THE SOLUTION EXISTENCE OF GENERAL INCLUSIONS USING GENERALIZED KKM THEOREMS WITH APPLICATIONS TO MINIMAX PROBLEMS

Phan Quoc Khanh[1] and Nguyen Hong Quan[2]

Abstract. Applying the generalized KKM-type theorems established in our previous paper [22], we prove the solution existence of a general variational inclusion problems, which contains most of the existing results of this type. Then we obtain as applications minimax theorems in various settings and saddle-point theorems in particular. Examples are given to explain advantages of our results.

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1. Introduction

In the last decade, two features can be recognized in the increasingly intensive study of optimization-related problems, in particular of the solution existence of such problems. Firstly, the problem settings have been being getting more and more general and tending to unified studies for large classes of problems which may occur in practice. We explain the second feature according to the purpose of this paper, in terms of the solution existence issue. The assumptions imposed for getting sufficient conditions for existence are becoming more and more relaxed. One of the most powerful and frequently used tools is the celebrated KKM theorem. Hence it has been much generalized by many authors.

[1]Department of Mathematics, International University of Hochiminh City, Linh Trung, Thu Duc, Hochiminh City, Vietnam, E-mail pqkhanh@hcmiu.edu.vn

[2]Department of Mathematics, Information Technology College of Hochiminh City, Hoa Thanh, Tan Phu, Hochiminh City, Vietnam, E-mail nguyenhongquan1978@gmail.com

The aim of this paper is to apply very recent generalized KKM type theorems to establish sufficient conditions for the solution existence of a general variational inclusion problem, which encompasses most of problems of this type in the literature. Let Y, Z, Ω and D be nonempty sets. Let $S_1 : Z \to 2^Z$, $S_2 : Z \to 2^Y$, $T : Z \times Y \to 2^\Omega$, $f : \Omega \times Z \times Y \to 2^D$ and $g : \Omega \times Z \to 2^D$ be multifunctions. One of the often-considered general variational inclusion problems is of

$$\text{Finding } \bar{z} \in S_1(\bar{z}) \text{ such that,} \forall y \in S_2(\bar{z}), \forall t \in T(\bar{z}, y),$$

$$f(t, \bar{z}, y) \subseteq g(t, \bar{z}). \tag{1}$$

However, in some practical situations, the term $\forall t \in T(\bar{z}, y)$ should be replaced by $\exists t \in T(\bar{z}, y)$. Similarly, (1) may be replaced by the request of the negation relation $\not\subseteq$, or of "$f(t, \bar{z}, y) \cap g(t, \bar{z}) \neq \emptyset$" or its negation (with $= \emptyset$). Thus, we have eight problems, which may have quite different practical meanings but may be mathematically treated very similarly. In [1-6, 14, 16, 17, 20] we proposed a general problem setting to encompass these problems, but still not all of them. Now we modify this setting to include all eight problems as follows. For any sets U and V we use the notations

$$r_1(U, V) \text{ means } U \subseteq V;\ r_2(U, V) \text{ means } U \cap V \neq \emptyset;$$

$$r_3(U, V) \text{ means } U \not\subseteq V;\ r_4(U, V) \text{ means } U \cap V = \emptyset;$$

$$\alpha_1(x, U) \text{ means } \forall x \in U;\ \alpha_2(x, U) \text{ means } \exists x \in U,$$

and the conventions that $r_5 = r_1$, $\alpha_3 = \alpha_1$ and, for $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2\}$, $\overline{r}_i = r_{i+2}$, $\overline{\alpha}_j = \alpha_{j+1}$. Now with the data of problem (1) and each $r \in \{r_1, r_2, r_3, r_4\}$, $\alpha \in \{\alpha_1, \alpha_2\}$ we consider the inclusion problem

$$(\mathrm{IP}_{r\alpha}) \qquad \text{Find } \bar{z} \in S_1(\bar{z}) \text{ such that,} \forall y \in S_2(\bar{z}), \alpha(t, T(\bar{z}, y)),$$

$$r(f(t, \bar{z}, y), g(t, \bar{z})).$$

This setting includes the encountered eight problems. Although it looks seemingly a purely mathematical formalism and not explicit, but it helps to shorten remarkably the presentation of results. Note further that several authors consider also quasivariational inclusion problems (or corresponding special cases of quasi-equilibirum problems and quasivariational inequalities), where a common point

$\bar{t} \in T(\bar{z})$ is required for all $y \in S_2(\bar{z})$ as follows. Let $Z, \Omega, D, S_1, S_2, f, g$ be as above. Let $Y = Z$ and $T : Z \to 2^{\Omega}$. The problem is to find $\bar{z} \in S_1(\bar{z})$ such that $\exists \bar{t} \in T(\bar{z}), \forall y \in S_2(\bar{z}), r(f(\bar{t}, \bar{z}, y), g(\bar{t}, \bar{z}))$. To express this seemingly different problem as a special case of (IP$_{r\alpha}$) we set $Z' = \Omega \times Z$, $S'_1 : Z' \to 2^{Z'}$ defined by $S'_1(z') = T(z) \times S_1(z)$, $S'_2 : Z' \to 2^Y$ by $S'_2(t, z) = S_2(z)$, $T' : Z' \times Y \to 2^{\Omega}$ by $T'(t, z, y) = \{t\}$ (then T' can be omitted from the setting), $f' : Z' \times Y \to 2^D$ by $f'(z', y) = f(t, z, y)$ and $g' : Z' \to 2^D$ by $g'(z') = g(t, z)$. Then this problem is reduced to the particular case of (IP$_{r\alpha}$): Find $\bar{z'} \in S'_1(\bar{z'})$ such that, $\forall y \in S'_2(\bar{z'})$, $r(f'(\bar{z'}, y), g'(\bar{z'}))$.

Of course, the sets and mappings in the data of problem $(IP_{r\alpha})$ should be given mathematical structures and properties when we study it. It is known that variational inclusion problems contain as special cases a wide range of optimization-related problems, see e.g the afore-mentioned papers among numerous others. In section 3 we will show that the main result of this paper improves - or includes as special cases- recent ones in the literature. Possible applications are rather clear. Hence in section 4 we restrict ourselves to those in minimax problems.

