$$\partial_C^2 g(0,0) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in [-1,1] \right\},$$

$$\partial^2 g(0,0) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

$$B_g(0,0) = \left\{ \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} -1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \right\}.$$

Example 2.5. The mapping $g : \mathbb{R}^2 \to \mathbb{R}$ given by

$$g(x,y) = \tfrac{2}{3}|x|^{3/2} + \tfrac{1}{2}y^2$$

is in C^1 but not in $C^{1,1}$. So $\partial_C^2 g$ does not exist and the other two second-order derivatives can be taken as

$$\partial^2 g(0,0) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} : \alpha \geq 0 \right\},$$

$$B_g(0,0) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1/2 \end{pmatrix} : \alpha > 0 \right\}.$$

3. Optimality Conditions for Unconstrained Optimization

Consider problem (P1) stated in Section 1:

$$\text{(P1)} \qquad \min f(x), \text{ s.t. } x \in X.$$

Here by min we mean the minimization in one of following three classical notions of local solutions in vector optimization.

Definition 3.1

(i) $x_0 \in X$ is called a local ideal solution of (P1) if there exists a neighborhood U of x_0 such that, $\forall x \in U$,

$$f(x) - f(x_0) \in C.$$

(ii) $x_0 \in X$ is said to be a local efficient solution of (P1) if there exists a neighborhood U of x_0 such that, $\forall x \in U$,

$$f(x) - f(x_0) \notin (-C) \setminus C.$$

(iii) $x_0 \in X$ is termed a local weakly efficient solution of (P1) if there exists a neighborhood U of x_0 such that, $\forall x \in U$,

$$f(x) - f(x_0) \notin -\text{int } C.$$

We propose the following relaxed compactness which will be needed for establishing optimality conditions in the sequel.

Definition 3.2

(i) Let f_α and f belong to $L(X,Y)$. We say that the net f_α pointwisely converges to f and write $f_\alpha \xrightarrow{\text{p}} f$ or $f = \text{p-lim}\, f_\alpha$ if $\lim f_\alpha(x) = f(x)$ for all $x \in X$. A similar definition is adopted for $f_\alpha, f \in B(X,X,Y)$.

(ii) A subset $A \subseteq L(X,Y)$ ($B \subseteq B(X,X,Y)$, respectively) is said to be relatively p-compact if each net $(f_\alpha) \subseteq A$ ($\subseteq B$, respectively) with a bounded image $\bigcup\limits_\alpha f_\alpha(x)$ for each $x \in X$ ($\bigcup\limits_\alpha f_\alpha(x,y)$ for each $(x,y) \in X \times Y$, respectively) has a subnet (f_β) and $f \in L(X,Y)$ ($f \in B(X,X,Y)$, respectively) such that $f = \text{p-lim}\, f_\beta$.

(iii) A subset $A \subseteq L(X,Y)$ is called asymptotically p-compact if for each net $(f_\alpha) \subseteq A$ with $\lim \|f_\alpha\| = \infty$, the net $(f_\alpha/\|f_\alpha\|)$ has a subnet which pointwisely converges to some $f \in L(X,Y) \setminus \{0\}$. For $B \subseteq B(X,X,Y)$ a similar definition is adopted. If $A \subseteq L(X,Y)$ and $B \subseteq B(X,X,Y)$ are both relatively p-compact (or asymptotically p-compact, respectively) we call (A,B) relatively p-compact (or asymptotically p-compact, respectively).

For $A \subseteq L(X,Y)$ and $B \subseteq B(X,X,Y)$ we adopt the notations:

$$\text{p-cl}A = \{f \in L(X,Y) : \exists (f_\alpha) \subseteq A, f = \text{p-lim}\, f_\alpha\},$$

$$\text{p-cl}B = \{g \in B(X,X,Y) : \exists (g_\alpha) \subseteq B, g = \text{p-lim}\, g_\alpha\},$$

$$\text{p-}A_\infty = \{f \in L(X,Y) : \exists (f_\alpha) \subseteq A, \exists t_\alpha \to 0^+, f = \text{p-lim}\, t_\alpha f_\alpha\},$$

$$\text{p-}B_\infty = \{g \in B(X,X,Y) : \exists (g_\alpha) \subseteq B, \exists t_\alpha \to 0^+, g = \text{p-lim}\, t_\alpha g_\alpha\}.$$

p-A_∞ and p-B_∞ are called p-recession cone of A and B, respectively.

Remark 3.1

(i) The pointwise convergence in $L(X,\mathbb{R}) = X^*$ coincides with the $*$-weak convergence and for $A \subseteq X^*$, p-cl$A$ coincides with $*$-cl$A$ (the $*$-weak closure of A).

(ii) If $f_\alpha \to f$ in $L(X,Y)$ or in $B(X,X,Y)$, then $f_\alpha \xrightarrow{\text{p}} f$. If X and Y are finite dimensional then the converse does hold. However, the converse is no longer true for the infinite dimensional case, see Example 3.1 below.

