Optimality conditions under relaxed quasiconvexity assumptions using star and adjusted subdifferentials

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Abstract

A set-constrained optimization problem and a mathematical programming problem are considered. We assume that the sublevel sets of the involving functions are convex only at the point under question, so these functions are still not quasiconvex. Using the two star subdifferentials and the adjusted subdifferential, we establish optimality conditions for usual minima and strict minima. Our results contain and improve some recent ones in the literature. Examples are provided to explain the advantages of each of our results.

Key words: Optimality conditions; Convex sublevel sets; Normal cone; Star subdifferentials; Adjusted subdifferentials

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1 Introduction

Optimality conditions for nonconvex-nonsmooth problems have been intensively studied for a long time (see e.g. important books [6, 22-24, 30] and some papers of our group [11-20] among numerous works of other authors), since convexity and/or differentiability conditions are often not satisfied for optimization-related problems in practice. A large number of classes of such problems have been proposed and investigated due to demands of practical applications and also to motivations for mathematical researchers. Quasiconvex problems constitute the most important ever-considered class of nonconvex optimization problems, because the extent of their applications is vast, see e.g. [1], and their structures are convenient for employing mathematical tools, including convex analysis. The reader can refer to [3, 4, 5, 8, 21, 25-28] and references therein for recent developments in quasiconvex optimization. A function f from a normed space X to $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is called quasiconvex if $\forall x \in X$ its sublevel set at x, i.e. $L_f(x) := \{u \in X : f(u) \leq f(x)\}$, is convex or, equivalently, if for each $r \in \mathbb{R}$ the strict sublevel set $\{u \in X : f(u) < r\}$ is convex. Hence, f is quasiconvex if and only if, $\forall x \in X$, the strict sublevel set $L_f^<(x) := \{u \in X : f(u) < f(x)\}$ is convex. Another equivalent statement, which is often met in the literature, is that f is quasiconvex if for all $x, y \in \mathrm{dom} f := \{x \in X : f(x) < +\infty\}$, all $t \in [0,1], f((1-t)x + ty) \leq \max\{f(x), f(y)\}$. An optimization problem is quasiconvex if the objective is quasiconvex and the constraint set is convex. The generalized subdifferentials used in [21, 26] were the lower subdifferential, or Plastria subdifferential [29], and the infradifferential or Gutiérrez subdifferential [10]. These subdifferentials enjoy many helpful properties and hence are convenient to be applied. However, they may be empty in a number of cases (many simple cases are

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given in examples of Sects. 3 and 4). There are even differentiable quasiconvex functions with the Plastria (Gutiérrez) subdifferentials which are empty at each point. [21] even required f to be a Plastria or Gutiérrez function (see the definitions below). [25, 27] made use additionally of the Greenberg-Pierskalla subdifferential [9], a kind of normal-cone subdifferentials, i.e. those with values being normal cones. In [5, 28] the two star subdifferentials, which are similar to the Greenberg-Pierskalla subdifferential, were introduced. They are nonempty under weak conditions and simple (e.g. they are closed convex cones, and quite different from the classical Fenchel subdifferential, which is very often bounded). However, these two star subdifferentials are in general neither quasimonotone nor cone-upper semicontinuous [3, 5]. (Roughly speaking, a cone-value mapping is cone-upper semicontinuous, cone-u.s.c. in short, if the corresponding mapping with values being the bases of the values is Berge u.s.c..) Hence, they are not suitable for relating minimization problems to variational inequalities. Motivated by this the authors of [3] introduced the adjusted sublevel set $L_f^a(x)$, which is between $L_f^<(x)$ and $L_f(x)$, and the corresponding adjusted subdifferential, defined similarly as the star subdifferentials, and used them to deal with the solution existence of a quasiconvex set-constraint minimization problem. [4] applied the adjusted subdifferential together with the limiting (or Mordukhovich) subdifferential [23] to study the solution existence and optimality conditions for quasiconvex problem with a locally starshaped constraint set with applications to mathematical programming with equilibrium constraints, where the equality constraint is quasiconvex and the equality and equilibrium constraints are quasiaffine. There have been also a number of papers using other kind of generalized derivatives to deal with quasiconvex optimization problems. For instance, in a recent one [8] the Dini directional derivatives are employed. But in this paper we are concerned only with the afore-mentioned subdifferentials. We observe that the changes of the properties of quasiconvex functions, when we assume that the sublevel sets are convex only at a point $\bar{x}$ under consideration, not at each point, can be controlled when considering optimality conditions. then the assumptions become remarkably less restrictive. This observation motivates the purpose of this paper, which is examining optimality conditions for constrained optimization problems with sublevel sets of objective and constraint functions assumed to be convex only at considered points. We do not impose any differentiability assumption and use the star and adjusted subdifferentials.

The layout of this paper is as follows. Section 2 contains definitions and preliminaries needed in the sequel. Section 3 is devoted to optimality conditions for a minimization problem with a convex constraint set. In section 4 optimality conditions are established for the mathematical programming problem.

2 Preliminaries

Throughout the paper, let X be a normed space. For $A \subseteq X$ int$A$, cl$A$, co$A$ and cone$A$ denote the interior, closure, convex hull of A and the conical hull (called also the cone generated by A), i.e. cone$A := \{\lambda x : x \in A, \lambda \in \mathbb{R}_+\}$, respectively. The distance from $x \in X$ to A is $\mathrm{dist}(x, A) = \inf\{\|x-y\| : y \in A\}$. X^* is the topological dual of X and $\langle ., . \rangle$ is the duality pairing. The normal cone at x to A, denoted by $N(A, x)$, is defined by

$$N(A, x) := \{x^* \in X^* : \forall u \in A, \langle x^*, u - x\rangle \leq 0\}.$$

If $x \notin A$ we adopt that $N(A, x) = \emptyset$. The contingent cone of A at $x \in X$, denoted by $T(A, x)$, is the following cone

$$T(A, x) := \{v \in X : \exists (r_n) \to 0_+, \exists (v_n) \to v, \forall n, x + r_n v_n \in A\}.$$