In [21, 22] we proposed the definition of a generalized KKM mapping (see Definition 2.2 below) on a GFC-space. We defined this general space as a generalization of FC-spaces [9], G-convex spaces [31] and many other spaces with generalized convexity structures or without any such structures. But as for FC-spaces, we can study numerous phenomena related to (generalized) convexity properties in GFC-spaces. In particular, we developed generalized KKM theorems and closely related theorems about intersections, coincidence and fixed points and maximal elements in GFC-spaces in [21, 22]. These generalized KKM-type theorems proved to contain many theorems of this type in [7, 10, 11, 12, 24] among others.

In this paper we will apply these generalized KKM-type theorems of [22] to establish a sufficient condition for the solution existence of the inclusion problem (IP$_{r\alpha}$) and discuss their consequences in some particular cases. Our results improve - or include as special cases - those in [8, 11, 12, 14, 17-19, 23, 25, 26, 28, 29, 33]. From our results on the existence of its solutions we can derive consequences for various problems of nonlinear analysis and optimization such as fixed or coincidence-point problems, quasivariational inclusion problems, quasiequilibrium problems, quasivariational inequalities, vector quasioptimization problems, etc. In this paper, as applications we focus only on minimax and saddle-point problems.

The layout of the paper is as follows. Section 2 is devoted to preliminaries, including basic elements of GFC-spaces, generalized quasiconvexity and the KKM-type Theorem 2.3, which will be the main tool for later applications. In section 3 the main existence theorem for general inclusion problems is established (Theorem 3.1) with illustrating examples, and some of its particular cases in the literature are discussed. Section 4 contains minimax and saddle-point theorems, which also generalize and improve recent results in the literature.

2. Preliminaries

Let Z be a topological space and $A, B \subseteq Z$. int$A$, cl$A$ (or $\overline{A}$), int$_B A$ and cl$_B A$ (or $\overline{A^B}$) stand for the interior, closure, interior in B and closure in B of A, respectively. A is said to be compactly open (compactly closed, resp.) if for any nonempty compact subset $K \subseteq Z$, $A \cap K$ is open (closed, resp.) in K. The compact interior of A and compact closure of A are defined, respectively, as

$$\mathrm{cint}A = \bigcup\{C \subseteq Z : C \subseteq A \text{ and } C \text{ is compactly open in } Z\},$$

$$\mathrm{ccl}A = \bigcap\{C \subseteq Z : C \supseteq A \text{ and } C \text{ is compactly closed in } Z\}.$$

It is clear that cint$A$ (ccl$A$, resp.) is compactly open (compactly closed, resp.) in Z and for each nonempty compact subset $K \subseteq Z$ with $A \cap K \neq \emptyset$ one has $K \cap \mathrm{cint}A = \mathrm{int}_K(K \cap A)$ and $K \cap \mathrm{ccl}A = \mathrm{cl}_K(K \cap A)$. It is equally obvious that $A \subseteq Z$ is compactly open (compactly closed, resp.) if and only if $\mathrm{cint}A = A$ ($\mathrm{ccl}A = A$, resp.). Let Y be a nonempty set, a multifunction $F : Y \to 2^Z$ is called transfer open-valued (transfer closed-valued, resp.) if, for each $y \in Y$ and $z \in F(y)$ ($z \notin F(y)$, resp.), there exists $y' \in Y$ such that $z \in \mathrm{int}F(y')$ ($z \notin \mathrm{cl}F(y')$, resp.). F is said to be transfer compactly open-valued (transfer compactly closed-valued, resp.) if for each $y \in Y$, each nonempty compact subset $K \subseteq Z$ and each $z \in F(y) \cap K$ ($z \notin F(y) \cap K$, resp.), there is $y' \in Y$ such that $z \in \mathrm{int}_K(F(y') \cap K)$ ($z \notin \mathrm{cl}_K(F(y') \cap K)$, resp.). A transfer open-valued (transfer closed-valued, resp.) multifunction is evidently also transfer compactly open-valued (transfer compactly closed-valued, resp.). $\langle Y \rangle$ will denote the class of all finite subsets of Y. By Δ_n we denote the standard n- dimensional simplex in R^{n+1} with vertices e_1, e_2, ..., e_{n+1}, which form a basis of R^{n+1}.

Definition 2.1 (see [21])

(i) Let X be a topological space, Y be a nonempty set and Φ be a family of continuous mappings $\varphi : \Delta_n \to X, n \in \mathbb{N}$. Then a triple (X, Y, Φ) is said to be a generalized finitely continuous topological space (GFC-space in short) if for each finite subset $N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$, there is $\varphi_N : \Delta_n \to X$ of the family Φ. Later we also use $(X, Y, \{\varphi_N\})$ to denote (X, Y, Φ).

(ii) Let $D, C \subseteq Y$ and $S : Y \to 2^X$ be given. D is called an S-subset of Y (S-subset of Y wrt C) if $\forall N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$, $\forall \{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N \cap D$ ($\subseteq N \cap C$, respectively), $\varphi_N(\Delta_k) \subseteq S(D)$, where Δ_k is the face of Δ_n corresponding to $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\}$.

Note that this definition encompasses most of the existing ones in the vein. If $Y = X$ (we then write simply (X, Φ)) the GFC-space collapses to the FC-space [9]. The S-subset of Y becomes the FC-subspace of Y [9] if in addition $S = I$, the identity map. The G-convex space (X, Y, Γ) defined in [31] is a special case of GFC-space (X, Y, Φ), if one takes $\varphi_N : \Delta_n \to \Gamma(N)$, where Γ is the generalized convex hull operator [31]. Note further that the G-convex space and the FC-space are incomparable and include many spaces with general convexity structures.

Definition 2.2 (see [22])

(i) Let (X, Y, Φ) be a GFC-space and Z be a topological space. Let $F : Y \to 2^Z$ and $H : X \to 2^Z$ be set-valued mappings. F is said to be a generalized KKM mapping with respect to (wrt) H (H-KKM mapping in short) if, for each $N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$ and each $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$, one has $H(\varphi_N(\Delta_k)) \subseteq \bigcup_{j=0}^{k} F(y_{i_j})$, where $\varphi_N \in \Phi$ is corresponding to N and Δ_k is the face of Δ_n corresponding to $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\}$.

(ii) We say that a set-valued mapping $H : X \to 2^Z$ has the generalized KKM property if, for each H-KKM mapping $F : Y \to 2^Z$, the family $\{\overline{F(y)} : y \in Y\}$ has the finite intersection property, i.e. each finite intersection of sets of this family is nonempty. By KKM(X, Y, Z) we denote the class of all the mappings $H : X \to 2^Z$ which enjoy the generalized KKM property.

The following generalized KKM-type theorem contains many existing KKM-type theorems as mentioned in Section 1.