(iii) If $A \subseteq L(X,Y)$ ($B \subseteq B(X,X,Y)$) is relatively compact, then it is relatively p-compact but the converse is not valid following Example 3.1 below. If X and Y are finite dimensional then every subset of $L(X,Y)$ and $B(X,X,Y)$ are relatively p-compact.

(iv) If A is asymptotically compact in $L(X,Y)$ or in $B(X,X,Y)$, then it is asymptotically p-compact. If X and Y are finite dimensional then these two notions coincide and every subset of $L(X,Y)$ and $B(X,X,Y)$ are asymptotically compact. However, for general spaces this is no longer true as shown by Example 3.2 below.

(v) It is obvious that $A_\infty \subseteq$ p-A_∞, where A_∞ is the recession cone of $A \subseteq L(X,Y)$ (or $A \subseteq B(X,X,Y)$) defined by $A_\infty = \{f \in L(X,Y) : \exists (f_\alpha) \subseteq A, \exists t_\alpha \to 0^+, f = \lim t_\alpha f_\alpha\}$ (or by $A_\infty = \{f \in B(X,X,Y) : \exists (f_\alpha) \subseteq A, \exists t_\alpha \to 0^+, f = \lim t_\alpha f_\alpha\}$, respectively).

(vi) If $x_\alpha \to x$ in X and $A_\alpha \xrightarrow{\text{p}} A$ in $L(X,Y)$, then $A_\alpha x_\alpha \to Ax$ in Y. Similarly, if $x_\alpha \to x$, $y_\alpha \to y$ in X and $B_\alpha \xrightarrow{\text{p}} B$ in $B(X,X,Y)$, then $B_\alpha(x_\alpha, y_\alpha) \to B(x,y)$ in Y.

Recall that $g : X \to Y$ is called C-convex at $x_0 \in X$ if there is a convex neighborhood U of x_0 such that, $\forall x \in U$, $\forall \alpha \in [0,1]$,

$$(1-\alpha)f(x_0) + \alpha f(x) - f((1-\alpha)x_0 + \alpha x) \in C.$$

We propose the following relaxed property. $g : X \to Y$ is said to be generalized C-quasiconvex at $x_0 \in X$ if there exists a convex neighborhood U of x_0 such that, $\forall x \in U \setminus \{x_0\}$, $\exists \alpha_0 \in (0,1)$, $\forall \alpha \in (0, \alpha_0)$,

$$[g(x) - g(x_0) \in Y \backslash \text{int}\, C] \Longrightarrow [g((1-\alpha)x_0 + \alpha x) - g(x_0) \in Y \backslash \text{int}\, C].$$

3.1 Optimality Conditions for Ideal Solutions

Theorem 3.1. Assume that $(A_f(x_0), B_f(x_0))$ is a second-order approximation of f at x_0, which is both relatively p-compact and asymptotically p-compact. Assume further that x_0 is a local ideal solution of (P_1). Then

(i) $\forall h \in X$, $\exists M \in$ p-cl$A_f(x_0) \bigcup$(p-$A_f(x_0)_\infty \setminus \{0\}$), $Mh \in C$;

(ii) $\forall h \in \text{Ker}A_f(x_0) := \{v \in X : Mv = 0, \forall M \in A_f(x_0)\}$, $\exists N \in$p- cl$B_f(x_0)$ $\bigcup$(p-$B_f(x_0)_\infty \setminus \{0\}$), $N(h,h) \in C$.

Proof. (i) Let $h \in X$ and $x_i = x_0 + h/i$, $i = 1, 2, ...$ By the two assumptions, for sufficiently large i one has, for some $M_i \in A_f(x_0)$,

$$f(x_i) - f(x_0) \in C, \tag{1}$$

$$f(x_i) - f(x_0) = M_i(x_i - x_0) + o(\|x_i - x_0\|). \tag{2}$$

Consequently,

$$M_i(x_i - x_0) + o(\|x_i - x_0\|) \in C. \tag{3}$$

If the sequence (M_i) is bounded, by the relative p-compactness of $A_f(x_0)$ we can assume, by taking a subsequence if necessary, the existence of $M \in$ p-cl$A_f(x_0)$ such that $M_i \xrightarrow{\text{p}} M$. Dividing (3) by $1/i$ and letting $i \to \infty$ one obtains $Mh \in C$.

If (M_i) is unbounded, by the asymptotic p-compactness of $A_f(x_0)$ we can assume that $\|M_i\| \to \infty$ and $M_i/\|M_i\| \xrightarrow{\text{p}} M$ for some $M \in$ p-$A_f(x_0)_\infty \setminus \{0\}$. Dividing (3) by $\|M_i\|/i$ and letting $i \to \infty$ one gets also $Mh \in C$.