To see relationships between $N(A,x)$ and $T(A,x)$, recall that the polar cones of cones $B \subseteq X$ and $D \subseteq X^*$ are

$$B^- := \{x^* \in X^* : \forall x \in B, \langle x^*, x\rangle \le 0\},$$

$$D^- := \{x \in X : \forall x^* \in D, \langle x^*, x\rangle \le 0\}.$$

Clearly $N(A,x) = [\operatorname{clcone}(A-x)]^-$. Setting, in the definition of $T(A,x)$, $x_n = x + r_n v_n$, we see that

$$T(A,x) = \{v : \exists (r_n) \to 0, \exists (x_n) \subseteq A \to x,\ \lim \frac{x_n - x}{r_n} = v\} \subseteq \operatorname{clcone}(A-x).$$

Hence, $T(A,x)^- \supseteq N(A,x)$. Furthermore, if $v \in T(A,x)$, i.e. v is of the form $\lim \frac{x_n - x}{r_n}$, and $x^* \in N(A,x)$, then $\langle x^*, v\rangle \le 0$. Therefore, $T(A,x) \subseteq N(A,x)^-$. Moreover, if A is convex then the above containments become equalities. $A \subseteq X$ is called strictly convex at $\bar{x}$ if $\langle x^*, x - \bar{x}\rangle < 0$ for every $x \in A \setminus \{\bar{x}\}$ and $x^* \in N(A,\bar{x}) \setminus \{0\}$. If $N(A,\bar{x}) \setminus \{0\} \neq \emptyset$, this strict convexity implies that $\bar{x}$ is an extreme point of A. The converse is not true. For instance, the set $\{(x,y) \in \mathbb{R}^2 : y \ge e^{|x|}\}$ is strictly convex at the point (0,1), but $\mathbb{R}^2_+$ is not strictly convex at the origin, although this point is an extreme point.

Let $f : X \to \bar{\mathbb{R}}$ be an arbitrary function, which is finite at $\bar{x}$. We recall the definitions of the important subdifferentials, encountered in the Instruction. The lower subdifferential or Plastria subdifferential [29] is defined by

$$\partial^< f(\bar{x}) := \left\{x^* \in X^* : \forall x \in L_f^<(\bar{x}), f(x) - f(\bar{x}) \ge \langle x^*, x - \bar{x}\rangle\right\}.$$

The infradifferential or Gutiérrez subdifferential [10] is

$$\partial^{\le} f(\bar{x}) := \{x^* \in X^* : \forall x \in L_f(\bar{x}), f(x) - f(\bar{x}) \ge \langle x^*, x - \bar{x}\rangle\}.$$

The Greenberg-Pierskalla subdifferential [9], which is akin to the normal cone, is defined by

$$\partial^* f(\bar{x}) := \left\{x^* \in X^* : \forall x \in L_f^<(\bar{x}), \langle x^*, x - \bar{x}\rangle < 0\right\}.$$

So we say that it is a kind of normal-cone subdifferentials. The star subdifferentials [5, 28], are the following normal-cone subdifferentials

$$\partial^{\nu} f(\bar{x}) := N(L_f(\bar{x}), \bar{x}),$$

$$\partial^{\circledast} f(\bar{x}) := N(L_f^{<}(\bar{x}), \bar{x}).$$

The adjusted sublevel set of f at $\bar{x}$ [3] is

$$L_f^a(\bar{x}) = L_f(\bar{x}) \cap \mathrm{cl}B(L_f^{<}(\bar{x}), \rho_{\bar{x}})$$

if x is not a global minimizer of f and $L_f^a(\bar{x}) = L_f(\bar{x})$ otherwise, where $B(A, \rho) = \{x \in X : \mathrm{dist}(x, A) < \rho\}$ and $\rho_x = \mathrm{dist}(x, L_f^{<}(x))$. The adjusted subdifferential [3] is

$$\partial^a f(\bar{x}) := N(L_f^a(\bar{x}), \bar{x}).$$

It is obvious that

$$\partial^{<} f(\bar{x}) \subseteq \partial^{*} f(\bar{x}) \subseteq \partial^{\circledast} f(\bar{x}), \tag{1}$$

$$\partial^{\leq} f(\bar{x}) \subseteq \partial^{\nu} f(\bar{x}) \subseteq \partial^a f(\bar{x}) \subseteq \partial^{\circledast} f(\bar{x}). \tag{2}$$

For details about the calculus of these subdifferentials the reader is referred to [3, 28]. Although they are defined for arbitrary functions (finite at $\bar{x}$), they possess good properties only under additional conditions. In the literature the sublevel sets are usually assumed to be convex. In this paper we relax remarkably this assumption to the convexity only at $\bar{x}$.

By (1) and (2) it is clear that $\mathbb{R}_+\partial^{<} f(\bar{x}) \subseteq \partial^{\circledast} f(\bar{x})$ and $\mathbb{R}_+\partial^{\leq} f(\bar{x}) \subseteq \partial^{\nu} f(\bar{x})$. Therefore, the following definitions are natural. A function f is said to be a Plastria function at $\bar{x}$ if its strict sublevel set $L_f^{<}(\bar{x})$ is convex and

$$\mathbb{R}_+\partial^{<} f(\bar{x}) = \partial^{\circledast} f(\bar{x})$$

and to be a Gutiérrez function at $\bar{x}$ if $L_f(\bar{x})$ is convex and

$$\mathbb{R}_+\partial^{\leq} f(\bar{x}) = \partial^{\nu} f(\bar{x}).$$

Since $\partial^< f(\bar{x})$ (and also $\partial^\leq f(\bar{x})$) are shady, i.e. $\forall \gamma \geq 1, \gamma \partial^< f(\bar{x}) \subseteq \partial^< f(\bar{x})$, the relations in the above definition can be written as $[0, r)\partial^< f(\bar{x}) = \partial^\circledast f(\bar{x})$ and $[0, r)\partial^\leq f(\bar{x}) = \partial^\nu f(\bar{x})$, respectively, for any positive scalar r.