Theorem 2.3 (see [22]). *Let* (X, Y, Φ) *be a GFC-space*; Z *be a topological space*; $S : Y \to 2^X$, $H : X \to 2^Z$ *and* $F : Y \to 2^Z$ *be multifunctions, where* $H \in \mathrm{KKM}(X, Y, Z)$. *Assume that*

(i) *for each compact subset* $X_0 \subseteq X$, $\mathrm{cl}H(X_0)$ *is compact* ;

(ii) F *is H-KKM and transfer compactly closed-valued*;

(iii) *there exists a compact subset* $K \subseteq Z$ *such that for each* $N \in \langle Y \rangle$, *there is an S-subset* L_N *of* Y, *containing* N *with* $S(L_N)$ *being compact and*

$$\mathrm{cl}H(S(L_N)) \cap \bigcap_{\mathrm{y \in L_N}} \mathrm{ccl}F(y) \subseteq K.$$

Then

$$\mathrm{cl}H(S(Y)) \cap \bigcap_{y \in Y} F(y) \neq \emptyset.$$

Now we pass to discuss generalized convexity in connection with KKM properties. From now on, if not otherwise stated, let (X, Y, Φ) be a GFC-space, Z be a topological space, Ω and D be nonempty sets, $H : X \to 2^Z$, $T : Z \times Y \to 2^\Omega$, $h : \Omega \times Z \times Y \to 2^D$ and $k : \Omega \times Z \to 2^D$ be multivalued mappings. For $r \in \{r_1, r_2, r_3, r_4\}$ and $\alpha \in \{\alpha_1, \alpha_2\}$ define another multifunction $h_{r\alpha} : Y \to 2^Z$ by

$$h_{r\alpha}(y) = \{z \in Z \mid \alpha(t, T(z, y)), r(h(t, z, y), k(t, z))\}.$$

Definition 2.4. Multifunction h is called $(k, T, r\alpha)$-quasiconvex wrt H if, $\forall N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$, $\forall \{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$, $\forall z \in H(\varphi_N(\Delta_k))$, $\exists j \in \{0, 1, ..., k\} \subseteq N$, $\alpha(t, T(z, y_{i_j}))$, $r(h(t, z, y_{i_j}), k(t, z))$, where φ_N and Δ_k are defined as in Definition 2.1.

This property is very relaxed and general. If X, Y and Z are convex subsets of topological vector spaces and $H = I$, then this definition collapses to the k-quasiconvexity wrt T or k-quasiconvexlikeness wrt T (depending on r,α) defined in section 2 of [14]. If, furthermore, $T(z, y) = \{z\}$ and $k(t, z) := k(z)$, it becomes the strong k-diagonal quasiconvexity or strong k-diagonal quasiconcavity proposed in [26]. If $Y = X$ (the GFC-space becomes an FC-space), $\Omega = Z$, $T(z, y) = \{z\}$, $h(t, z, y) = h(z, y)$ and $k(t, z) = k(z)$, this definition contains the k-diagonal quasiconvexity wrt H of type I, II and III stated in Definition 4.2 of [11]. As explained in [11, 14, 26], all these definitions generalize and relax the classical quasiconvexity.

Proposition 2.5. *h is $(k, T, r\alpha)$-quasiconvex wrt H if and only if $h_{r\alpha}$ is H-KKM.*

Proof. "*Only if*". For each $N = \{y_0, y_1, ..., y_n\} \in \langle Y\rangle$, each $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$ and each $z \in H(\varphi_N(\Delta_k))$, there is $j \in \{0, 1, ..., k\}$ such that

$$\alpha(t, T(z, y_{i_j})), r(h(t, z, y_{i_j}), k(t, z)). \tag{2}$$

If $h_{r\alpha}$ was not $H{-}KKM$, there would be $\overline{N} = \{\overline{y}_0, \overline{y}_1, ..., \overline{y}_m\} \in \langle Y\rangle$, $\{\overline{y}_{i_0}, \overline{y}_{i_1}, ..., \overline{y}_{i_l}\} \subseteq \overline{N}$ such that $H(\varphi_{\overline{N}}(\Delta_l)) \not\subseteq \bigcup_{j=0}^{l} h_{r\alpha}(\overline{y}_{i_j})$. This means the existence of $\overline{z} \in H(\varphi_{\overline{N}}(\Delta_l))$ such that, $\forall j \in \{0, 1, ..., l\}$,, $\overline{z} \notin h_{r\alpha}(\overline{y}_{i_j})$, i.e.

$$\overline{\alpha}(t, T(\overline{z}, \overline{y}_{i_j})), \overline{r}(h(t, \overline{z}, \overline{y}_{i_j}), k(t, \overline{z})),$$

which contradicts (2).
"*If*". Assume the H-KKM property of $h_{r\alpha}$, i.e. for each $N \in \langle Y\rangle$, each $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$, one has

$$H(\varphi_N(\Delta_k)) \subseteq \bigcup_{j=0}^{k} h_{r\alpha}(y_{i_j}). \tag{3}$$

Supposing h is not $(k, T, r\alpha)$-quasiconvex wrt H we easily use the definition of this quasiconvexity to get a contradiction to (3). $\square$

Note that if $X = Y$ (and the GFC-space becomes an FC-space), $\Omega = Z$, $T(z, y) = \{z\}$, $h(t, z, y) = h(z, y)$ and $k(t, z) = k(z)$, this proposition collapses to Proposition 4.1 of [11].

Proposition 2.6. *If a multifunction $S : Y \to 2^Z$ exists such that, for each $z \in Z$, $Y \backslash S^{-1}(H^{-1}(z))$ is S-subset of Y wrt $Y \backslash h_{r\alpha}^{-1}(z)$, then h is $(k, T, r\alpha)$-quasiconvex wrt H.*

Proof. We show that $h_{r\alpha}$ is H-KKM by supposing to the contrary the existence of $N \in \langle Y \rangle$ and $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$ so that $H(\varphi_N(\Delta_k)) \not\subseteq \bigcup_{j=0}^k h_{r\alpha}(y_{i_j})$. This means the existence of $\overline{x} \in \varphi_N(\Delta_k)$ and $\overline{z} \in H(\overline{x})$ with $\overline{z} \notin h_{r\alpha}(y_{i_j})$ for all $j \in \{0, 1, ..., k\}$, i.e. $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N \cap (Y \backslash h_{r\alpha}^{-1}(\overline{z}))$. As $Y \backslash S^{-1}(H^{-1}(\overline{z}))$ is S-subset of Y wrt $Y \backslash h_{r\alpha}^{-1}(\overline{z})$, one has $\overline{x} \in \varphi_N(\Delta_k) \subseteq S(Y \backslash S^{-1}(H^{-1}(\overline{z})))$. Consequently, $\overline{y} \in Y \backslash S^{-1}(H^{-1}(\overline{z}))$ exists such that $\overline{z} \in H(\overline{x}) \subseteq H(S(\overline{y}))$, i.e. $\overline{y} \in S^{-1}(H^{-1}(\overline{z}))$, which is impossible. Applying now Proposition 2.5 completes the proof. □

For the special case as afore-mentioned for Proposition 2.5, Proposition 2.6 is reduced to Proposition 4.2 of [11].