(ii) Let $h \in \mathrm{Ker}A_f(x_0)$ and $x_i = x_0 + h/i$, $i = 1, 2, \ldots$. By the definition of $(A_f(x_0), B_f(x_0))$, for each i large enough there is (M_i, N_i) in this set such that

$$f(x_i) - f(x_0) = M_i(x_i - x_0) + N_i(x_i - x_0, x_i - x_0) + o(\|x_i - x_0\|^2). \tag{4}$$

Hence, by (1) and the fact that $h \in \mathrm{Ker}A_f(x_0)$,

$$N_i(x_i - x_0, x_i - x_0) + o(\|x_i - x_0\|^2) \in C. \tag{5}$$

Arguing with (N_i) similarly as with (M_i) above, dividing (5) by $1/i^2$ if (N_i) is bounded and by $\|N_i\|/i^2$ otherwise, one gets N as stated in the theorem. □

Remark 3.2

(i) Assume that $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$. If $f \in C^{0,1}$ and $A_f(x_0) = \partial_C f(x_0)$, then Theorem 3.1(i) collapses to Theorem 4.1(i) of Ref. 7. If $f \in C^{1,1}$ and $(A_f(x_0), B_f(x_0)) = (f'(x_0), \frac{1}{2}\partial_C^2 f(x_0))$, then Theorem 3.1(ii) becomes Theorem 4.1(ii) of Ref. 7.

(ii) If $Y = \mathbb{R}, C = [0, +\infty)$ and $(A_f(x_0),\ B_f(x_0))$ is compact, then Theorem 3.1 drops to Theorem 3.1.1 of Ref. 2.

The following two examples make it clear that our relaxed compactness assumptions are really weaker than the compactness assumptions.

Example 3.1. Let $f_i : l^2 \to l^2$ be defined by $f_i(x) = (x_1, 0, ..., 0, x_i, 0, ...)$, where $x = (x_1, ..., x_i, ...) \in l^2$ and $A = \{f_i : i \geq 2\} \subset L(l^2, l^2)$. Then $\|f_i\| = 1$. Let $f_0 \in L(l^2, l^2)$ be defined by $f_0(x) = (x_1, 0, ...)$, then $f_i \xrightarrow{\text{p}} f_0$, but (f_i) does not contain convergent subsequence. Therefore, A is relatively p-compact but not relatively compact.

Example 3.2. Let $A = \{if_i : i \geq 2\} \subset L(l^2, l^2)$, where f_i be defined as in Example 3.1. Then $\|if_i\| = i$ and $if_i/\|if_i\| \xrightarrow{\text{p}} f_0$ with f_0 as in Example 3.1. However, $(if_i/\|if_i\|)$ has no convergent subsequence. Thus, A is asymptotically p-compact but not asymptotically compact.

Theorem 3.2. If $(A_f(x_0), B_f(x_0))$ is a second-order approximation of f at x_0, which is relatively p-compact as well as asymptotically p-compact. Then each of the following conditions is sufficient for x_0 to be a local ideal solution of (P1):

(i) f is C-convex at x_0 and for each $h \in X \setminus \{0\}$ one has $Mh \in C$ if $M \in$ p-cl $A_f(x_0)$ and $Mh \in C \setminus (-C)$ if $M \in$ p-$A_f(x_0)_\infty \setminus \{0\}$;

(ii) f is generalized C-quasiconvex at x_0 and for each $h \in X \setminus \{0\}$ one has $Mh \in \text{int } C$ if $M \in$ p-cl$A_f(x_0) \bigcup($p-$A_f(x_0)_\infty \setminus \{0\})$;

(iii) $X = \mathbb{R}^n$ and for each $h \in \mathbb{R}^n \setminus \{0\}$ one has $Mh \in C$ if $M \in A_f(x_0)$

and $N(h,h) \in \text{int}\, C$ if $N \in \text{p-cl}B_f(x_0) \bigcup (\text{p-}B_f(x_0)_\infty \setminus \{0\})$.

Proof. (i) Assume that U is a convex neighborhood of x_0 where f is C-convex and $(A_f(x_0), B_f(x_0))$ satisfies the definition of the second-order approximation. Then, for $x \in U \setminus \{x_0\}$ and $i = 1, 2, ...,$

$$f(x) - f(x_0) - i[f(x_0 + \tfrac{1}{i}(x - x_0)) - f(x_0)] \in C.$$

Hence, with $M_i \in A_f(x_0)$ such that

$$f(x_0 + \tfrac{1}{i}(x - x_0)) - f(x_0) = M_i(\tfrac{1}{i}(x - x_0)) + o(1/i),$$

one has

$$f(x) - f(x_0) - M_i(x - x_0) - io(1/i) \in C. \tag{6}$$

If (M_i) is bounded, one can assume that $M_i \xrightarrow{\text{p}} M \in \text{p-cl}A_f(x_0)$. Hence,

$$f(x) - f(x_0) \in M(x - x_0) + C \subseteq C.$$

Now suppose that $\|M_i\| \to \infty$ and $M_i/\|M_i\| \xrightarrow{\text{p}} M \in \text{p-}A_f(x_0)_\infty \setminus \{0\}$. Dividing (6) by $\|M_i\|$ and letting $i \to \infty$ one gets $-M(x - x_0) \in C$, i.e. $M(x - x_0) \in -C$, which is impossible.