Now we develop several simple properties of the star subdifferentials. The first one deals with transformable functions.

Proposition 2.1 *Let $f := h \circ g$, where $g : X \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function with $h(+\infty) = +\infty$ and $\bar{x} \in \mathrm{dom} g$. Then*

(i) *h is a Plastria function at $r \in \mathrm{dom} h$ with $\mathbb{R}_+ \partial^< h(r) = \partial^\circledast h(r) = \mathbb{R}_+$ provided $\partial^< h(r) \neq \emptyset$;*

(ii) *h is a Gutiérrez function at $r \in \mathrm{dom} h$ with $\mathbb{R}_+ \partial^\leq h(r) = \partial^\nu h(r) = \mathbb{R}_+$ provided $\partial^\leq h(r) \neq \emptyset$;*

(iii) *$\partial^\nu f(x) = \partial^\nu g(x)$ and $\partial^\circledast f(x) = \partial^\circledast g(x)$ for all $x \in X$;*

(iv) *f is a Plastria function at $\bar{x}$ provided that g is a Plastria function at $\bar{x}$ and $\partial^< h(g(\bar{x})) \neq \emptyset$;*

(v) *f is a Gutiérrez function at $\bar{x}$ provided that g is a Gutiérrez function at $\bar{x}$ and $\partial^\leq h(g(\bar{x})) \neq \emptyset$.*

Proof. (i) and (ii) are obvious.

(iii) Since $L_h^<(r) = (-\infty, r)$, we get $\partial^\circledast h(r) = \mathbb{R}_+$. By definition, $x^* \in \partial^< h(r)$ if and only if $h(r) - h(x) \leq x^*(r - x)$, $\forall x \in (-\infty, r)$. Hence, $x^* > 0$ and we are done.

(iv) Setting $r := g(\bar{x})$, using (iii) and Proposition 3.5 of [28] we get

$$\partial^\circledast f(\bar{x}) = \partial^\circledast g(\bar{x}) = \mathbb{R}_+ \partial^< g(\bar{x}) = \mathbb{R}_+ \partial^< h(r) \partial^< g(\bar{x}) \subset \mathbb{R}_+ \partial^< f(\bar{x}).$$

(v) Similarly as in (iv) we have

$$\partial^\nu f(\bar{x}) = \partial^\nu g(\bar{x}) = \mathbb{R}_+ \partial^\leq g(\bar{x}) = \mathbb{R}_+ \partial^\leq h(r) \partial^\leq g(\bar{x}) \subset \mathbb{R}_+ \partial^\leq f(\bar{x}).$$

□

The following result characterizes the star subdifferentials of Gateaux differentiable functions.

Proposition 2.2 *Let $f : X \to \mathbb{R}_\infty$ be Gateaux differentiable at $\bar{x}$ with a nonzero derivative. Then*

(i) $\partial^{\circledast} f(\bar{x}) = \mathbb{R}_+ f'(\bar{x})$ *provided that* $L_f^<(\bar{x})$ *is convex;*

(ii) $\partial^{\nu} f(\bar{x}) = \mathbb{R}_+ f'(\bar{x})$ *provided that* $L_f(\bar{x})$ *is convex.*

Proof

(i) $L_f^<(\bar{x})$ is nonempty since $f'(\bar{x}) \neq 0$. By the definition of the directional derivative, we have $f'(\bar{x})^{-1}((-\infty, 0)) \subseteq T(L_f^<(\bar{x}), \bar{x}) \subseteq f'(\bar{x})^{-1}(-\infty, 0]$. Since $f'(\bar{x}) \neq 0$, we can find some $w \in X$ with $f'(\bar{x})w < 0$. Then, for any $v \in f'(\bar{x})^{-1}(-\infty, 0]$ and any sequence $(r_n) \to 0_+$, we have $v_n := v + r_n w \in f'(\bar{x})^{-1}(-\infty, 0) \subseteq T(L_f^<(\bar{x}), \bar{x})$. Since $(v_n) \to v$ and $T(L_f^<(\bar{x}), \bar{x})$ is closed, $v \in T(L_f^<(\bar{x}), \bar{x})$. Thus, $T(L_f^<(\bar{x}), \bar{x}) = f'(\bar{x})^{-1}(-\infty, 0]$. Then, $\partial^{\circledast} f(\bar{x}) = [f'(\bar{x})^{-1}(-\infty, 0]]^-$. This means that $x^* \in \partial^{\circledast} f(\bar{x})$ if and only if $\langle x^*, v\rangle \leq 0$ and $\langle f'(\bar{x}), v\rangle \leq 0$. By the Farkas lemma the latter is equivalent to $x^* = rf'(\bar{x})$ and $r \geq 0$. Thus, we are done.

(ii) It is proved similarly as for (i).

□

3 Optimality conditions for set-constrained problems

Consider the minimization problem

$$\text{minimize } f(x) \text{ subject to } x \in C, \tag{3}$$

where $f : X \to \overline{\mathbb{R}}$ and C is a convex subset of X.

Apart from usual solutions we consider also strict solutions to (3), i.e., points $\bar{\mathrm{x}} \in C$ such that $f(\bar{\mathrm{x}}) < f(x)$ for all $x \in C \setminus \{\bar{\mathrm{x}}\}$.