Proposition 2.7. *If a multifunction* $S : Y \to 2^Z$ *exists such that*

(i) *for each* $y \in Y$ *and* $z \in H(S(y))$, $z \in h_{r\alpha}(y)$;

(ii) *for each* $z \in Z$, $Y \backslash h_{r\alpha}^{-1}(z)$ *is an* S*-subset of* Y,

then h *is* $(k, T, r\alpha)$*-quasiconvex wrt* H.

Proof. Suppose, ab absurdo, the existence of $N \in \langle Y \rangle$, $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$ and $\overline{z} \in H(\varphi_N(\Delta_k))$ such that, for all $j \in \{0, 1, ..., k\}$,

$$\overline{\alpha}(t, T(\overline{z}, y_{i_j})), \ \overline{r}(h(t, \overline{z}, y_{i_j}), k(t, \overline{z})),$$

i.e. $y_{i_j} \in Y \backslash h_{r\alpha}^{-1}(\overline{z})$. Since the last set is an S-subset of Y, $\overline{z} \in H(S(Y \backslash h_{r\alpha}^{-1}(\overline{z})))$. Hence, $\overline{z} \in H(S(\overline{y}))$ for some $\overline{y} \in Y \backslash h_{r\alpha}^{-1}(\overline{z})$. By (i), $\overline{z} \in h_{r\alpha}(\overline{y})$ and we arrive at the contradiction that $\overline{y} \in Y \backslash h_{r\alpha}^{-1}(\overline{z})$. □

For the above-encountered special case, this proposition collapses to Proposition 4.3 of [11].

3. The solution existence for problem (IP$_{r\alpha}$)

In this section, besides (X, Y, Φ), Z, Ω, D, H, T, h, k, $h_{r\alpha}$ defined as in section 2, let be given additionally $S_1 : Z \to 2^Z$, $S_2 : Z \to 2^Y$, $f : \Omega \times Z \times Y \to 2^D$ and $g : \Omega \times Z \to 2^D$. Let E stand for the set of fixed points of S_1, i.e. $E = \{z \in Z \mid z \in$

$S_1(z)\}$.

Theorem 3.1. *Assume for problem* $(\mathrm{IP}_{r\alpha})$ *that*

(i) *for each compact subset* $X_0 \subseteq X$, $\mathrm{cl}H(X_0)$ *is compact*;

(ii) E *is nonempty, compactly closed*; *for each* $z \in Z\backslash E$, $S_2(z) \neq \emptyset$ *and for each* $N \in \langle Y\rangle$, *each* $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N \cap S_2(z)$, $H(\varphi_N(\Delta_k)) \subseteq S_1(z)$;

(iii) $\forall (z, y) \in E \times S_2(z)$, $[\alpha(t, T(z, y)),\ r(h(t, z, y), k(t, z))]$ *implies* $[\alpha(t, T(z, y)),$ $r(f(t, z, y), g(t, z))]$;

(iv) h *is* $(k, T, r\alpha)$*-quasiconvex wrt* H;

(v) S_2^{-1} *and* $h^*_{r\alpha}$ *are transfer compactly open-valued, where for* $y \in Y$,
$$h^*_{r\alpha}(y) = \{z \in S_2^{-1}(y) \mid \overline{\alpha}(t, T(z, y)), \overline{r}(h(t, z, y), k(t, z))\};$$

(vi) *there exist* $S : Y \to 2^X$ *and compact subset* $K \subseteq Z$ *such that*, $\forall N \in \langle Y\rangle$, *there is an* S*-subset* L_N *of* Y *containing* N *with* $S(L_N)$ *being compact and*

(a) $\forall z \in \mathrm{cl}H(S(L_N))\backslash K$, $\exists y \in L_N$, $z \in \mathrm{cint}S_2^{-1}(y)$;

(b) $\forall z \in (\mathrm{cl}H(S(L_N))\backslash K) \cap E$, $\exists y \in L_N$, $z \in \mathrm{cint}(h^*_{r\alpha}(y))$.

Then $(\mathrm{IP}_{r\alpha})$ *has solutions.*

Proof. For $y \in Y$, set
$$Q(y) = (Z\backslash S_2^{-1}(y)) \cup (E \cap h_{r\alpha}(y)).$$
To apply Theorem 2.3 for Q in the place of F we have to check its assumption (ii) and (iii). We show first that Q is H-KKM. Let $N \in \langle Y\rangle$, $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$ and $z \in H(\varphi_N(\Delta_k))$. If $z \in E$, by (iv) and Proposition 2.5 there is $\overline{j} \in \{0, 1, ..., k\}$ such that $z \in E \cap h_{r\alpha}(y_{i_{\overline{j}}}) \subseteq Q(y_{i_{\overline{j}}})$ and we are done. If $z \in (Z\backslash E) \cap S_2^{-1}(y_{i_j})$, $\forall j \in \{0, 1, ..., k\}$, then $y_{i_j} \in S_2(z)$. According to assumption (ii) of this theorem, $z \in H(\varphi_N(\Delta_k)) \subseteq S_1(z)$, i.e. $z \in E$, a contradiction. Thus there is $\overline{j} \in \{0, 1, ..., k\}$ such that $z \in Z\backslash S_2^{-1}(y_{i_{\overline{j}}}) \subseteq Q(y_{i_{\overline{j}}})$. Hence Q is H-KKM.

