QUALIFICATION AND OPTIMALITY CONDITIONS FOR DC PROGRAMS WITH INFINITE CONSTRAINTS

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Dedicated to Van Hien Nguyen in honor of his 65th birthday.

Abstract The paper is devoted to the study of a new class of optimization problems with objectives given as differences of convex (DC) functions and constraints described by infinitely many convex inequalities. We consider such problems in the general framework of locally convex topological vector spaces, although the major results obtained in the paper are new even in finite dimensions when the problems under consideration reduce to DC semi-infinite programs. The main attention is paid to deriving qualified necessary optimality conditions as well as necessary and sufficient optimality conditions for DC infinite and semi-infinite programs and to establishing relations between various qualification conditions. The results obtained are applied to and specified for particular classes of DC programs involving polyhedral convex functions in DC objectives, programs with cone constraints as well as those with positive semi-definite constraints described via the Löwer partial order.

Keywords Convex and variational analysis · Topological vector spaces · Generalized differentiation · Differences of convex functions · Semi-infinite and infinite programming · Cone constraints · Semi-definite constraints · Constraint qualifications · Optimality conditions

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1 Introduction

This paper deals with a new class of *DC infinite programs* given in the form:

$$\begin{cases} \text{minimize } \ \vartheta(x)-\theta(x) \ \text{ subject to} \\ \vartheta_t(x)\le 0, \ \ t\in T, \ \ \text{and} \ \ x\in\Theta, \end{cases} \tag{1.1}$$

where $\vartheta\colon X\to\overline{I\!R}:=(-\infty,\infty]$ and $\vartheta_t\colon X\to I\!R$ as $t\in T$ are *proper, convex, lower semicontinuous* (l.s.c.) functions with values in the *extended* real line $\overline{I\!R}$, where $\theta\colon X\to I\!R$ is also a proper, l.s.c., *convex* function while real-valued, and where $\Theta\subset X$ is a closed and *convex* subset of a a locally convex Hausdorff *topological vector space.* The above assumptions are *standing* throughout the whole paper. An important feature of problem (1.1) is that the index set T is arbitrary, i.e., may be *infinite.* When the space X is finite-dimensional, optimization problems with infinite inequality constraints belong to the area well known as *semi-infinite programming*, where the word "semi-infinite" refers to the finite number

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of decision variables and infinite number of constraints; see, e.g., the now classical book [15] and the references therein. Similarly, the word "infinite" in the context of (1.1) refers to optimization problems with *infinitely many* variables as well as constraints. The latter terminology has been recently introduced in [10] in the context of *convex* infinite programs.

Problems of *DC* (*difference of convex*) *programming* are highly important from both viewpoints of optimization theory and applications. They have been extensively studied in the literature, mainly for problems with finitely many convex inequality constraints; see, e.g., [11, 12, 18, 19] and the references therein. On one hand, such problems—being heavily nonconvex—can be considered as a special class in nondifferentiable programming (in particular, quasidifferentiable programming [8]) and thus are suitable for applying advanced techniques of variational analysis and generalized differentiation developed, e.g., in [7, 8, 29, 30, 32, 33]. On the other hand, the special *convex* structure of both "*plus*" function ϑ and "*minus*" function θ in the objective of (1.1) offers the possibility to use powerful tools of convex analysis in the study of DC programs.

Infinite DC programs of type (1.1) have been considered in the framework of Banach spaces X in our recent paper [13], where we introduced a new *closeness qualification condition* (CQC) and employed it to deriving *qualified* (i.e., of normal/KKT form) *necessary* optimality conditions for *local minimizers* to (1.1) by using refined techniques and results of convex analysis. In this paper we extend these results to DC programs in *topological vector spaces* and also derive new *necessary and sufficient* conditions for *global minimizers* to (1.1) under the CQC requirement. Furthermore, we establish various characterizations of the CQC property and its relations with some other qualification conditions in DC and convex infinite programming. Finally, we obtain implementations and specifications of the obtained optimality conditions and constraint qualifications in particular cases of DC programs involving *cone constraints*, *polyhedral convex* "minus" functions in their objectives, and also with *positive semi-definite* constraints defined via the *Löwer partial order*.

The rest of the paper is organized as follows. In Section 2 we present basic definitions as well as some preliminary, less standard facts of convex analysis broadly employed in the paper. Section 3 is devoted to a detail study of a new closedness qualification condition (CQC) for DC infinite programs in topological vector spaces, which plays a crucial role for the subsequent results of the paper. In particular, in this section we obtain various characterizations of the CQC and related Farkas-Minkowski (FM) constraint qualifications and consider their applications to calculus rules involving infinite constraints. In Section 4 we compare the CQC and FM qualification conditions with other major ones, in both primal and dual spaces, known in this and related areas. Section 5 concerns deriving new necessary optimality conditions as well as necessary and sufficient optimality conditions for local and global minimizers in DC infinite and semi-infinite programs of type (1.1). Section 6 presents some specifications of the general results obtained for DC problems with infinite constraints to an important class of cone-constrained DC programs. In Section 7 we develop efficient implementations of the general results obtained to a special case of DC programs whose "minus" objective functions θ are given in the polyhedral convex form. Finally, Section 8 contains new results for DC programs with positive semi-definite constraints defined via the

Löwer partial order in the framework of finite-dimensional spaces.

Our notation is basically standard in convex and variational analysis except special symbols introduced where they are defined; cf. [19, 29, 32]. Recall that cl^* signifies the *closure* in the *weak** *topology* w^* of dual spaces.