Theorem 3.1 *Let $L_f^{<}(\bar{x}) \cup \{\bar{x}\}$ be convex and $\bar{x} \in C$ be a solution to* (3)*, which is not a local minimizer of f on X. Assume either of the following conditions holds*

(i) $\mathrm{int} L_f^{<}(\bar{x}) \neq \emptyset$ *or f is u.s.c. at a point of $L_f^{<}(\bar{x})$;*

(ii) $\mathrm{int} C \neq \emptyset$;

(iii) *X is finite dimensional.*

Then

$$\partial^{\circledast} f(\bar{x}) \cap (-N(C, \bar{x})) \neq \{0\}. \tag{4}$$

If furthermore f is lower semicontinuous (l.s.c.) at $\bar{x}$, then

$$\partial^{a} f(\bar{x}) \cap (-N(C, \bar{x})) \neq \{0\}. \tag{5}$$

Proof. It is clear that $\bar{x}$ is a solution to (3) if and only if the sublevel set $L_f^{<}(\bar{x})$ is disjoint from C. For (i) observe first that, if f is u.s.c. at some point of $L_f^{<}(\bar{x})$, then $\mathrm{int} L_f^{<}(\bar{x}) \neq \emptyset$. So assume that $\mathrm{int} L_f^{<}(\bar{x}) \neq \emptyset$. By the Hahn-Banach separation theorem, this implies the existence of $c \in \mathbb{R}$ and $u^* \neq 0$ in X^* such that the following inequalities hold for all $w \in \mathrm{int} L_f^{<}(\bar{x})$ and $x \in C$

$$\langle u^*, x - \bar{x} \rangle \geq c \geq \langle u^*, w - \bar{x} \rangle.$$

Since $\mathrm{int} L_f^{<}(\bar{x}) \neq \emptyset$, we get $L_f^{<}(\bar{x}) \subseteq \mathrm{clint} L_f^{<}(\bar{x})$. So, the above inequalities hold for all $w \in L_f^{<}(\bar{x})$ and $x \in C$. Taking $x = \bar{x}$, we see that $c \leq 0$. Moreover, since $\bar{x}$ is not a local minimizer of f, there exists a sequence $(w_n) \to \bar{x}$ such that $w_n \in L_f^{<}(\bar{x})$ for each n. Therefore $c = 0$. Hence the left inequality means $u^* \in -N(C, \bar{x})$, and the right one means $u^* \in \partial^{\circledast} f(\bar{x})$ and we get the result.

If (ii) or (iii) holds, we also get the mentioned inequalities for all $w \in L_f^<(\bar{x})$ and $x \in C$. Therefore, the proof is completed by the same arguments.

Now assume the lower semicontinuity of f at $\bar{x}$. As $L_f^<(\bar{x}) \cup \{\bar{x}\}$ is convex, $\rho_{\bar{x}} = 0$. Consequently, $L_f^a(\bar{x}) = L_f(\bar{x}) \cap \mathrm{cl} L_f^<(\bar{x})$. Hence, $L_f^a(\bar{x}) \subseteq \mathrm{cl} L_f^<(\bar{x})$. To see the reverse inclusion, let (x_n) is in $L_f^<(\bar{x})$ and $x = \lim x_n$. By the lower semicontinuity $f(x) \geq f(\bar{x})$. As $x \in L_f(\bar{x})$ one has $f(x) = f(\bar{x})$. Therefore, $x \in L_f^a(\bar{x})$. Now looking at the inequalities by the beginning of the proof of the proposition, one sees that they hold for all $w \in L_f^a(\bar{x})$. Thus (5) is true. □

Corollary 3.2 *Let $\bar{x}$ be a solution of* (3) *but not a local minimizer of f on X.*

(i) *(Proposition 2.2 (i) of [25]) If X is finite dimensional and f is quasiconvex, then $\partial^{\circledast} f(\bar{x}) \cap (-N(C, \bar{x})) \neq \{0\}$.*

(ii) *(Proposition 2.2 (ii) of [25]) If f is quasiconvex and upper semicontinuous (u.s.c.) at each point of $L_f^<(\bar{x})$, then $\partial^* f(\bar{x}) \cap (-N(C, \bar{x})) \neq \{0\}$.*

(iii) *(Proposition 4 of [21]) If f is u.s.c. at each point of $L_f^<(\bar{x})$ and a Plastria function at $\bar{x}$, then $\partial^< f(\bar{x}) \cap (-N(C, \bar{x})) \neq \{0\}$.*

Proof. (i) is clear. To see (ii) note that in this case we have $\partial^{\circledast} f(\bar{x}) = \partial^* f(\bar{x}) \cup \{0\}$ and that the assumed upper semicontinuity implies the openness of $L_f^<(\bar{x})$. For (iii) observe that if f is u.s.c. at each point of $L_f^<(\bar{x})$ and a Plastria function at $\bar{x}$, then $L_f^<(\bar{x})$ is nonempty, open, and $\partial^{\circledast} f(\bar{x}) = \mathbb{R}_+ \partial^< f(\bar{x})$. □

In the following example, the above-mentioned results of [21, 25] and the ones of [26, 27] cannot be applied, but Theorem 3.1 can.

Example 3.3 *Let* $C = [0, 1]$, $\bar{x} = 0$ *and* $f : \mathbb{R} \to \mathbb{R}$ *be given by*

$$f(x) = \begin{cases} -2, & \text{for } x \leq -1, \\ -1, & \text{for } -1 < x \leq -\frac{2}{3}, -\frac{1}{3} \leq x < 0, \\ -\frac{1}{2}, & \text{for } -\frac{2}{3} < x < -\frac{1}{3}, \\ x, & \text{for } 0 \leq x. \end{cases}$$

Then $L_f^<(\bar{x}) = (-\infty, 0)$, f *is not u.s.c. at* $-1 \in L_f^<(\bar{x})$, *not quasiconvex and* $\partial^< f(\bar{x}) = \emptyset$. *So the mentioned results of [21, 25] cannot be applied. But the assumptions of Theorem* 3.1 *are satisfied. Easy direct calculations give* $\partial^{\circledast} f(\bar{x}) = \mathbb{R}_+$ *and* $N(C, \bar{x}) = -\mathbb{R}_+$. *So the conclusion of our theorem holds. Furthermore,* $\partial^{\leq} f(\bar{x}) = \emptyset$. *Hence, the results of [26, 27], using* $\partial^< f$ *and* $\partial^{\leq} f$, *do not work either.*