To prove (ii) of Theorem 2.3 it is more convenient to show that the complement $Z\backslash Q(\cdot)$ is transfer compactly open-valued. For any $y \in Y$, one has

$$\begin{aligned}Z\backslash Q(y) &= [S_2^{-1}(y)\cap(Z\backslash E)]\cup[S_2^{-1}(y))\cap(Z\backslash h_{r\alpha}(y))]\\ &= [S_2^{-1}(y)\cap(Z\backslash E)]\cup h^*_{r\alpha}(y).\end{aligned}$$

Now consider any nonempty compact subset $K\subseteq Z$ and any $z\in(Z\backslash Q(y))\cap K$. If $z\in(Z\backslash E)\cap S_2^{-1}(y)\cap K$, by the compact closedness of E and transfer compact open-valuedness of S_2^{-1}, there is $y'\in Y$ such that

$$\begin{aligned}z &\in (Z\backslash E)\cap \mathrm{int}_K(S_2^{-1}(y')\cap K)\\ &= \mathrm{int}_K((Z\backslash E)\cap S_2^{-1}(y')\cap K)\\ &\subseteq \mathrm{int}_K((Z\backslash Q(y'))\cap K).\end{aligned}$$

If $z\in h^*_{r\alpha}(y)\cap K$, by the transfer compact open-valuedness of $h^*_{r\alpha}$, one has $y'\in Y$ such that

$$\begin{aligned}z &\in \mathrm{int}_K(h^*_{r\alpha}(y')\cap K)\\ &\subseteq \mathrm{int}_K((Z\backslash Q(y'))\cap K).\end{aligned}$$

As

$$(Z\backslash Q(y))\cap K=((Z\backslash E)\cap S_2^{-1}(y)\cap K)\cup(h^*_{r\alpha}(y)\cap K)$$

we see that $Q(\cdot)$ is transfer compactly closed-valued as required in (ii).

For (iii) of Theorem 2.3, we first show, for S, K and L_N given in (vi),

$$\mathrm{cl}H(S(L_N))\cap\bigcap\nolimits_{y\in L_N}\mathrm{ccl}Q(y)\subseteq K.$$

Suppose the existence of z in the left-hand side such that $z\notin K$. If $z\in(Z\backslash E)$, then, by (vi), there is $y\in L_N$ such that $z\in\mathrm{cint}S_2^{-1}(y)\cap(Z\backslash E)$. As $Z\backslash E$ is compactly open (by (ii)), the last set is contained in $\mathrm{cint}(Z\backslash Q(y))=Z\backslash\mathrm{ccl}Q(y)$, a contradiction. If $z\in E$, (vi) gives $y\in L_N$ such that $z\in\mathrm{cint}h^*_{\mathrm{r}\alpha}(y)\subseteq\mathrm{cint}(Z\backslash Q(y))$ $=Z\backslash\mathrm{ccl}Q(y)$, another contradiction.

Now that the assumptions of Theorem 2.3 have been checked, one has $\overline{z}\in\bigcap_{y\in Y}Q(y)$. If $\overline{z}\in Z\backslash E$, the definition of Q shows that $\overline{z}\in Z\backslash S_2^{-1}(y)$, $\forall y\in Y$, which implies the contradiction that $S_2(\overline{z})=\emptyset$ (see (ii)). Thus $\overline{z}\in E$. For each $y\in S_2(\overline{z})$, i.e. $\overline{z}\notin Z\backslash S_2^{-1}(y)$, as $\overline{z}\in Q(y)$ one should have $\overline{x}\in h_{r\alpha}(y)$. Now employing assumption (iii) of this theorem completes proof. $\square$

Remark 3.2. In assumption (vi), if $K=Z$ then (a) and (b) are satisfied. Furthermore, in (v) we can replace the condition on $h^*_{r\alpha}$ by the transfer compact closed-valuedness of $h_{r\alpha}$. If in (IP$_{r\alpha}$), $X = Y = Z$ is a convex subset of a topological vector space, the four problems considered in [14, 17] become special cases of four cases among the eight ones of this special cases of (IP$_{r\alpha}$). Taking H and S being the identity maps, these four special cases of Theorem 3.1 still have assumptions weaker than the corresponding ones in the main theorems of [14, 17] (assumptions about convexity, openness, closedness, coercivity, etc). Hence this theorem generalizes and improves also the corresponding results in [18, 19, 23, 26, 27]. Consider Theorem 3.1 for the special case, where $X = Y$ (and the GFC-space becomes an FC-space), $S_1(z) = Z$, $S_2(z) = Y$, $\Omega = Z$, $T(z, y) = \{z\}$. If assumption (iv) is replaced by the equivalent assertion given in Proposition 2.7 and assumption (vi) is imposed particularly with $S = I$, then Theorem 3.1 collapses to Theorem 4.2 of [12].

To compare Theorem 3.1 with Theorems 4.1 and 4.2 of [11] we need the lemma below. First we explain some notations. If $T(z, y) = \{z\}$, T and α can be omitted from problem (IP$_{r\alpha}$). Then the $(k, T, r\alpha)$-quasiconvex wrt H is called simply (k, r) - quasiconvex wrt H. Similarly, $h^*_{r\alpha}$ becomes h^*_r.

Lemma 3.3. *Let (Y, Φ) be an FC-space, Z be a topological space, D be a set, $H : Y \to 2^Z$, $h, M : Z \times Y \to 2^D$ and $k : Z \to 2^D$. Assume that*

(i) $\forall y \in Y$, $\forall z \in H(y)$, $r(M(z, y), k(z))$;

(ii) *either of the following conditions holds*

(ii$_1$) *h is (k, r)-quasiconvex wrt $M_r : Y \to 2^Z$ defined by*

$$M_r(y) = \{z \in Z \mid r(M(z, y), k(z))\};$$

(ii$_2$) *$\forall z \in Z$, $Y \backslash M_r^{-1}(z)$ is an FC-subspace of Y wrt $h_r^{*-1}(z)$.*

Then h is (k, r)-quasiconvex wrt H.

Proof. (ii$_1$). $\forall N \in \langle Y \rangle$, $\forall \{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$, $\forall z \in M_r(\varphi_N(\Delta_k))$, $\exists j \in \{0, 1, ..., k\}$, $r(h(z, y_{i_j}), k(z))$ by assumption (ii$_1$). Now (i) implies that $\forall z \in H(\varphi_N(\Delta_k))$, $z \in M_r(\varphi_N(\Delta_k))$. Thus, h is (k, r)-quasiconvex wrt H.
(ii$_2$). It follows from (ii$_1$) and Proposition 2.6. $\square$

Taking Lemma 3.3 (and some facts from Propositions 4.1-4.6 of [11] for detailed checking, if needed) into account we obtain the following consequences, for the particular case of an FC-space with $S_1(z) = Z$, $S_2(z) = Y$, $\Omega = Z$, $T(z, y) = \{z\}$ (hence $f(t, z, y) = f(z, y)$, $g(t, z) = g(z)$) investigated in [7]. Let us impose in Theorem 3.1 assumption (vi) with $S = I$. With (iv) replaced by (i) and (ii$_1$) of Lemma 3.3, Theorem 3.1 is reduced to Theorem 4.1 of [11]. While with (iv) replaced by (i) and (ii$_2$) we receive Theorem 4.2 of [11].