(ii) Suppose there is a sequence x_i converging to x_0 such that, for $i = 1, 2, ...,$

$$f(x_i) - f(x_0) \in Y \setminus C. \tag{7}$$

Fix an i_0 such that x_{i_0} lies in the neighborhood of x_0 stated in the definition of the generalized C-quasiconvexity (for f) and set $h = x_{i_0} - x_0 \neq 0$. By this definition, for all large i,

$$f(x_0 + \tfrac{1}{i}h) - f(x_0) \in Y \backslash \text{int } C.$$

Hence, for i large enough, there is $M_i \in A_f(x_0)$ such that

$$M_i(\tfrac{1}{i}h) + o(1/i) \in Y \backslash \text{int } C.$$

By arguing both the case where (M_i) is bounded and the opposite case, similarly as in the proof of Theorem 3.1, one arrives at the contradiction that an $M \in$ p-cl$A_f(x_0) \bigcup (\text{p-}A_f(x_0)_\infty \setminus \{0\})$ exists such that $Mh \in Y \backslash \text{int } C$.

(iii) Suppose also the contrary (7). We can assume $(x_i - x_0)/\|x_i - x_0\| \to k$, for some $k \neq 0$. For all large i, there are $M_i \in A_f(x_0)$ and $N_i \in B_f(x_0)$ satisfying (4). Again considering both cases, where (N_i) is bounded or unbounded similarly as in the proof of Theorem 3.1, one gets a contradiction. □

Remark 3.3. Part (iii) contains Theorem 4.2 of Ref. 7 as a special case where $X = \mathbb{R}^n, Y = \mathbb{R}^m, f \in C^{1,1}, A_f(x_0) = \{f^{'}(x_0)\} = \{0\}$ and $B_f(x_0) = \frac{1}{2}\partial^2_C f(x_0)$. Parts (i) and (ii) are new.

3.2 Optimality Conditions for Weakly Efficient and Efficient Solutions

Theorem 3.3. Assume that $(A_f(x_0), B_f(x_0))$ is a second-order approximation of f at x_0, which is both relatively p-compact and asymptotically p-compact. If x_0 is a local weakly efficient solution of (P1), then

(i) $\forall h \in X$, $\exists M \in$ p-cl$A_f(x_0)\bigcup$(p-$A_f(x_0)_\infty \setminus \{0\}$), $Mh \in Y \setminus -\text{int } C$;

(ii) $\forall h \in X : A_f(x_0)(h) \subseteq -(C\setminus \text{ int } C)$, $\exists N \in$ p-cl$B_f(x_0)\bigcup$(p-$B_f(x_0)_\infty \setminus \{0\}$), $N(h,h) \in Y \setminus -\text{int } C$.

Proof. (i) Let $h \in X$ and $x_i = x_0 + h/i$, $i = 1, 2, ...$Then, for large i,

$$f(x_i) - f(x_0) \in Y \setminus -\text{int } C. \tag{8}$$

Then, there exists $M_i \in A_f(x_0)$ with

$$M_i(x_i - x_0) + o(\|x_i - x_0\|) \in Y \setminus -\text{int } C. \tag{9}$$

Similarly as in the proof of part (i) of Theorem 3.1, we have $M \in$ p-cl$A_f(x_0)\bigcup$(p-$A_f(x_0)_\infty \setminus \{0\}$) such that $Mh \in Y \setminus -\text{int } C$.

(ii) Let $h \in X$ be such that $A_f(x_0)(h) \subseteq -(C\setminus\text{int } C)$ and $x_i = x_0 + h/i$ satisfy (8). By the definition of the second-order approximation, $M_i \in A_f(x_0)$ and $N_i \in B_f(x_0)$ exist fulfilling (4). Therefore,

$$N_i(x_i - x_0, x_i - x_0) + o(\|x_i - x_0\|^2) \in Y \setminus -\text{int } C.$$

Similarly as part (ii) of Theorem 3.1 we get $N(h,h) \in Y \setminus -\text{int } C$ for some $N \in$ p-cl$B_f(x_0)\bigcup$(p-$B_f(x_0)_\infty \setminus \{0\}$). □

Remark 3.4. Theorem 5.1 of Ref. 7 is a particular case of Theorem 3.3.

Of course, an assertion similar to Theorem 5.1 of Ref. 7 for the finite dimensional case, using approximate Jacobians and approximate Hessians, can be formulated as a corollary of Theorem 3.3. However, for infinite dimensional cases as in the following example, Theorem 3.3 does work but the known results do not.