2 Basic Definitions and Preliminaries

As mentioned, throughout this paper we suppose that the underlying space X in the DC problem (1.1) is a *locally convex Hausdorff topological vector space*, where X^* stands for the topologically dual space X^* endowed with the weak* topology. We always use the notation $\langle\cdot,\cdot\rangle$ for the *canonical paring* between X and X^*.

Given a nonempty subset $\Omega\subset X$, the symbol $\mathrm{aff}\,\Omega$ signifies the *affine hull* of Ω, while the *core* of Ω is defined by

$$\mathrm{core}\,\Omega=\big\{x\in\Omega\big|\ \text{ for all }\ y\in X\ \text{ there is }\ \varepsilon>0\ \text{ with }\ x+\nu y\in\Omega\ \text{ whenever }\ \nu\in[-\varepsilon,\varepsilon]\big\}.$$

The core of Ω relative to the affine hull $\mathrm{aff}\,\Omega$ is called the *intrinsic core* of Ω and is denoted by $\mathrm{icr}\,\Omega$. If Ω is *convex*, the collections of all $x\in\Omega$ for which the conic closure $\mathrm{cl}[\mathrm{cone}(\Omega-x)]$ of $\Omega-x$ is a linear subspace of X is known as the *quasi relative interior* of Ω denoted by $\mathrm{qri}\,\Omega$. Note [6] that $\mathrm{qri}\,\Omega\neq\emptyset$ for any convex subset of a separable Banach space. The collection of al $x\in\Omega$ for which the conic hull $\mathrm{cone}(\Omega-x)$ is a closed linear subspace is called the *strong quasi relative interior* of Ω denoted by $\mathrm{sqri}\,\Omega$; see [24] for more discussions.

Having the generally *infinite* index set T in (1.1), consider the *product space* $I\!R^T$ of multipliers $\lambda=(\lambda_t|\ t\in T)$ with $\lambda_t\in I\!R$ for all $t\in T$ and denote by $\widetilde{I\!R}^T$ the collection of $\lambda\in I\!R^T$ with $\lambda_t\neq 0$ for *finitely many* $t\in T$. Let $\widetilde{I\!R}^T_+$ be the *positive cone* in $\widetilde{I\!R}^T$ defined by

$$\widetilde{I\!R}^T_+:=\big\{\lambda\in\widetilde{I\!R}^T\big|\ \lambda_t\geq 0\ \text{ for all }\ t\in T\big\}. \tag{2.1}$$

Given $u\in I\!R^T$ and $\lambda\in\widetilde{I\!R}^T$ and denoting $\mathrm{supp}\,\lambda:=\{t\in T|\ \lambda_t\neq 0\}$, we have

$$\langle\lambda,u\rangle:=\sum_{t\in T}\lambda_t u_t=\sum_{t\in\mathrm{supp}\,\lambda}\lambda_t u_t.$$

Considering further an extended-real-valued function $\varphi\colon X\to\overline{I\!R}$ with the *domain* $\mathrm{dom}\,\varphi:=\{x\in X|\ \varphi(x)<\infty\}$, we always assume that it is *proper*, i.e., $\varphi(x)\not\equiv\infty$ on X. The *conjugate function* $\varphi^*\colon X^*\to\overline{I\!R}$ to φ is defined by

$$\varphi^*(x^*):=\sup\big\{\langle x^*,x\rangle-\varphi(x)\big|\ x\in X\big\}=\sup\big\{\langle x^*,x\rangle-\varphi(x)\big|\ x\in\mathrm{dom}\,\varphi\big\}. \tag{2.2}$$

For any $\varepsilon\geq 0$, the ε-*subdifferential* (or *approximate subdifferential* if $\varepsilon>0$) of a *convex* function $\varphi\colon X\to\overline{I\!R}$ at $\bar x\in\mathrm{dom}\,\varphi$ is

$$\partial_\varepsilon\varphi(\bar x):=\big\{x^*\in X^*\big|\ \langle x^*,x-\bar x\rangle\leq\varphi(x)-\varphi(\bar x)+\varepsilon\ \text{ for all }\ x\in X\big\},\quad \varepsilon\geq 0. \tag{2.3}$$

When $\bar x\notin\mathrm{dom}\,\varphi$ we put $\partial_\varepsilon\varphi(\bar x):=\emptyset$. If $\varepsilon=0$ in (2.3), the set $\partial\varphi(\bar x):=\partial_0\varphi(\bar x)$ is the classical *subdifferential of convex analysis*. It is clear that

$$\partial_\varepsilon\varphi(\bar x)\subset\partial_\eta\varphi(\bar x)\ \text{ whenever }\ 0\leq\varepsilon\leq\eta. \tag{2.4}$$

The following representation [20] of the *epigraph of the conjugate function* (2.2) to a l.s.c. convex function $\varphi\colon X \to \overline{I\!R}$ via the ε-subdifferential (2.3) of φ at any point $x \in \mathrm{dom}\,\varphi$ is useful in our further development:

$$\mathrm{epi}\,\varphi^* = \bigcup_{\varepsilon \geq 0} \Big\{ \big(x^*, \langle x^*, x\rangle + \varepsilon - \varphi(x)\big) \Big|\ x^* \in \partial_\varepsilon \varphi(x) \Big\}. \tag{2.5}$$

Further, it is well known in convex analysis that the *conjugate epigraphical rule*

$$\mathrm{epi}\big(\varphi_1 + \varphi_2\big)^* = \mathrm{cl}^*\big(\mathrm{epi}\,\varphi_1^* + \mathrm{epi}\,\varphi_2^*\big) \tag{2.6}$$

is satisfied for any l.s.c. convex functions $\varphi_i\colon X \to \overline{I\!R}$, $i = 1, 2$, where the *weak* closure operation* on the right-hand side of (2.6) can be *omitted* provided that one of the functions φ_i is *continuous* at some point $\bar{x} \in \mathrm{dom}\,\varphi_1 \cap \mathrm{dom}\,\varphi_2$. More general results in this direction implying the fundamental subdifferential sum rule of convex analysis have been recently established in [5, 14] and the references therein.