For sufficient conditions we have

Theorem 3.4 *Assume that either of relations* (4) *and* (5) *is satisfied at* $\bar{x} \in C$. *Then* $\bar{x}$ *is a solution of* (3) *if either of the following conditions holds*

(i) $L_f^<(\bar{x})$ *is open;*

(ii) $C \setminus \{\bar{x}\}$ *is open.*

Proof. We have to deal only with the weaker relation (4), which means the existence of $u^* \in X^* \setminus \{0\}$ such that, $\forall w \in L_f^<(\bar{x}), \forall x \in C$,

$$\langle u^*, x - \bar{x} \rangle \geq 0 \geq \langle u^*, w - \bar{x} \rangle.$$

Suppose $L_f^<(\bar{x}) \cap C \neq \emptyset$. The preceding inequalities imply that $\langle u^*, v - \bar{x} \rangle = 0$ for any $v \in L_f^<(\bar{x}) \cap C$.

(i) Let $h \in X$ be arbitrary. Since $L_f^<(\bar{x})$ is open, there exists $t > 0$ small enough such that $v + th \in L_f^<(\bar{x})$. Hence

$$t\langle u^*, h\rangle = \langle u^*, v - \bar{x} + th\rangle - \langle u^*, v - \bar{x}\rangle \leq 0.$$

Consequently, $u^* = 0$, a contradiction. Thus $L_f^<(\bar{x}) \cap C = \emptyset$, i.e. $\bar{x}$ is a solution of (3).

(ii) Let $h \in X$ be arbitrary. There exists $t > 0$ small enough such that $v + th \in C \setminus \{\bar{x}\}$ as C is open. Therefore,

$$t\langle u^*, h\rangle = \langle u^*, v - \bar{x} + th\rangle - \langle u^*, v - \bar{x}\rangle \geq 0,$$

which implies that $u^* = 0$, again a contradiction. $\square$

The following example illustrates advantages of Theorem 3.4.

Example 3.5 *Let $C = \{(x_1, x_2) : x_2 < 0\} \cup \{(0,0)\}$, $\bar{x} = (0,0)$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be given by*

$$f(x_1, x_2) = \begin{cases} -1, & \text{if } (x_1, x_2) \neq (0,0) \text{ and } x_2 \geq 0, \\ 0, & \text{if } (x_1, x_2) = (0,0) \text{ or } x_2 < 0. \end{cases}$$

Then $C \setminus \{\bar{x}\}$ is open. Since $\partial^* f(\bar{x}) = \partial^< f(\bar{x}) = \emptyset$, Propositions 5 of [21], Proposition 2.1 of [25] and the results of [26, 27] using these two subdifferentials cannot be in use. But the assumptions of Theorem 3.4 (ii) are fulfilled, since $(0, -1) \in \partial^{\circledast} f(\bar{x}) \cap -N(C, x)$. It is easy to see directly that $\bar{x}$ is a minimizer of f on C.

Now we prove a necessary condition for strict solutions to (3).

Theorem 3.6 *Let $\bar{x}$ be a strict solution to (3) and an extreme point of C. Assume that $L_f(\bar{x})$ is convex, C is not reduced to $\{\bar{x}\}$ and either of the conditions* *(i)-(iii) of Theorem 3.1 holds. Then*

$$\partial^{\nu} f(\bar{x}) \cap (-N(C,\bar{x})) \neq \{0\}\,. \tag{6}$$

Proof. Since $\bar{x}$ is an extreme point of C and $C \neq \{\bar{x}\}$, the set $C \setminus \{\bar{x}\}$ is convex and nonempty. As $\bar{x}$ is a strict solution to (3), $C \setminus \{\bar{x}\}$ and $L_f(\bar{x})$ are disjoint. For (i), by the Hahn-Banach separation theorem, there exists some $c \in \mathbb{R}$ and $0 \neq u^* \in X^*$ such that the following inequalities hold, for all $w \in L_f(\bar{x})$ and $x \in C \setminus \{\bar{x}\}$,

$$\langle u^*, x - \bar{x}\rangle \geq c \geq \langle u^*, w - \bar{x}\rangle.$$

Since x can be arbitrarily close to $\bar{x}$, we have $c \leq 0$. On the other hand, since we can take $w = \bar{x}$, $c \geq 0$ and hence $c = 0$. Therefore, the left inequality means $u^* \in -N(C,\bar{x})$ and the right one means $u^* \in \partial^{\nu} f(\bar{x})$, and the result follows.

For (ii) and (iii), we can also apply the separation theorem to get the above inequalities for all $w \in L_f(\bar{x})$ and $x \in C \setminus \{\bar{x}\}$. Similar arguments complete the proof. □

Corollary 3.7 *Let f be a Gutiérrez function at a strict solution $\bar{x}$ to* (3)*. Assume that C is not reduced to $\{\bar{x}\}$, $\bar{x}$ is an extreme point of C, and either of the conditions (i)-(iii) of Theorem 3.1 holds. Then*

$$0 \in \partial^{\leq} f(\bar{x}) + N(C,\bar{x}).$$

Proof. By Theorem 3.6, there exists $u^* \neq 0$ such that $-u^* \in N(C,\bar{x})$ and $u^* \in \partial^{\nu} f(\bar{x}) = \mathbb{R}_+ \partial^{\leq} f(\bar{x})$, since f is a Gutiérrez function at $\bar{x}$. So, one can find $s > 0$ and $x^* \in \partial^{\leq} f(\bar{x})$ such that $u^* = s x^*$ and $-x^* \in N(C,\bar{x})$. □

Corollary 3.7 (iii) is the necessary condition of Proposition 6 in [21]. However, being a Gutiérrez function may be a severe restriction as shown by the following example.