Example 3.3. Let $f : l^2 \to \mathbb{R}^2$ be defined as, for $x = (x_1, x_2, ...) \in l^2$,

$$f(x) = (-\tfrac{2}{3}|x_1|^{3/2} - \tfrac{2}{3}|x_2|^{3/2} - \tfrac{1}{2}x_2^2,\ -\tfrac{2}{3}|x_1|^{3/2} - \tfrac{2}{3}|x_2|^{3/2} + \tfrac{1}{2}x_2^2)$$

and $C = \mathbb{R}^2_+$. Then $(A_f(0), B_f(0))$ is a second-order approximation of f at 0, where $A_f(0) = \{f'(0)\} = \{0\}$, $B_f(0) = \{N_{\alpha\beta} \in B(l^2, l^2, \mathbb{R}^2) : \alpha, \beta < -1\}$, where, for $x, y \in l^2$ and $y = (y_1, y_2, ...)$,

$$N_{\alpha\beta}(x, y) = (\alpha x_1 y_1 + \beta x_2 y_2 - \tfrac{1}{2}x_2 y_2,\ \alpha x_1 y_1 + \beta x_2 y_2 + \tfrac{1}{2}x_2 y_2).$$

Observe that, for all $\alpha, \beta < -1$, $N_{\alpha\beta} \in \text{spand}\{N_1, N_2, N_3\}$, where spand$\{.\}$ stands for the linear space generated by $N_1, N_2, N_3 \in B(l^2, l^2, \mathbb{R}^2)$, where

$$N_1(x, y) = (x_1 y_1, x_1 y_1),$$

$$N_2(x, y) = (x_2 y_2, x_2 y_2),$$

$$N_3(x, y) = (-\tfrac{1}{2}x_2 y_2, \tfrac{1}{2}x_2 y_2).$$

Being contained in a finite dimensional space, $(A_f(0), B_f(0))$ is both relatively p-compact and asymptotically p-compact. We have further

$$\text{p-cl}B_f(0) = \{N_{\alpha\beta} \in B(l^2, l^2, \mathbb{R}^2) : \alpha, \beta \leq -1\},$$

$$\text{p-}B_f(0)_\infty = \{\lambda N_1 + \gamma N_2 : \lambda, \gamma \leq 0\}.$$

To show that $0 \in l^2$ is not a local weakly efficient solution of (P1) we choose $h = (h_1, h_2, h_3, ...) \in l^2$ with $h_1 \neq 0 \neq h_2$. Then $\forall N_{\alpha\beta} \in \text{p-cl}B_f(0)$,

$$N_{\alpha\beta}(h, h) = (\alpha h_1^2 + (\beta - \tfrac{1}{2})h_2^2,\ \alpha h_1^2 + (\beta + \tfrac{1}{2})h_2^2) \in -\text{ int } C.$$

$\forall N_{\alpha\beta} \in \text{p-}B_f(0)_\infty \setminus \{0\}$: $N_{\alpha\beta} = \lambda N_1 + \gamma N_2$, $\lambda^2 + \gamma^2 > 0$, $\lambda, \gamma \leq 0$,

$$N_{\alpha\beta}(h, h) = (\lambda h_1^2 + \gamma h_2^2, \lambda h_1^2 + \gamma h_2^2) \in -\text{int } C.$$

Thus, (ii) of Theorem 3.3 is violated.

Theorem 3.4. Assume that $(A_f(x_0), B_f(x_0))$ is as in Theorem 3.3. Then x_0 is a local efficient solution of (P1), if one of (i) or (ii) below holds:

(i) $X = \mathbb{R}^n$ and $\forall h \in \mathbb{R}^n \setminus \{0\}$, $\forall M \in \text{p-cl}A_f(x_0) \bigcup (\text{p-}A_f(x_0)_\infty \setminus \{0\})$, $Mh \in Y \setminus -C$;

(ii) $X = \mathbb{R}^n$ and

(a) $\forall h \in \mathbb{R}^n \setminus \{0\}$, $\forall M \in \text{p-cl}A_f(x_0) \bigcup (\text{p-}A_f(x_0)_\infty \setminus \{0\})$, $Mh \in (Y \setminus - C) \cup (C \cap (-C))$;

(b) $\forall h \in \mathbb{R}^n \setminus \{0\} : \exists M \in$ p-cl$A_f(x_0) \bigcup($p-$A_f(x_0)_\infty \setminus \{0\})$, $Mh \in C \cap$

$(-C)$; $\forall N \in$ p-cl$B_f(x_0) \bigcup($p-$B_f(x_0)_\infty \setminus \{0\})$; $N(h,h) \in \text{int}\, C$.

Proof. (i) Suppose to the contrary that a sequence $x_i \to x_0$ with

$$f(x_i) - f(x_0) \in (-C) \setminus C. \tag{10}$$

We can assume that $(x_i - x_0)/\|x_i - x_0\| \to h$, for some $h \neq 0$. For all large i, there is $M_i \in A_f(x_0)$ which satisfies (2). Then, $M_i(x_i - x_0) + o(\|x_i - x_0\|) \in -C$. Therefore, by a routine argument as before one obtains $Mh \in -C$ for some $M \in$ p-cl$A_f(x_0) \bigcup($p-$A_f(x_0)_\infty \setminus \{0\})$.