Since the above subdifferential definitions and results are given for any extended-real-valued (l.s.c. and convex) functions, they encompass the case of *sets* by considering the *indicator function* $\delta(x; \Omega)$ of a set $\Omega \subset X$ equal to 0 when $x \in \Omega$ and ∞ otherwise. In this way, the *normal cone* to a convex set Ω at $\bar{x} \in \Omega$ is defined by

$$N(\bar{x}; \Omega) := \partial\delta(\bar{x}; \Omega) = \big\{ x^* \in X^* \big|\ \langle x^*, x - \bar{x}\rangle \leq 0 \ \text{ for all } \ x \in \Omega \big\}. \tag{2.7}$$

For $\varepsilon > 0$, the collection of ε-*normals* to a convex set Ω at $\bar{x} \in \Omega$ is naturally defined by

$$N_\varepsilon(\bar{x}; \Omega) := \partial_\varepsilon \delta(\bar{x}; \Omega) = \big\{ x^* \in X^* \big|\ \langle x^*, x - \bar{x}\rangle \leq \varepsilon \ \text{ for all } \ x \in \Omega \big\}. \tag{2.8}$$

Finally in this section, let us consider the following *system of linear inequalities*

$$\sigma_L := \big\{ \langle a_t, x\rangle \leq \beta_t,\ t \in T \big\}, \tag{2.9}$$

where $a_t \in X^*$ and $\beta_t \in I\!R$ are given, and where T is an arbitrary index set. A linear inequality $\langle a, x\rangle \leq \beta$ is called a *consequence relation of* σ_L if this inequality is satisfied for every solutions to σ_L. Moreover, the linear system σ_L in (2.9) is called a *linearly Farkas-Minkowski* (*FM*) *system* if every linear consequence relation of σ_L is a consequence relation of a *finite subsystem* of σ_L; see [15, 16, 27] for more details and discussions. Recall the following remarkable fact established in [16] for $X = I\!R^n$: a system σ_L from (2.9) is linearly FM *if and only if* the conic hull

$$\mathrm{cone}\,\big\{ \{(a_t, \beta_t) \in I\!R^n \times I\!R |\ t \in T\} \cup (0, 1) \big\} \ \text{ with } \ 0 \in I\!R^n \tag{2.10}$$

is a *closed* subset of the space $I\!R^n \times I\!R$.

3 Closedness Qualification Conditions for Infinite Programs

This section is devoted to *qualification conditions* for DC and convex *infinite programs* in topological vector spaces. We pay the main attention to *dual-type* qualification conditions formulated in dual spaces. The basic qualification condition employed in this paper is the following *closedness qualification condition-CQC* introduced in our previous paper [13] in the case of Banach spaces. Consider the cone

$$K := \text{cone}\Big\{\bigcup_{t\in T}\text{epi}\,\vartheta_t^*\Big\} + \text{epi}\,\delta^*(\cdot;\Theta) \tag{3.1}$$

built upon the constraint data of the underlying DC infinite program (1.1).

Definition 3.1 (closedness qualification condition-CQC). *We say that problem* (1.1) *satisfies the* CLOSEDNESS QUALIFICATION CONDITION*, CQC in brief, if the set*

$$\text{epi}\,\vartheta^* + K$$

is weak closed in the space $X^*\times I\!R$, where the cone K is defined in* (3.1)*.*

The following related while generally different qualification condition for the *infinite convex constraint system*

$$\sigma := \big\{\vartheta_t(x)\le 0,\ t\in T;\ x\in\Theta\big\} \tag{3.2}$$

in (1.1) has been recently introduced in [10] and employed therein to stability issues for convex infinite programs. We say that the constraint system (3.2) satisfies the *Farkas-Minkowski* (*FM*) *constraint qualification* if

$$\text{the cone } K \text{ defined in (3.1) is weak}^* \text{ closed in } X^*\times I\!R. \tag{3.3}$$

Observe that, in contrast to CQC, the FM qualification condition *does not* involve the cost function of (1.1) depending only on the constraints system (3.4). In the case of *linear* functions ϑ_t, $t\in T$, the FM qualification condition reduces to the now classical constraint qualification of the Farkas-Minkowski type developed in [15, 27]. Recall also the relation

$$\text{epi}\,\delta^*(\cdot;\Xi) = \text{cl}^*K$$

established in [10], where Ξ stands for the set of *feasible solutions* to (1.1), i.e.,

$$\Xi := \Theta\cap\big\{x\in X\big|\ \vartheta_t(x)\le 0\ \text{ for all }\ t\in T\big\}. \tag{3.4}$$