Consider now an example, where we apply Theorem 3.1.

Example 3.4. Let Y = [-2, 0], Z = [0, 2], $\Omega = D = R$. Let $S_1 : Z \to 2^Z$, $S_2 : Z \to 2^Y$, $T : Z \times Y \to 2^\Omega$, $f : \Omega \times Z \times Y \to 2^D$ and $g : \Omega \times Z \to 2^D$ be given by

$$S_1(z) = \begin{cases} [0, z] & \text{if} \quad 0 \le z \le 1, \\ (z, 2] & \text{if} \quad 1 < z \le 2, \end{cases}$$

$$S_2(z) = (-z, 0),$$

$$T(z, y) = (-1, 1),$$

$$f(t, z, y) = [\min\{tz,\ ty\}, \max\{tz,\ ty\}],$$

$$g(t, z) = (-\infty, z].$$

Find $\bar{z} \in S_1(\bar{z})$ such that,$\forall y \in S_2(\bar{z}), \forall t \in T(\bar{z}, y)$,

$$[\min\ \{t\bar{z}, ty\},\ \max\{t\bar{z}, ty\}] \subseteq (-\infty, \bar{z}].$$

This is a variational inclusion problem (IP$_{r_1\alpha_1}$). To apply Theorem 3.1, choosing X = [0, 1], we define a GFC-space $(X, Y, \{\varphi_N\})$ as follows. $\forall N \in \langle Y \rangle$, take $\varphi_N : \Delta_n \to [0, 1]$ as the canonical projection on the first coordinate axis (among n + 1 axes). Next we choose $H \in KKM(X, Y, Z)$ by setting $H(x) = \{2\}$, $\forall x \in X$ (then clearly $H \in KKM(X, Y, Z)$). Now we verify the assumptions of Theorem 3.1. (i) and (iii) are obvious. For (ii) we see that $E = [0, 1] \cup \{2\}$ is closed and hence compactly closed. Further, $\forall z \in Z \backslash E$ = (1, 2), $\forall N \in \langle Y \rangle$, $\forall \{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N \cap S_2(z)$, $z \in H(\varphi_N(\Delta_k)) = \{2\}$, we see that, $\forall t \in T(z, y_{i_j}) = (-1, 1)$,

$$f(t, z, y_{i_j}) = [\min\ \{2t, ty_{i_j}\},\ \max\{2t, ty_{i_j}\}]$$

$$= \begin{cases} [ty_{i_j}, 2t] & \text{if} \quad 0 \le t < 1, \\ [2t, ty_{i_j}] & \text{if} -1 < t < 0, \end{cases}$$

$$\subseteq \begin{cases} [ty_{i_j}, 2] & \text{if} \quad 0 \le t < 1, \\ [2t, 2] & \text{if} -1 < t < 0, \end{cases}$$

$$= (-\infty, 2]$$

$$= g(t, 2),$$

i.e. f is $(g, T, r_1\alpha_1)$-quasiconvex wrt H. Passing to (v) we first compute $S_2^{-1}(y)$. Clearly $S_2^{-1}(y) = \emptyset$ for $y = 2$ and $y = 0$. For other $y \in [-2, 0]$, one has

$$S_2^{-1}(y) = \{z \in Z \mid -z < y < 0\} = (-y, 2].$$

For calculating $h^*_{r_1\alpha_1}$, we rewrite the above formula of f as follows, $\forall t \in T(z, y) =$ (-1, 1),

$$f(t, z, y) = \begin{cases} [ty, tz] & \text{if} \quad 0 \le t < 1, \\ [zt, ty] & \text{if} -1 < t < 0. \end{cases}$$

As $tz \le z$, $\forall t \in (-1, 1)$ we obtain

$$\{z \in Z \mid f(t, z, y) \not\subseteq g(t, z)\} = \{z \in Z \mid \exists t \in (-1, 1), z < ty\}$$

$$\subseteq [0, -y).$$

Therefore,

$$h^*_{r_1\alpha_1}(y) \subseteq S_2^{-1}(y) \cap [0, -y) = \emptyset.$$

Thus, both S_2^{-1} and $h^*_{r_1\alpha_1}$ are transfer compactly open-valued in Y.

Finally for (vi) we simply take $K = Z$ and S defined by $S(y) = X$, $\forall y \in Y$. According to Theorem 3.1, problem ($\mathrm{IP}_{r_1\alpha_1}$) in question has solutions. By direct checking we see that $\overline{z} = \frac{1}{2}$ is a solution.

The results in [14, 17, 20] (and heme all early results in the literature mentioned in these papers) cannot be applied for Example 3.4, since many assumptions are not satisfied, e.g. S_1 is not closed, $\mathrm{co}S_2(z) \not\subseteq S_1(z)$ for $z \in Z \backslash E$, etc.

The problem given in the next example can hardly be considered by known results in FC-spaces, G-convex spaces, but is easily investigated by Theorem 3.1.

Example 3.5. Let $\mathbb{N}$ be the set of the natural numbers, $Y = \mathbb{N} \cup (-\mathbb{N})$, $Z = [-1, 1]$, $D = R$, $f(z, y) = [zy, +\infty)$ and $g(z) = [z, +\infty)$. Find $\overline{z} \in Z$ such that $\forall y \in Y$, $f(\overline{z},\ y) \subseteq [\overline{z}, +\infty)$.

This is a special case of $(\mathrm{IP}_{r_1\alpha_1})$, but α_1 needs not be included as, in this case, $S_1(z) = Z$, $S_1(z) = Y$, $\Omega = Z$, $T(z, y) = \{z\}$. If one uses an FC-space $(Y, \{\varphi_N\})$, one can hardly choose a suitable topology on Y and the corresponding $\varphi_N : \Delta_n \to Y$. Now we define a GFC-space by taking $X = R$ and, $\forall N \in \langle Y \rangle$, $\varphi_N(e) = \sum_{i=0}^{n} \lambda_i y_i$, where $e = \sum_{i=0}^{n} \lambda_i e_i \in \Delta_n$. We choose $H : X \to 2^Z$ as $H(x) = \{0\}$, $\forall x \in X$. Clearly $H \in KKM(X, Y, Z)$ and assumptions (i)-(iii) of Theorem 3.1 are satisfied.