(ii) Suppose (10), by (i) we have $Mh \in -C$ for some $M \in$ p-cl$A_f(x_0) \bigcup($p-$A_f(x_0)_\infty \setminus \{0\})$. By assumption (a), $Mh \in C \cap (-C)$. Employing a similar argument as in Theorem 3.2(iii), (b) and (4) we get

$$f(x_i) - f(x_0) \in (Y \setminus -C) \cup (C \cap (-C)) + \text{int}\, C \subseteq Y \setminus -C,$$

which contradicts (10). □

Remark 3.5. Theorem 3.4 contains Theorem 5.2 of Ref. 7 as a special case in the same way as Theorem 3.3 contains Theorem 5.1 of Ref. 7.

4. Optimality Conditions for Constrained Optimization

Now pass to the constrained nonsmooth vector problem

(P2) $\min f(x)$, s.t. $x \in S$.

We first recall that all the solution notions specified in Definition 3.1 for the unconstrained problem (P1) are extended naturally to the problem (P2) with "$\forall x \in U$" replaced by "$\forall x \in U \cap S$". Recall also the definitions of the first and second-order contingent cones of S at x_0, respectively,

$$T_1(S, x_0) = \{u \in X : \exists t_i \to 0^+, x_i = x_0 + t_i u + o(t_i) \in S\},$$

$$T_2(S, x_0) = \{(u, v) \in X \times X : \exists t_i \to 0^+, x_i = x_0 + t_i u + \tfrac{1}{2} t_i^2 v + o(t_i^2) \in S\}. \tag{11}$$

The positive polar cone of C is defined as

$$C^* = \{y^* \in Y^* : \langle y^*, x \rangle \geq 0, \forall x \in C\},$$

and the truncated cone generated by S at x_0 is

$$S_\delta(x_0) = \{t(x - x_0) : t \geq 0, x \in S, \|x - x_0\| \leq \delta\}.$$

Denote further $\Lambda = \{\lambda \in C^* : \|\lambda\| = 1\}$.

Theorem 4.1. Assume that $(A_f(x_0), B_f(x_0))$ is the same as in Theorem 3.3. If $x_0 \in S$ is a local weakly efficient solution of (P2), then, $\forall (u, v) \in T_2(S, x_0)$,

(i) $\exists M \in \text{p-cl}A_f(x_0) \bigcup (\text{p-}A_f(x_0)_\infty \setminus \{0\})$, $\exists \lambda \in \Lambda$, $\langle \lambda, Mu \rangle \geq 0$;

(ii) If $A_f(x_0) = \{f'(x_0)\}$ and $f'(x_0)(u) = 0$, then there is $\bar{\lambda} \in \Lambda$ such that

$N \in$ p-cl$B_f(x_0)$ exists such that $\langle \bar{\lambda}, \frac{1}{2} f'(x_0)(v) + N(u,u) \rangle \geq 0$ or $\bar{N} \in$ p-$B_f(x_0)_\infty \setminus \{0\}$ such that $\langle \bar{\lambda}, \bar{N}(u,u) \rangle \geq 0$.

Proof. (i) Let $(u,v) \in T_2(S, x_0)$, i.e. we have (11). Then, for all large i, we have (8) and (9). Dividing (9) by t_i if (M_i) is bounded and by $t_i \|M_i\|$ if $\lim_{i \to \infty} \|M_i\| = \infty$ we obtain $M \in$ p-cl$A_f(x_0) \bigcup$(p-$A_f(x_0)_\infty \setminus \{0\}$) such that $Mu \in Y \setminus -\text{int } C$. Hence $\lambda \in \Lambda$ exists such that $\langle \lambda, Mu \rangle \geq 0$.

(ii) Assume that $A_f(x_0) = \{f'(x_0)\}$ and $f'(x_0)(u) = 0$. By (8), there is $N_i \in B_f(x_0)$ for all large i such that,

$$f(x_i) - f(x_0) = f'(x_0)(x_i - x_0) + N_i(x_i - x_0, x_i - x_0) + o(\|x_i - x_0\|^2)$$

$$= \tfrac{1}{2} t_i^2 f'(x_0)(v) + t_i^2 N_i(u,u) + o(t_i^2) \in Y \setminus -\text{int } C.$$

Dividing this by t_i^2 if (N_i) is bounded and by $t_i^2 \|N_i\|$ if $\|N_i\| \to \infty$, similarly as have been done several times we complete the proof. □

Remark 4.1. (i) If $X = \mathbb{R}^n, Y = \mathbb{R}^m$, f is continuously differentiable, $\partial^2 f$ is upper semicontinuous at x_0 and $(A_f(x_0), B_f(x_0)) = (f'(x_0), \frac{1}{2}\text{co}\partial^2 f(x_0))$, then Theorem 4.1 collapses to Theorem 3.1 of Ref. 18.

(ii) If $X = \mathbb{R}^n, Y = \mathbb{R}^m, f \in C^{1,1}$ and $(A_f(x_0), B_f(x_0)) = (f'(x_0), \frac{1}{2}\partial_C^2 f(x_0))$, then Theorem 4.1 drops to Theorem 3.1 of Ref. 8.