Thus the FM qualification condition (3.3) requires in fact that the cone K in (3.1) is equal to the epigraph of the *support function* $\delta^*(\cdot;\Xi)$ of the feasible set (3.4). Dealing similarly with the CQC property from Definition 3.1 and employing the *conjugate epigraphical rule* (2.6), we have the equalities

$$\text{epi}\,\big(\vartheta+\delta(\cdot;\Xi)\big)^* = \text{cl}^*\big(\text{epi}\,\vartheta^* + \text{epi}\,\delta^*(\cdot;\Xi)\big) = \text{cl}^*(\text{epi}\,\vartheta^* + \text{cl}^*K) = \text{cl}^*(\text{epi}\,\vartheta^* + K), \tag{3.5}$$

where the last equality follows from the general fact that $\text{cl}(A+B)=\text{cl}(A+\text{cl}B)$ for any nonempty subsets of locally convex Hausdorff topological vector spaces; this can be checked directly by definitions. Thus we conclude from (3.5) that the CQC property of Definition 3.1

is *equivalent* to the requirement that the set $\operatorname{epi}\vartheta^* + K$ coincides with the epigraph of the conjugate function $(\vartheta + \delta(\cdot;\Xi))^*$.

The next theorem, which is the main result of this section, provides several *characterizations* of the CQC property for DC infinite programs that seem to be new in general infinite-dimensional as well as finite finite-dimensional settings.

Theorem 3.2 (characterizations of the CQC property). *In addition to the standing assumptions of Section* 1, *suppose that* $\Xi \cap \operatorname{dom}\vartheta \neq \emptyset$ *in the DC infinite program* (1.1). *Then the following are equivalent:*

(i) *The CQC property holds for* (1.1).

(ii) *For all* $x^* \in X^*$ *we have the equality*

$$\big(\vartheta + \delta(\cdot;\Xi)\big)^*(x^*) = \min_{\lambda \in \widetilde{I\!R}^T_+} \min_{\substack{u, v_t \in X^* \\ t \in \operatorname{supp}\lambda}} \Big[\vartheta^*(u) + \sum_{t \in \operatorname{supp}\lambda} \lambda_t \vartheta_t^*(v_t) + \delta^*(\cdot;\Theta)\Big(x^* - u - \sum_{t \in \operatorname{supp}\lambda} \lambda_t v_t\Big)\Big], \tag{3.6}$$

where both minima are realized at some $\lambda \in \widetilde{I\!R}^T_+$, $t \in \operatorname{supp}\lambda$ *and* $u, v_t \in X^*$.

(iii) *For any* $\bar{x} \in \Xi \cap \operatorname{dom}\vartheta$ *and* $\varepsilon \geq 0$ *we have the equality*

$$\partial_\varepsilon\big(\vartheta + \delta(\cdot;\Xi)\big)(\bar{x}) = \bigcup_{\lambda \in \widetilde{I\!R}^T_+} \bigcup_{\substack{\eta, \nu, \varepsilon_t \geq 0,\ t \in \operatorname{supp}\lambda \\ \eta + \sum_{t \in \operatorname{supp}\lambda} \lambda_t[\varepsilon_t - \vartheta_t(\bar{x})] + \nu = \varepsilon}} \Big\{ \partial_\eta \vartheta(\bar{x}) + \sum_{t \in \operatorname{supp}\lambda} \lambda_t \partial_{\varepsilon_t}\vartheta_t(\bar{x}) + N_\nu(\bar{x};\Theta)\Big\}, \tag{3.7}$$

where $N_\nu(\bar{x};\Theta)$ *is the* ν*-normal cone to* Θ *at* $\bar{x}$ *defined in* (2.8).

Proof. To justify the implication **(i)**$\Longrightarrow$**(ii)**, suppose that the CQC property holds in (1.1) and pick any $x^* \in X^*$. Then it follows directly from the corresponding definitions that

$$\begin{cases} \vartheta^*(u) \geq \langle u, x\rangle - \vartheta(x), \vartheta_t^*(v_t) \geq \langle v_t, x\rangle - \vartheta_t(x) \geq \langle v_t, x\rangle, \\ \delta^*(\cdot;\Theta)\Big(x^* - u - \sum_{t \in \operatorname{supp}\lambda} \lambda_t v_t\Big) \geq \Big\langle x^* - u - \sum_{t \in \operatorname{supp}\lambda} \lambda_t v_t, x\Big\rangle \end{cases}$$

for each $\lambda \in \widetilde{I\!R}^T_+$, $t \in \operatorname{supp}\lambda$, $u, v_t \in X^*$, and $x \in \Xi \cap \operatorname{dom}\vartheta$. The latter implies that

$$\vartheta^*(u) + \sum_{t \in \operatorname{supp}\lambda} \lambda_t \vartheta_t^*(v_t) + \delta^*(\cdot;\Theta)\Big(x^* - u - \sum_{t \in \operatorname{supp}\lambda} \lambda_t v_t\Big) \geq \langle x^*, x\rangle - \vartheta(x),$$

and hence we get the inequality

$$\vartheta^*(u) + \sum_{t \in \operatorname{supp}\lambda} \lambda_t \vartheta_t^*(v_t) + \delta^*(\cdot;\Theta)\Big(x^* - u - \sum_{t \in \operatorname{supp}\lambda} \lambda_t v_t\Big) \geq \big(\vartheta + \delta(\cdot;\Xi)\big)^*(x^*). \tag{3.8}$$