Example 3.8 *Let* $C = [0, 1]$*,* $\bar{x} = 0$ *and* $f : \mathbb{R} \to \mathbb{R}$ *be given by*

$$f(x) = \begin{cases} -1, & \text{for } x < 0, \\ x, & \text{for } x \geq 0. \end{cases}$$

Then $L_f(\bar{x}) = (-\infty, 0]$ *and* $\partial^{\leq} f(\bar{x}) = \emptyset$*. So Corollary* 3.7 *cannot be applied. But* $\partial^{\nu} f(\bar{x}) = \mathbb{R}_+$ *and* $N(C, \bar{x}) = -\mathbb{R}_+$ *and hence the conclusion of Theorem* 3.6 *is true. The assumptions of Theorem* 3.6 *are also easily seen to be verified. Furthermore,* $\partial^{<} f(\bar{x})$ *is also empty and then the results involving these two subdifferentials of [21, 25, 26, 27] are out of use.*

Passing to sufficient conditions for strictly solutions we have

Theorem 3.9 $\bar{x}$ *is a strict solution of* (3) *if* (6) *is satisfied and either of the following conditions holds*

(i) *either* C *is strictly convex at* $\bar{x}$ *or* $C \setminus \{\bar{x}\}$ *is open;*

(ii) $L_f(\bar{x}) \setminus \{\bar{x}\}$ *is open.*

Proof. Suppose, ab absurdo, $(L_f(\bar{x}) \cap C) \setminus \{\bar{x}\} \neq \emptyset$. The relation (6) implies that there exists $0 \neq u^* \in X^*$ such that

$$\langle u^*, x - \bar{x} \rangle \geq 0 \geq \langle u^*, w - \bar{x} \rangle, \qquad \forall w \in L_f(\bar{x}),\ x \in C,$$

then $\langle u^*, v - \bar{x} \rangle = 0$ for any $v \in (L_f(\bar{x}) \cap C) \setminus \{\bar{x}\}$. For (i), observe first that $C \setminus \{\bar{x}\}$ is open implies that C is strictly convex at $\bar{x}$. Indeed, if $C \setminus \{\bar{x}\}$ is open, it is equal to int$C$. Hence $\langle x^*, x - \bar{x} \rangle < 0$ for every $x \in C \setminus \{\bar{x}\}$ and $x^* \in N(C, \bar{x}) \setminus \{0\}$, i.e. C is strictly convex at $\bar{x}$. Now we have to consider only the case where C is strictly convex at $\bar{x}$. Then, by the strict convexity, $\langle -u^*, x - \bar{x} \rangle < 0$ for all $x \in C \setminus \{\bar{x}\}$, a contradiction. For (ii), let $h \in X$ be arbitrary. As $L_f(\bar{x}) \setminus \{\bar{x}\}$ is open, there exists $t > 0$ small enough such that $v + th \in L_f(\bar{x})$. Then

$$t\langle u^*, h\rangle = \langle u^*, v - \bar{x} + th\rangle - \langle u^*, v - \bar{x}\rangle \leq 0.$$

Hence $u^* = 0$, again a contradiction. $\square$

Note that, like Theorem 3.4, in this sufficient condition no convexity condition is imposed on f. Note further that Theorem 3.9 (i) sharpens the sufficient condition of Proposition 6 in [21], where the Gutiérrez subdifferential is employed. However, Gutiérrez subdifferentials may be empty in many cases as shown in Example 3.8, where $\partial^\nu f(\bar{x}) \cap N(C, x) = \mathbb{R}_+$, and so Theorem 3.9 is directly verified (and can be applied).

The condition that $L_f(\bar{x}) \setminus \{\bar{x}\}$ is open looks restrictive. So we illustrate it in the following example.

Example 3.10 *Let $C = 0 \times (-\infty, 0]$, $(\bar{x}_1, \bar{x}_2) = (0, 0)$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be given by*

$$f(x_1, x_2) = \begin{cases} -1, & \text{if } x_1 \neq 0 \text{ and } x_2 > 0, \\ x_2^2, & \text{if } x_2 < 0 \text{ or } x_1 = 0, \\ x_1^2, & \text{if } x_2 = 0. \end{cases}$$

Then $L_f(\bar{x}) = \{(x_1, x_2) : x_2 > 0, x_1 \neq 0\} \cup \{\bar{x}\}$. C is not strictly convex at $\bar{x}$ since $\langle (1, 0), x\rangle = 0$ for all $x \in C$. So Proposition 6 of [21] does not work. But $L_f(\bar{x}) \setminus \{\bar{x}\}$ is open. $\partial^\nu f(\bar{x}) = 0 \times (-\infty, 0]$ and $(0, -1) \in \partial^\nu f(\bar{x}) \cap -N(C, \bar{x})$ and hence Theorem 3.9 concludes that $\bar{x}$ is a strict solution.

A natural question is whether we can replace (6) by the following weaker relation

$$\partial^a f(\bar{x}) \cap (-N(C, \bar{x})) \neq \{0\},$$

in Theorem 3.9. The following example yields a negative answer.

Example 3.11 *Let* $X = \mathbb{R}, C = R^{+}, \bar{x} = 0$ *and*

$$f(x) = \begin{cases} 0, & \text{if } x \geq 0, \\ \\ -1, & \text{if } x < 0. \end{cases}$$

Then, $\partial^{a} f(\bar{x}) = [0, +\infty)$ *and* $-N(C, \bar{x}) = [0, +\infty)$ *and the mentioned modified conditions are fulfilled but* $\bar{x}$ *is not a strict solution.*

4 Optimality conditions for the mathematical programming problem

Let us consider now the case in which the constraint set C is defined by a finite family of inequalities, so that problem (3) turns into the mathematical programming problem

$$\text{minimize } f(x) \quad \text{subject to } \; g_1(x) \leq 0, ..., g_n(x) \leq 0. \tag{7}$$

We denote $g = \max_{1 \leq i \leq n} g_i$, $C = g^{-1}(-\infty, 0]$, $I = \{i : g_i(\bar{x}) = 0\}$ and $h = \max_{i \in I} g_i$.