Theorem 4.2. Let $(A_f(x_0), B_f(x_0))$ be as in Theorem 3.3 and $x_0 \in S$. Then each of the following conditions is sufficient for x_0 to be a locally unique efficient solution of (P2):

(i) $X = \mathbb{R}^n$ and $\forall u \in T_1(S, x_0) \setminus \{0\}$, $\forall M \in$ p-cl$A_f(x_0) \bigcup($p-$A_f(x_0)_\infty \setminus \{0\})$, $\exists \lambda \in \Lambda$, $\langle \lambda, Mu \rangle > 0$;

(ii) $X = \mathbb{R}^n, Y = \mathbb{R}^m$ and $\exists \delta > 0$, $\forall v \in S_\delta(x_0)$, $\forall u \in T_1(S, x_0)$,

(a) $\forall M \in A_f(x_0)$, $\exists \lambda \in \Lambda$, $\langle \lambda, Mv \rangle \geq 0$ and

(b) $\forall N \in$ p-cl$B_f(x_0) \bigcup($p-$B_f(x_0)_\infty \setminus \{0\})$, $\forall \lambda \in \Lambda$, $\langle \lambda, N(u, u) \rangle > 0$.

Proof. (i) Arguing by contradiction assume the existence of $x_i \in S$, $x_i \to x_0$, such that

$$f(x_i) - f(x_0) \in -C. \tag{12}$$

We can assume that $(x_i - x_0)/\|x_i - x_0\|$ converges to some $u \in T_1(S, x_0) \setminus \{0\}$. By (12), for all large i there exists $M_i \in A_f(x_0)$ such that

$$M_i(x_i - x_0) + o(\|x_i - x_0\|) \in -C.$$

By repeating an argument which has been employed several times previously we receive $M \in$ p-cl$A_f(x_0) \bigcup($p-$A_f(x_0)_\infty \setminus \{0\})$ with $Mu \in -C$, a contradiction.

(ii) Suppose again (12). For all large i, $x_i - x_0 \in S_\delta(x_0)$ and there is $(M_i, N_i) \in$

$(A_f(x_0), B_f(x_0))$ satisfying (4). On other hand, (a) implies the existence of $\lambda_i \in \Lambda$ such that $\langle \lambda_i, M_i(x_i - x_0) \rangle \geq 0$. Hence, by (12) and (4)

$$\langle \lambda_i, N_i(x_i - x_0, x_i - x_0) + o(\|x_i - x_0\|^2) \rangle \leq 0. \tag{13}$$

Since Λ is compact we can assume that $\lambda_i \to \lambda_0 \in \Lambda$. Again by considering the two cases depending on whether (N_i) is bounded or not, by (13) $N \in$ p-cl$B_f(x_0)$ $\bigcup$(p-$B_f(x_0)_\infty \setminus \{0\}$) exists with $\langle \lambda_0, N(u,u) \rangle \leq 0$, contradicting (b). $\square$

Remark 4.2. (i) If $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, f is continuously differentiable, $\partial^2 f$ is upper semicontinuous at x_0 and $(A_f(x_0), B_f(x_0)) = (f^{'}(x_0), \frac{1}{2}\text{co}\partial^2 f(x_0))$, then Theorem 4.2 coincides with Theorem 4.1 of Ref. 18.

(ii) If $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, $f \in C^{1,1}$ and $(A_f(x_0), B_f(x_0)) = (f^{'}(x_0), \frac{1}{2}\partial^2_C f(x_0))$ Theorem 4.2 collapses to Theorem 3.3 of Ref. 8.

References

1. JOURANI, A. and THIBAULT, L., *Appoximations and Metric Regularity in Mathematical Programming in Banach Spaces*, Mathematics of Operations Research, Vol. 18, pp. 390 - 400, 1993.

2. ALLALI, K. and AMAHROQ, T., *Second-Order Approximations and Primal and Dual Necessary Optimality Conditions*, Optimization, Vol. 40, pp. 229 - 246, 1997.

3. PENOT, J. -P., *Second-Order Generalized Derivatives: Comparison of Two Types of Epiderivatives*, Advances in Optimization, Edited by W. Oettli and D. Pallaschke, Springer Verlag, Berlin, Germany, pp. 52 - 76, 1992.

4. HIRIART-URRUTY, J. B., STRODIOT, J. J., and NGUYEN, V. H., *Generalized Hessian Matrix and Second-Order Optimality Conditions for Problems with $C^{1,1}$ Data*, Applied Mathematics and Optimization, Vol. 11, pp. 43 - 56, 1984.

5. KLATTE, D., and TAMMER, K., *On the Second-Order Sufficient Optimality Conditions for $C^{1,1}$ Optimization Problems*, Optimization, Vol. 19, pp. 169 - 180, 1988.

6. LIU, L. P., *The Second-Order Conditions of Nondominated Solutions for $C^{1,1}$ Generalized Multiobjective Mathematical Programming*, Systems Sciences and Mathematical Sciences, Vol. 4, pp. 128 - 138, 1991.

7. GUERRAGGIO, A., and LUC, D. T., *Optimality Conditions for $C^{1,1}$ Vector Optimization Problems*, Journal of Optimization Theory and Applications, Vol. 109, pp. 615 - 629, 2001.