If $x^* \notin \operatorname{dom}(\vartheta + \delta(\cdot;\Xi))^*$, then $(\vartheta + \delta(\cdot;\Xi))^*(x^*) = \infty$. Thus, by taking (3.8) into account, the required relation in (3.6) holds trivially in this case. Otherwise, suppose that $x^* \in \operatorname{dom}(\vartheta + \delta(\cdot;\Xi))^*$. Combining now (3.5) and the CQC property in (i), we get

$$\operatorname{epi}\big(\vartheta + \delta(\cdot;\Xi)\big)^* = \operatorname{cl}^*(\operatorname{epi}\vartheta^* + K) = \operatorname{epi}\vartheta^* + K. \tag{3.9}$$

Since $\big(x^*,(\vartheta+\delta(\cdot;\Xi))^*(x^*)\big)\in\text{epi}\,(\vartheta+\delta(\cdot;\Xi))^*$, relations (3.9) and (3.1) yield that

$$\big(x^*,(\vartheta+\delta(\cdot;\Xi))^*(x^*)\big)\in\text{epi}\,\vartheta^*+\text{cone}\Big\{\bigcup_{t\in T}\text{epi}\,\vartheta_t^*\Big\}+\text{epi}\,\delta^*(\cdot;\Theta).$$

In turn, the latter inclusion ensures the existence of $\lambda=(\lambda_t|\,t\in T)\in\widetilde{I\!\!R}^T_+$, $(u,r)\in\text{epi}\,\vartheta^*$, $(w,k)\in\text{epi}\,\delta^*(\cdot;\Xi)$, and $(v_t,s_t)\in\text{epi}\,\vartheta_t^*$ as $t\in\text{supp}\,\lambda$ satisfying

$$\big(x^*,(\vartheta+\delta(\cdot;\Xi))^*(x^*)\big)=(u,r)+\sum_{t\in\text{supp}\,\lambda}\lambda_t(v_t,s_t)+(w,k),$$

which gives $x^*=u+\sum_{t\in\text{supp}\,\lambda}\lambda_tv_t+w$ and

$$(\vartheta+\delta(\cdot;\Xi))^*(x^*)\ge\vartheta^*(u)+\sum_{t\in\text{supp}\,\lambda}\lambda_t\vartheta_t^*(v_t)+\delta^*(\cdot;\Xi)(w).$$

The last two expressions imply that

$$\big(\vartheta+\delta(\cdot;\Xi)\big)^*(x^*)\ge\vartheta^*(u)+\sum_{t\in\text{supp}\,\lambda}\lambda_t\vartheta_t^*(v_t)+\delta^*(\cdot;\Xi)\Big(x^*-u-\sum_{t\in\text{supp}\,\lambda}\lambda_tv_t\Big).$$

Combining the latter inequality with that in (3.8), we conclude that (3.6) holds in the case under consideration with both minima attained at $\lambda\in\widetilde{I\!\!R}^T_+$, $t\in\text{supp}\,\lambda$, and $u,v_t\in X^*$. This fully justifies the implication (i)$\Longrightarrow$(ii) in the theorem.

Next let us prove the implication **(ii)$\Longrightarrow$(iii)**. Suppose that (ii) holds and take arbitrary elements $\bar x\in\Xi\cap\text{dom}\,\vartheta$ and $\varepsilon\ge0$. Pick further a dual vector $x^*\in X^*$ belonging to the right-hand side set in (3.7) and find by definition elements $\lambda\in\widetilde{I\!\!R}^T_+$, $\eta,\nu,\varepsilon_t\ge0$, $u\in\partial_\eta\vartheta(\bar x)$, $v_t\in\partial_{\varepsilon_t}\vartheta(\bar x)$ as $t\in\text{supp}\,\lambda$, and $w\in\partial_\nu\delta(\cdot;\Theta)(\bar x)$ such that

$$\eta+\sum_{t\in\text{supp}\,\lambda}\lambda_t\big(\varepsilon_t-\vartheta_t(\bar x)\big)+\nu=\varepsilon\ \text{ and }\ x^*=u+\sum_{t\in\text{supp}\,\lambda}\lambda_tv_t+w.$$

It follows from these equalities and the definition of ε-subdifferentials in (2.3) that

$$\vartheta(x)-\vartheta(\bar x)+\sum_{t\in\text{supp}\,\lambda}\lambda_t\vartheta_t(x)\ge\langle x^*,x-\bar x\rangle-\varepsilon\ \text{ for all }\ x\in\Theta\cap\text{dom}\,\vartheta.$$

Thus for all $x\in\Xi\cap\text{dom}\,\vartheta$ the inequality,

$$\vartheta(x)-\vartheta(\bar x)\ge\langle x^*,x-\bar x\rangle-\varepsilon,$$

which shows that $x^*\in\partial_\varepsilon(\vartheta+\delta(\cdot;\Xi))(\bar x)$. This therefore implies the inclusion

$$\partial_\varepsilon\big(\vartheta+\delta(\cdot;\Xi)\big)(\bar x)\supset\bigcup_{\lambda\in\widetilde{I\!\!R}^T_+}\ \bigcup_{\substack{\eta,\nu,\varepsilon_t\ge0,\ t\in\text{supp}\,\lambda\\ \eta+\sum_{t\in\text{supp}\,\lambda}\lambda_t[\varepsilon_t-\vartheta_t(\bar x)]+\nu=\varepsilon}}\Big\{\partial_\eta\vartheta(\bar x)+\sum_{t\in\text{supp}\,\lambda}\lambda_t\partial_{\varepsilon_t}\vartheta_t(\bar x)+N_\nu(\bar x;\Theta)\Big\}.$$