For (iv) (with $h = f$ and $k = g$), with any $N \in \langle Y \rangle$, any $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$ and $z \in H(\varphi_N(\Delta_k)) = \{0\}$ we have $f(z, y_{i_j}) = [0, +\infty) = g(z)$, $\forall j \in \{0, 1, ..., k\}$. Hence f is (g, r_1)-quasiconvex wrt H. To check (v) we compute

$$f^*_{r_1}(y) = \{z \in Z \mid f(z, y) \not\subseteq g(z)\}.$$

$$= \{z \in [-1, 1] \mid zy < z\}$$

$$= \begin{cases} \emptyset & \text{if } \ y = 1, \\ [-1, 0) & \text{if } \ y > 1, \\ (0, 1] & \text{if } \ y < 1, \end{cases}$$

which is open in Z, for each $y \in Y$. Therefore, $f^*_{r_1}$ is transfer compactly open-valued.

For (vi) we take $K = Z$, $S : Y \to 2^X$ as $S(y) = [-|y|, |y|]$, $\forall y \in Y$ and, $\forall N \in \langle Y \rangle$, $L_N = N$. Then it is not hard to see that L_N is an S-subset of Y and $S(L_N))$ is compact. Furthermore, $\mathrm{cl}(H(S(L_N))) \backslash K = \emptyset$. Then (vi) is fulfilled. Theorem 3.1 concludes that the considered problem has solutions. We easily see directly that $\overline{z} = 0$ is a solution.

4. Applications to minimax problems

We first propose a particular case of $(k, T, r_1\alpha_1)$-quasiconvexity wrt H, which is suitable for minimax problems.

Definition 4.1. Let $(X, Y, \{\varphi_N\})$ be a GFC-space, Z be a topological space, $H : X \to 2^Z$, $h : Z \times Y \to R \cup \{\pm\infty\}$ and $\lambda \in R$. h is called λ-quasiconvex (λ-quasiconcave, resp.) wrt H in y if $\forall N \in \langle Y \rangle$, $\forall \{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$, $\forall z \in H(\varphi_N(\Delta_k))$, $\min_{0\le j\le k} h(z, y_{i_j}) \le \lambda$ ($\max_{0\le j\le k} h(z, y_{i_j}) \ge \lambda$, resp.). If $Z = X$ and $H = I$ we omit "wrt H" in the terminology.

Note that if h is λ-quasiconvex (λ-quasiconcave, resp.) wrt H in y then h is also $(k, T, r_1\alpha_1)$-quasiconvexity wrt H, where $\Omega = Z$, $T(z, y) = \{z\}$, $D = R \cup \{\pm\infty\}$, and $k(t, z) = (-\infty, \lambda]$ ($[\lambda, +\infty)$, resp.). Note further that Definition 4.1 generalizes Definitions 2.4 and 2.5 of [8] of λ-generalized G-diogonal quasiconvexity and λ-generalized S-diogonal quasiconvexity, and the corresponding quasiconcavity for G-convex spaces. Hence, this definition contains as special cases also the corresponding notions in Definition 7 of [25] and Definition 1.7 of [33].

We need also the following notion of Definition 2.6 in [8].

Definition 4.2 (see [8]). Let Y be a nonempty set, Z be a topological space, $h : Z \times Y \to R \cup \{\pm\infty\}$ and $\lambda \in R$. h is called λ-transfer compactly lower (upper, resp.) semicontinuous in z if for each compact subset $K \subseteq Z$ and $z \in K$,

$$[\exists y,\ h(z, y) > \lambda \ (< \lambda, \text{resp.})] \Rightarrow [\exists U(z), \text{ open neighborhood}, \exists y_0 \in Y,$$
$$\forall z' \in U(z),\ h(z', y_0) > \lambda \ (< \lambda, \text{resp.})].$$

Let us define $h^* : Y \to 2^Z$ by $h^*(y) = \{z \in Z \mid h(z, y) > \lambda\}$ ($< \lambda$, resp.). Then h is λ-transfer compactly lower (upper, resp.) semicontinuous in z if and only if h^* is transfer compactly open-valued.

Theorem 4.3. *Let $(X, Y, \{\varphi_N\})$ be a GFC-space, X being a Hausdorff topological space, $h, f : X \times Y \to R \cup \{\pm\infty\}$, $\lambda, \mu \in R$. Assume that*

(i) $\forall (x, y) \in X \times Y$, $h(x, y) \le \lambda \Rightarrow f(x, y) \le \mu$;

(ii) *h is λ-quasiconvex in y and λ-transfer compactly lower semicontinuous in x;*

(iii) *there exist $S : Y \to 2^X$ and a compact subset $K \subseteq X$ such that, $\forall N \in \langle Y \rangle$,* *there is an S-subset L_N of Y containing N with $S(L_N)$ being compact and $S(L_N) \cap \bigcap_{y \in L_N} \mathrm{ccl}\{x \in X \mid h(x,y) \leq \lambda\} \subseteq K$.*

Then $\overline{x} \in X$ exists such that $f(\overline{x}, y) \leq \mu$, $\forall y \in Y$. Hence $\inf_{x \in X} \sup_{y \in Y} f(x,y) \leq \mu$.

Proof. To apply Theorem 3.1 we verify its assumptions for the case of $(\mathrm{IP}_{r_1\alpha_1})$, where $Z = \Omega = X$, $S_1(x) = X$, $S_2(x) = Y$, $H = I$, $T(x,y) = \{x\}$, $D = R \cup \{\pm\infty\}$, $k(t,x) = [-\infty, \lambda]$ and $g(t,x) = [-\infty, \mu]$. (i)-(ii) are obviously fulfilled. (iii) and (iv) follow from (i) and (ii) of Theorem 4.3. For (v) we have

$$h^*_{r_1\alpha_1}(y) = \{z \in Z \mid h(z,y) \notin [-\infty, \lambda]\} = h^*(y).$$

Hence $h^*_{r_1\alpha_1}$ is transfer compactly open-valued by (ii) of Theorem 4.3. Finally, for (vi) we see that if $x \in [\mathrm{cl}S(L_N) \backslash K] \cap E = S(L_N) \backslash K$ then, by (iii) of Theorem 4.3,

$$x \in \bigcup_{y \in L_N} (X \backslash \mathrm{ccl}\{x' \in X \mid h(x', y) \leq \lambda\}).$$