8. GUERRAGGIO, A., and LUC, D. T., *Optimality Conditions for $C^{1,1}$ Constrained Multiobjective Problems*, Journal of Optimization Theory and Applications, Vol. 116, pp. 117 - 129, 2003.

9. DEMYANOV, V. F., and RUBINOV, A. M., *Constructive Nonsmooth Analysis*, Peter Lang, Frankfurt am Main, Germany, 1995.

10. STUDNIARSKI, M., *Second-Order Necessary Conditions for Optimality in Nonsmooth Nonlinear Programming*, Journal of Mathematical Analysis and Applications, Vol. 154, pp. 303 - 317, 1991.

11. GINCHEV, I. and GUERRAGGIO, A., *Second-Order Optimality Conditions in Nonsmooth Unconstrained Optimization*, Pliska Studia Mathematica Bulgarica, Vol. 12, pp. 39 - 50, 1998.

12. GINCHEV, I., *Higher-Order Optimality Conditions in Nonsmooth Optimiza-*

tion, Optimization, Vol. 51, pp. 47 - 72, 2002.

13. GINCHEV, I., GUERRAGGIO, A., and ROCCA., M., *First-Order Conditions for $C^{0,1}$ Constrained Vector Optimization*, Variational Analysis and Applications, Nonconvex Optimization Application, Springer, New York, Vol. 79, pp. 427 - 450, 2005.

14. GINCHEV, I., GUERRAGGIO, A., and ROCCA., M., *From Scalar to Vector Optimization* (submitted).

15. KHANH, P. Q., LUC, D. T., and TUAN, N. D., *Local Uniqueness of Solutions of an Equilibrium Problem* (submitted).

16. JEYAKUMAR, V., and LUC, D. T., *Approximate Jacobian Matrices for Nonsmooth Continuous Maps and C^1 -Optimization*, SIAM Journal on Control and Optimization, Vol. 36, pp. 1815 - 1832, 1998.

17. JEYAKUMAR, V., and WANG, X., *Approximate Hessian Matrices and Second-Order Optimality Conditions for Nonlinear Programming Problem with C^1 Data*, Journal of Australian Mathematical Society, Vol. 40B, pp. 403 - 420, 1999.

18. GUERRAGGIO, A., LUC., D. T., and MINH., N. B., *Second-Order Optimality Conditions for C^1 Multiobjective Programming Problems*, Acta Math-

ematica Vietnamica, Vol. 26, pp. 257 - 268, 2001.

19. LUC, D. T., *Second-Order Optimality Conditions for Problems with Continuously Differentiable Data*, Optimization, Vol. 51, pp. 497 - 510, 2002.

20. LUC, D., T., *The Fréchet Approximate Jacobian and Local Uniqueness in Variational Inequalities*, Journal of Mathematical Analysis and Applications, Vol. 268, pp. 629 - 646, 2002.

21. WANG, X., and JEYAKUMAR, V., *A Sharp Lagrange Multiplier Rule for Nonsmooth Mathematical Programming Problems Involving Equality Constraints*, SIAM Journal on Optimization, Vol. 10, pp. 1136 - 1148, 2000.

22. LUC, D. T., *A Multiplier Rule in Multiobjective Programming Problems with Continuous Data*, SIAM Journal on Optimization, Vol. 13, pp. 168 - 178, 2002.

23. AMAHROQ, T., and TAA, A., *On Lagrange - Kuhn - Tucker Multipliers for Multiobjective Optimization Problems*, Optimization, Vol. 41, pp. 159 - 172, 1997.

24. KHANH, P. Q., and LUU, L. M., *Necessary Optimality Conditions in Problems Involving Set - Valued Maps with Parameters*, Acta Mathematica Vietnamica, Vol. 26, pp. 279 - 295, 2001.

25. KHANH, P. Q., and LUU, L. M., *Multifunction Optimization Problems Involving Parameters: Necessary Optimality Conditions*, Optimization, Vol. 51, pp. 577 - 595, 2002.

26. AMAHROQ, T., and GADHI, N., *Second-Order Optimality Conditions for the Extremal Problem under Inclusion Constraints*, Journal of Mathematical Analysis and Applications, Vol. 285, pp. 74 - 85, 2003.

27. MORDUKHOVICH, B. S., *Maximum Principle in the Optimal Time Control Problem with Nonsmooth Constraints*, Soviet Journal of Applied Mathematics and Mechanics, Vol. 40, pp. 960 - 968, 1976.

28. MORDUKHOVICH, B. S., *Metric Approximations and Necessary Optimality Conditions for General Classes of Nonsmooth Extremal Problems*, Soviet Mathematical Doklady, Vol. 22, pp. 526 - 530, 1980.

29. IOFFE, A. D., *Nonsmooth Analysis: Differential Calculus of Nondifferentiable Mappings*, Transactions of American Mathematical Society, Vol. 266, pp. 1 - 56, 1981.

30. CLARKE, F. H., *Optimization and Nonsmooth Analysis*, Wiley Interscience, New York, NY, 1983.