To justify the opposite inclusion in (3.7), take any $x^*\in\partial_\varepsilon(\vartheta+\delta(\cdot;\Xi))(\bar x)$ and get by representation (2.5) that

$$\big(x^*,\langle x^*,\bar x\rangle+\varepsilon-\vartheta(\bar x)-\delta(\bar x;\Xi)\big)\in\text{epi}\,\big(\vartheta+\delta(\cdot;\Xi)\big)^*,$$

which implies in turn the inequality

$$\langle x^*, \bar{x}\rangle + \varepsilon - \vartheta(\bar{x}) \geq \big(\vartheta + \delta(\cdot;\Xi)\big)^*(x^*). \tag{3.10}$$

By the assumed property (ii) there are $\lambda \in \widetilde{I\!R}^T_+$ and $u, w, v_t \in X^*$ as $t \in \operatorname{supp}\lambda$ such that

$$x^* = u + \sum_{t\in\operatorname{supp}\lambda} v_t + w \;\text{ and }\; \big(\vartheta + \delta(\cdot;\Xi)\big)^*(x^*) = \vartheta^*(u) + \sum_{t\in\operatorname{supp}\lambda} \lambda_t\vartheta_t^*(v_t) + \delta^*(\cdot;\Theta)(w).$$

Combining this with (3.10), we arrive at the inequality

$$\langle x^*, \bar{x}\rangle + \varepsilon - \vartheta(\bar{x}) \geq \vartheta^*(u) + \sum_{t\in\operatorname{supp}\lambda} \lambda_t\vartheta_t^*(v_t) + \delta^*(\cdot;\Theta)(w), \tag{3.11}$$

which ensures that $u \in \operatorname{dom}\vartheta^*$, $w \in \operatorname{dom}\delta^*(\cdot;\Theta)$, and $v_t \in \operatorname{dom}\vartheta_t^*$ for all $t \in \operatorname{supp}\lambda$.

Defining further the number $\eta := \vartheta^*(u) - \langle u, \bar{x}\rangle + \vartheta(\bar{x})$ and using definition (2.2) of the conjugate function, we get $\eta \geq 0$. The representation formula (2.5) for the conjugate function gives $u \in \partial_\eta\vartheta(\bar{x})$. Similarly we have $w \in N_{\nu'}(\bar{x};\Theta)$ and $v_t \in \partial_{\varepsilon_t}\vartheta_t(\bar{x})$, where

$$\nu' := \delta^*(\cdot;\Theta)(w) - \langle w, \bar{x}\rangle \geq 0 \;\text{ and }\; \varepsilon_t := \vartheta_t^*(v_t) - \langle v_t, \bar{x}\rangle + \vartheta_t(\bar{x}) \geq 0, \quad t \in \operatorname{supp}\lambda.$$

Combining the latter facts with (3.11), we conclude that

$$\varepsilon + \langle x^*, \bar{x}\rangle - \vartheta(\bar{x}) \geq \eta + \langle u, \bar{x}\rangle - \vartheta(\bar{x}) + \sum_{t\in\operatorname{supp}\lambda} \lambda_t\big[\varepsilon_t + \langle v_t, \bar{x}\rangle - \vartheta_t(\bar{x})\big] + \nu' + \langle w, \bar{x}\rangle.$$

Since $x^* = u + \sum_{t\in\operatorname{supp}\lambda} v_t + w$, the last inequality gives

$$\varepsilon \geq \eta + \sum_{t\in\operatorname{supp}\lambda} \lambda_t\big[\varepsilon_t - \vartheta_t(\bar{x})\big] + \nu'.$$

Letting finally $\nu := \varepsilon - \eta - \sum_{t\in\operatorname{supp}\lambda} \lambda_t[\varepsilon_t - \vartheta_t(\bar{x})]$, observe that $\nu \geq \nu'$ and thus, by (2.4),

$$\begin{aligned} x^* &= u + \sum_{t\in\operatorname{supp}\lambda} \lambda_t v_t + w \in \partial_\eta\vartheta(\bar{x}) + \sum_{t\in\operatorname{supp}\lambda} \lambda_t\partial_{\varepsilon_t}\vartheta_t(\bar{x}) + N_{\nu'}(\bar{x};\Theta) \\ &\subset \partial_\eta\vartheta(\bar{x}) + \sum_{t\in\operatorname{supp}\lambda} \lambda_t\partial_{\varepsilon_t}\vartheta_t(\bar{x}) + N_\nu(\bar{x};\Theta), \end{aligned}$$

which completes the proof of the implication (ii)$\Longrightarrow$(iii).