Theorem 4.1 *Assume for problem* (7) *that*

(i) $L_f^{<}(\bar{x}) \cup \{\bar{x}\}$ *and* $g_i^{-1}(-\infty, 0]$ *are convex for i=1,...,n;*

(ii) g_i *are u.s.c. at* $\bar{x}$ *for* $i = 1, ..., n$*;*

(iii) *either of the following regularity conditions holds*

(a) there exists $k \in I$ *such that* $L_{g_k}(\bar{x}) \cap \{x \in X : x \in L_{g_i}^{<}(\bar{x}), \forall i \in I \setminus \{k\}\} \neq \emptyset$ *(Slater condition);*

(b) X *is complete,* $L_{g_i}(\bar{x})$ *is closed for each* $i \in I$ *and* $\mathbb{R}_+(\Delta - \prod_{i \in I} L_{g_i}(\bar{x})) = X^I$*, where* $\Delta = \{(x_i)_{i \in I} : \forall j, k \in I; x_j = x_k\}$ *is the diagonal*

of X^I, (Attouch-Brézis's regularity condition);

(iv) *either X is finite dimensional or f is u.s.c. at some point of $L_f^<(\bar{x})$.*

If $\bar{x}$ is a solution but not a local minimizer of f on X, then

$$\partial^{\circledast} f(\bar{x}) \cap \left(-\sum_{i \in I} \partial^{\nu} g_i(\bar{x}) \right) \neq \{0\} . \tag{8}$$

Hence there are $\lambda_1, ..., \lambda_n \in \mathbb{R}_+$, not all zero, such that

$$0 \in \partial^{\circledast} f(\bar{x}) + \sum_{j=0}^{n} \lambda_j \partial^{\nu} g_j(\bar{x}), \tag{9}$$

$$\lambda_j g_j(\bar{x}) = 0, j = 1, ..., n. \tag{10}$$

If additionally, f is l.s.c. at $\bar{x}$ then, in (8) *and* (9)*, $\partial^{\circledast} f(\bar{x})$ can be replaced by $\partial^a f(\bar{x})$.*

Proof. Observe that C is convex and contained in $L_h(\bar{x})$. So $N(L_h(\bar{x}), \bar{x}) \subseteq N(C, \bar{x})$.

To prove the reverse inclusion we show that $T(L_h(\bar{x}), \bar{x}) \subseteq T(C, \bar{x})$. By the assumed convexity we have $T(L_h(\bar{x}), \bar{x}) = \text{clcone}(L_h(\bar{x}) - \bar{x})$, i.e. any $v \in T(L_h(\bar{x}), \bar{\text{x}})$ is of the form lim $t_k(x_k - \bar{x})$, where $t_k > 0$ and $x_k \in L_h(\bar{x})$. On the other hand, let $x \in L_h(\bar{x})$ be arbitrary. If $i \in I$ and $x_t := \bar{x} + t(x - \bar{x})$, then $g_i(x_t) \leq 0$ for $t \in [0, 1]$ by the convexity. For $i \notin I$, $\bar{x} \in \text{int } g_i^{-1}(-\infty, 0]$ by the assumed upper semicontinuity. So, for $t > 0$ small enough, $g_i(x_t) \leq 0$. Hence $x_t \in C$. Therefore, $t(x - \bar{x}) \in \text{cone}(C - \bar{x})$ for any $x \in L_h(\bar{x})$ and any $t > 0$. It follows that the above-mentioned lim $t_k(x_k - \bar{x})$ belongs to clcone $(C - \bar{x}) = T(C, \bar{x})$. Thus, $T(L_h(\bar{x}), \bar{x}) \subseteq T(C, \bar{x})$, and then $N(C, \bar{x}) \subseteq N(L_h(\bar{x}), \bar{x})$. Thus we have equality.

Now we have $L_h(\bar{x}) = \cap_{i \in I} L_{g_i}(\bar{x})$, $N(C, \bar{x}) = N(L_h(\bar{x}), \bar{x}) = \text{clco} \cup_{i \in I} N(L_{g_i}(\bar{x}), \bar{x})$ and $L_{g_k}(\bar{x}) \cap (\cap_{i \in I \setminus \{k\}} \text{int} L_{g_i}(\bar{x})) \neq \emptyset$ (by the assumed upper semicontinuity). Then, in case (a) of the regularity condition, by the Moreau-Rockafellar theorem, and in case (b) by [2], we have

$$N(C,\bar{x}) = \sum_{i\in I} N(L_{g_i}(\bar{x}),\bar{x}).$$

By (iv) we can apply Theorem 3.1 to get (8).

Taking $\lambda_i \in \mathbb{R}\setminus\{0\}$ arbitrarily for $i \in I$ and $\lambda_i = 0$ for $i \notin I$ we obtain (9) and (10).

If f is l.s.c. at $\bar{x}$, applying the last assertion of Theorem 3.1, the counterpart of (8)-(10) in this theorem, involving $\partial^a f$, is proven. □

For the special case, where f is a Plastria function at $\bar{x}$ and g_i are Gutiérrez functions at $\bar{x}$ for all $i \in I$ (severe conditions), Theorem 4.1 with f being u.s.c. in (iv) collapses to Theorem 10 of [21] and relation (9) takes the form

$$0 \in \partial^{<} f(\bar{x}) + \sum_{i=0}^{n} \lambda_i \partial^{\leq} g_i(\bar{x}). \tag{11}$$

If, more specifically, f and g_i, $i = 1, ..., n$ are Gateaux differentiable, then (9) and (10) become the classical Kuhn-Tucker multiplier rule. It should be noted here that (9) and (10) together are still weaker than (8), but they look more similar to the mentioned classical rule. Theorem 4.1 with $\dim X$ being finite in (iv) is new and indicated in the following example to be conveniently applied in some cases where Theorem 10 of [21] fails in use.