Hence, $\exists y \in L_N$, $\exists x \in X \backslash \mathrm{ccl}\{x' \in X \mid h(x', y) \leq \lambda\} = \mathrm{cint}\{x' \in X \mid h(x', y) \notin [-\infty, \lambda]\} = \mathrm{cint}h^*(y)$. Thus, (vi) is checked. By virtue of Theorem 3.1, there exists $\overline{x} \in X$ such that $f(\overline{x}, y) \in [-\infty, \mu]$, $\forall y \in Y$. $\square$

Note that if $Y \subseteq X$, $\{\varphi_N\}$ is the family of continuous mappings $\varphi_N : \Delta_N \to \Gamma(N)$ defined for the G-convex space (X, Y, Γ), and $\lambda = \mu$, then Theorem 4.3 includes properly Theorem 4.1 of [8] (the former has weaker assumptions). If additionally, $h = f$, $\lambda = \mu = 0$ and Y is compact, a weaker counterpart of this case of Theorem 4.3 is Theorem 4.2 of [8]. Hence Theorem 9 of [25], Theorem 3.3 of [33] and the well-known Ky Fan minimax inequality are also proper special cases of Theorem 4.3.

Theorem 4.4. *Let $(X, Y, \{\varphi_N\})$ and $(Y, X, \{\varphi'_N\})$ be two GFC-spaces, X and Y being Hausdorff topological spaces, $h, k, f : X \times Y \to R \cup \{\pm\infty\}$ and $\lambda, \mu, \gamma \in R$. Assume that*

(i) $\forall (x,y) \in X \times Y$, $h(x,y) \leq \lambda \Rightarrow f(x,y) \leq \mu$ *and* $k(x,y) \geq \gamma \Rightarrow f(x,y) \geq \mu$;

(ii) *h is λ-quasiconvex in y and λ-transfer compactly lower semicontinuous in x; k is γ-quasiconcave in x and γ-transfer compactly upper semicontinuous in y;*

(iii$_1$) *there exist a compact subset* $X_0 \subseteq X$ *and* $S : Y \to 2^X$ *such that,* $\forall N \in \langle Y \rangle$, *there is an* S*-subset* L_N *of* Y *containing* N *with* $S(L_N)$ *being compact and*

$$S(L_N) \cap \bigcap_{y \in L_N} \mathrm{ccl}\{x \in X \mid h(x, y) \leq \lambda\} \subseteq X_0;$$

(iii$_2$) *there exist a compact subset* $Y_0 \subseteq Y$ *and* $S' : X \to 2^Y$ *such that,* $\forall N' \in \langle X \rangle$, *there is an* S'*-subset* $L_{N'}$ *of* X *containing* N' *with* $S'(L_{N'})$ *being compact and*

$$S'(L_{N'}) \cap \bigcap_{x \in L_{N'}} \mathrm{ccl}\{y \in Y \mid k(x, y) \geq \gamma\} \subseteq Y_0.$$

Then a saddle point $(\overline{x}, \overline{y})$ *of* f *exists, i.e.* $f(\overline{x}, y) \leq f(\overline{x}, \overline{y}) \leq f(x, \overline{y})$, *for all* $(x, y) \in X \times Y$. *Hence*

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y).$$

Proof. Apply first Theorem 4.3 to get $\overline{x} \in X$ with $f(\overline{x}, y) \leq \mu$, $\forall y \in Y$. Now invoking to Theorem 3.1 with exchanging the roles of X and Y and taking $r = r_1$, $\alpha = \alpha_1$, $S_1(y) = Y$, $S_2(y) = X$, $\Omega = Y = Z$, $H = I$, $T(y, x) = \{y\}$, $D = \mathbb{R} \cup \{\pm\infty\}$, $k(t, y) = [\gamma, +\infty]$, $g(t, y) = [\mu, +\infty]$ we obtain $\overline{y} \in Y$ such that $f(x, \overline{y}) \geq \mu$, $\forall x \in X$, and the conclusion is obtained. $\square$

Corollary 4.5. *Let* $(X, Y, \{\varphi_N\})$ *and* $(Y, X, \{\varphi'_N\})$ *be two GFC-spaces,* X *and* Y *being Hausdorff topological spaces,* $S : Y \to 2^X$, $S' : X \to 2^Y$, $f : X \times Y \to R \cup \{\pm\infty\}$ and $\mu \in R$. *Assume that*

(i) *in* y *function* f *is* μ*-quasiconvex,* μ*-transfer compactly upper semicontinuous, and in* x *function* f *is* μ*-quasiconcave and* μ*-transfer compactly lower semicontinuous*;

(ii) *there exist compact subsets* $X_0 \subseteq X$ *and* $Y_0 \subseteq Y$ *such that,* $\forall N \in \langle Y \rangle$, $\forall N' \in \langle X \rangle$, *there are* S*-subset* L_N *of* Y *containing* N and S'*-subset* $L_{N'}$ *of* X *containing* N' *with* $S(L_N)$ *and* $S'(L_{N'})$ *being compact and*

$$S(L_N) \cap \textstyle\bigcap_{y \in L_N} \mathrm{ccl}\{x \in X \mid f(x, y) \leq \mu\} \subseteq X_0,$$
$$S'(L_{N'}) \cap \textstyle\bigcap_{x \in L_{N'}} \mathrm{ccl}\{y \in Y \mid f(x, y) \geq \mu\} \subseteq Y_0.$$

Then there exists a saddle $(\overline{x}, \overline{y})$ *of* f, *i.e.* $f(\overline{x}, y) \leq f(\overline{x}, \overline{y}) \leq f(x, \overline{y})$, *for all* $(x, y) \in X \times Y$. *Hence*

$$\min_{x\in X}\max_{y\in Y}f(x, y) = \max_{y\in Y}\min_{x\in X}f(x, y).$$

Proof. We simply apply Theorem 4.4 with $h = k = f$ and $\lambda = \gamma = \mu$. □

We remark that for the special case, where X and Y are subsets of two G-convex spaces E and E', $\{\varphi_N\}$ and $\{\varphi_N^{'}\}$ are the families of continuous mappings defined in these G-convex spaces, and $\mu = 0$, Corollary 4.5 includes and improves Theorems 4.3 - 4.5 of [8], Theorems 4.1, 4.2 and 4.4 of [33], Theorem 10 of [25] and of course also the celebrated Von Neumann minimax theorem.

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