It remains to justify the implication **(iii)$\Longrightarrow$(i)** in the theorem. Assume that (iii) holds and $(x^*, r) \in \operatorname{cl}^*(\operatorname{epi}\vartheta^* + K)$, where the cone K is defined in (3.1). By (3.5) we have $(x^*, r) \in \operatorname{epi}(\vartheta + \delta(\cdot;\Xi))^*$ and, using the subdifferential representation (2.5), find $\varepsilon \geq 0$ such that $x^* \in \partial_\varepsilon(\vartheta + \delta(\cdot;\Xi))(\bar{x})$ and $r = \langle x^*, \bar{x}\rangle - \vartheta(\bar{x}) + \varepsilon$. Employing further the same technique as in the proof of the implication (ii)$\Longrightarrow$(iii) above, find elements $(u, s) \in \operatorname{epi}\vartheta^*$, $(v_t, k_t) \in \operatorname{epi}\vartheta_t^*$, and $(w, h) \in \operatorname{epi}\delta^*(\cdot;\Theta)$ such that

$$\begin{aligned} (x^*, r) &= (u, s) + \sum_{t\in\operatorname{supp}\lambda} \lambda_t(v_t, k_t) + (w, h) \in \operatorname{epi}\vartheta^* \\ &\quad + \operatorname{cone}\Big\{\bigcup_{t\in T}\operatorname{epi}\vartheta_t^*\Big\} + \operatorname{epi}\delta^*(\cdot;\Theta) = \operatorname{epi}\vartheta^* + K. \end{aligned}$$

This ensures the weak* closedness of the set $\mathrm{epi}\,\vartheta^* + K$ in $X^* \times I\!R$, which is the property CQC in (1.1). Thus we get (iii) $\Longrightarrow$(i) and complete the proof of the theorem. $\triangle$

The equivalence results of Theorem 3.2 extend to the general DC infinite program setting (1.1) those recently obtained in [11, Theorem 3.1] for cone-constrained programs considered in Section 6 in more detail.

Next we derive several consequences of Theorem 3.2. The first one is a *subdifferential calculus rule* involving *infinite* constraints in (1.1). The result obtained in the *equality* form in topological vector spaces significantly improves its previous inclusion version derived in [13, Corollary 3.3] by a different way in the Banach space setting. To proceed, define the set of *active constraint multipliers* for the original constraint system (3.2) by

$$A(\bar{x}) := \big\{\lambda \in \widetilde{I\!R}^T_+ \big|\ \lambda_t \vartheta_t(\bar{x}) = 0 \ \text{ for all } \ t \in \mathrm{supp}\,\lambda\big\}. \tag{3.12}$$

Corollary 3.3 (subdifferential sum rule involving convex infinite constraints). *Let $\bar{x} \in \Xi \cap \mathrm{dom}\,\vartheta$ in problem* (1.1)*, where the CQC property is satisfied. Then*

$$\partial\big(\vartheta + \delta(\cdot;\Xi)\big)(\bar{x}) = \partial\vartheta(\bar{x}) + \bigcup_{\lambda \in A(\bar{x})} \Big\{ \sum_{t \in \mathrm{supp}\,\lambda} \lambda_t \partial\vartheta_t(\bar{x}) \Big\} + N(\bar{x};\Theta). \tag{3.13}$$

Proof. Take $\varepsilon = 0$ in (3.7). Then $\eta = \nu = \varepsilon_t = 0$ and $\lambda_t \vartheta_t(\bar{x}) = 0$ for all $t \in \mathrm{supp}\,\lambda$, since $\eta, \nu, \varepsilon_t \geq 0$ and $\lambda_t \vartheta_t(\bar{x}) \leq 0$ for all $t \in \mathrm{supp}\,\lambda$. Thus (3.7) becomes (3.13) in this case. $\triangle$

The following corollary provides a new *conjugate sum rule* for the "plus" cost function and infinite constraints in the original DC problem (1.1).

Corollary 3.4 (conjugate sum rule involving convex infinite constraints). *Under the qualification condition CQC in problem* (1.1)*, for all $x^* \in X^*$ we have the equality*

$$\big(\vartheta + \delta(\cdot;\Xi)\big)^*(x^*) = \min_{\lambda \in \widetilde{I\!R}^T_+} \Big(\vartheta + \sum_{t \in \mathrm{supp}\,\lambda} \lambda_t \vartheta_t + \delta(\cdot;\Theta)\Big)^*(x^*). \tag{3.14}$$

Proof. Taking any $x^* \in X^*$ and applying the conjugate representation (3.6) of Theorem 3.2(ii) equivalent to the CQC, we find $\widetilde{\lambda} \in \widetilde{I\!R}^T_+$ and $u, v_t \in X^*$ such that

$$\big(\vartheta + \delta(\cdot;\Xi)\big)^*(x^*) = \vartheta^*(u) + \sum_{t \in \mathrm{supp}\,\widetilde{\lambda}} \widetilde{\lambda}_t \vartheta_t^*(v_t) + \delta^*(\cdot;\Theta)\Big(x^* - u - \sum_{t \in \mathrm{supp}\,\widetilde{\lambda}} \widetilde{\lambda}_t v_t\Big).$$

Thus for each $x \in X$ we have the inequalities

$$\begin{aligned}&\big(\vartheta + \delta(\cdot;\Xi)\big)^*(x^*) \geq \langle u, x\rangle - \vartheta(x) + \sum_{t \in \mathrm{supp}\,\widetilde{\lambda}} \widetilde{\lambda}_t \big[\langle v_t, x\rangle - \vartheta_t(x)\big] \\ &+\Big\langle x^* - u - \sum_{t \in \mathrm{supp}\,\widetilde{\lambda}} \widetilde{\lambda}_t v_t, x\Big\rangle - \delta(x;\Theta) \geq \langle x^*, x\rangle - \Big(\vartheta + \sum_{t \in \mathrm{supp}\,\widetilde{\lambda}} \widetilde{\lambda}_t \vartheta_t + \delta(\cdot;\Theta)\Big)(x),\end{aligned}$$

which imply that

$$\big(\vartheta + \delta(\cdot;\Xi)\big)^*(x^*) \geq \Big(\vartheta + \sum_{t \in \mathrm{supp}\,\widetilde{\lambda}} \widetilde{\lambda}_t \vartheta_t + \delta(\cdot;\Theta)\Big)^*(x^*). \tag{3.15}$$