Example 4.2 *Let $X = \mathbb{R}$, $\bar{x} = 0$, $g_1(x) = -x$ and $f : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} x & \text{if } x \leq 0, \\ x+1 & \text{if } x > 0. \end{cases}$$

Then, $L_f^{<}(\bar{x}) = (-\infty, 0)$ and $L_g(\bar{x}) = [0, +\infty)$. We can compute directly the subdifferentials $\partial^{<} f(\bar{x}) = [1, +\infty)$, $\partial^{\circledast} f(\bar{x}) = [0, +\infty)$, $\partial^{\leq} g_1(\bar{x}) = (-\infty, -1]$ and $\partial^{\nu} g_1(\bar{x}) = (-\infty, 0]$. Consequently, f is a Plastria function at $\bar{x}$ and g_1 is *a Gutiérrez functions at $\bar{x}$. The Slater condition is satisfied at $\bar{x}$. $\bar{x}$ is not a local minimizer of f on $\mathbb{R}$ but is a solution of problem* (7)*. Therefore, we can employ Theorem 4.1. In fact, taking $\lambda_1 = 1$ we get* (11) *and $\lambda_1 g_1(\bar{x}) = 0$. Note that Theorem 10 of [21] cannot be used as f is not u.s.c.*

The following theorem gives a simple sufficient condition for problem (7).

Theorem 4.3 *Let f be u.s.c. and $\bar{x}$ be a feasible solution of problem* (7)*. Then, relation* (8) *implies that $\bar{x}$ is a solution.*

Proof. Let $D = h^{-1}((-\infty, 0])$. Then, $C \subseteq D$. Observe that $\bar{x}$ is a solution to problem (7) if and only if $L_f^<(\bar{x}) \cap C = \emptyset$. We shall prove a stronger conclusion that $L_f^<(\bar{x}) \cap D = \emptyset$.

Since f is u.s.c., $L_f^<(\bar{x})$ is open. Applying Theorem 3.4, we see that $\bar{x}$ is a solution to the following set-constrained problem

$$\text{minimize } f(x) \text{ subject to } x \in D.$$

This completes the proof. $\square$

The following example yields a case where Theorem 4.3 is more advantageous than Theorem 12 of [21].

Example 4.4 *Let $\bar{x} = 0$, $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = -1$ for $x < 0$, $f(x) = x$ for $x \geq 0$, and $g_1 : \mathbb{R} \to \mathbb{R}$ be given by $g_1(x) = -x$. Then f is u.s.c. but $\partial^< f(\bar{x}) = \emptyset$. Hence, Theorem 12 of [21] cannot be applied. However, $\partial^{\circledast} f(\bar{x}) = \mathbb{R}_+$ and $\partial^\nu g_1(\bar{x}) = -\mathbb{R}_+$. So Theorem* 4.3 *concludes that $\bar{x}$ is a solution. (This is also easily checked directly.)*

Let us turn now to strict solutions of problem (7). A necessary condition is slightly stronger than that for usual solutions given in Theorem 4.1, under almost the same assumptions.

Theorem 4.5 *For problem* (7) *let conditions (i)-(iv) of Theorem 4.1 hold. If $\bar{x}$ is a strict solution, which is an extreme point but not a single point of the feasible set, then*

$$\partial^{\nu} f(\bar{x}) \cap \left(-\sum_{i \in I} \partial^{\nu} g_i(\bar{x}) \right) \neq \{0\} \,. \tag{12}$$

Hence there are $\lambda_1, ..., \lambda_n \in \mathbb{R}_+$, not all zero, such that

$$0 \in \partial^{\nu} f(\bar{x}) + \sum_{j=o}^{n} \lambda_j \partial^{\nu} g_j(\bar{x}), \tag{13}$$

$$\lambda_j g_j(\bar{x}) = 0, j = 1, ..., n. \tag{14}$$

Proof. By applying Theorem 3.6 and the same arguments as in the proof of Theorem 4.1, we get the result. □

Remark 4.6 *The condition that $\bar{x}$ is an extreme point of C can be guaranteed by a stronger but easier-to-be-checked one that it is an extreme point of $L_{g_{i*}}(\bar{x})$, for some $i^* \in I$. This condition in turn is weaker than the* strictly quasiconvexity *of g_{i*}.*

If f and g_i are Gutiérrez functions at $\bar{x}$, for all $i \in I$, relation (13) *takes the form*

$$0 \in \partial^{\leq} f(\bar{x}) + \sum_{i \in I} y_i \partial^{\leq} g_i(\bar{x}). \tag{15}$$

Theorem 4.7 *Let f be u.s.c. and $\bar{x}$ be a feasible solution of problem* (7) *such that $f^{-1}(f(\bar{x})) = \{\bar{x}\}$. Then, the relation* (12) *implies that $\bar{x}$ is a strict solution.*

Proof. Since $f^{-1}(f(\bar{x})) = \{\bar{x}\}$, we get $L_f(\bar{x}) \backslash \{\bar{x}\} = S_f^{<}(\bar{x})$. Hence $L_f(\bar{x}) \backslash \{\bar{x}\}$ is open (by the assumed upper semicontinuity). By applying Theorem 3.9 and the same arguments as in the proof of Theorem 4.3, the proof is complete. □

Example 4.8 *Let* $\bar{x} = 0$, $g_1(x) = -x$ *and*

$$f(x) = \begin{cases} -1, & \text{for } x < 0, \\ x, & \text{for } x \geq 0. \end{cases}$$

We have $\partial^\nu f(\bar{x}) = \mathbb{R}_+$ *and* $\partial^\nu g_1(\bar{x}) = -\mathbb{R}_+$ *and hence the assumptions of Theorem* 4.7 *are fulfilled. Hence,* $\bar{x}$ *is a strict solution (as is easily verified directly).*

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