On the other hand, it easily follows for each $\lambda \in \widetilde{I\!R}^T_+$ that

$$\begin{aligned}\Big(\vartheta + \sum_{t\in\operatorname{supp}\lambda}\lambda_t\vartheta_t + \delta(\cdot;\Theta)\Big)^*(x^*) &= \sup_{x\in\Theta}\Big\{\langle x^*,x\rangle - \Big(\vartheta + \sum_{t\in\operatorname{supp}\lambda}\lambda_t\vartheta_t\Big)(x)\Big\}\\ &\ge \sup_{x\in\Xi}\Big\{\langle x^*,x\rangle - \vartheta(x) - \sum_{t\in\operatorname{supp}\lambda}\lambda_t\vartheta_t(x)\Big\}\\ &\ge \sup_{x\in\Xi}\big\{\langle x^*,x\rangle - \vartheta(x)\big\} = \big(\vartheta + \delta(\cdot;\Xi)\big)^*(x^*).\end{aligned} \tag{3.16}$$

Combining finally (3.15) and (3.16), we get for all $x^* \in X^*$ that

$$\begin{aligned}\big(\vartheta + \delta(\cdot;\Xi)\big)^*(x^*) &= \min_{\lambda\in\widetilde{I\!R}^T_+}\Big(\vartheta + \sum_{t\in\operatorname{supp}\lambda}\lambda_t\vartheta_t + \delta(\cdot;\Theta)\Big)^*(x^*)\\ &= \Big(\vartheta + \sum_{t\in\operatorname{supp}\widetilde{\lambda}}\widetilde{\lambda}_t\vartheta_t + \delta(\cdot;\Theta)\Big)^*(x^*),\end{aligned}$$

which completes the proof of the corollary. △

The next result provides characterizations of the *Farkas-Minkowski constraint qualification* in problem (1.1).

Corollary 3.5 (characterizations of the FM constraint qualification). *The following properties are equivalent for the constraint system σ in* (3.2)*:*

(i) *The Farkas-Minkowski constraint qualification* (3.3) *is satisfied.*

(ii) *For each $x^* \in X^*$ we have the representation*

$$\delta^*(x^*;\Xi) = \min_{\lambda\in\widetilde{I\!R}^T_+}\ \min_{v_t\in X^*,\, t\in T}\Big[\sum_{t\in\operatorname{supp}\lambda}\lambda_t\vartheta_t^*(v_t) + \delta^*(\cdot;\Theta)\Big(x^* - \sum_{t\in\operatorname{supp}\lambda}\lambda_t v_t\Big)\Big].$$

(iii) *For any $\bar{x} \in \Xi$ and $\varepsilon \ge 0$ the collection of ε-normals to Ξ is given by*

$$N_\varepsilon(\bar{x};\Xi) = \bigcup_{\lambda\in\widetilde{I\!R}^T_+}\ \bigcup_{\substack{\nu,\varepsilon_t\ge 0,\ t\in T\\ \sum_{t\in\operatorname{supp}\lambda}\lambda_t[\varepsilon_t - \vartheta_t(\bar{x})]+\nu=\varepsilon}}\Big\{\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial_{\varepsilon_t}\vartheta_t(\bar{x}) + N_\nu(\bar{x};\Theta)\Big\}.$$

Proof. Follows directly from Theorem 3.2 with $\vartheta = 0$. △

The last corollary gives the *exact formula* for computing the normal cone to the set of feasible solutions (3.4) to the original problem (1.1) involving *infinite* constraints in topological vector spaces. It significantly improves the previous result of [13, Corollary 3.4] establishing just an upper estimate of $N(\bar{x};\Xi)$ in Banach spaces.

Corollary 3.6 (normal cone to convex infinite constraints). *Under the Farkas-Minkowski constraint qualification* (3.3) *we have*

$$N(\bar{x};\Xi) = \bigcup_{\lambda\in A(\bar{x})}\Big[\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial\vartheta_t(\bar{x})\Big] + N(\bar{x};\Theta)\ \ \textit{for all}\ \ \bar{x}\in\Xi,$$

where the set of active multipliers is defined in (3.12)*.*

Proof. Follows from Corollary 3.3 with $\vartheta = 0$. △

4 Relations between Various Qualification Conditions

In this section we establish relations between the CQC and FM qualification conditions and some other constraint qualifications for infinite and semi-infinite constraint systems known in the literature. First let us present two sufficient conditions, which can be derived from the results recently established in [9, 10], ensuring the fulfillment of the CQC property in Definition 3.1 for the DC infinite program (1.1) provided that the FM constraint qualification (3.3) holds for the corresponding constraint system (3.2).

Proposition 4.1 (sufficient conditions for the CQC property of DC infinite programs in the presence of the FM constraint qualification). *Suppose that the FM constraint qualification* (3.3) *holds for the constraint system* σ *in* (3.2). *Then the CQC property from Definition* 3.1 *also holds if one of the following assumptions is satisfied:*

(A1) *The "plus" function* ϑ *in the objective of* (1.1) *is continuous at some point of the feasible set* Ξ *defined in* (3.4).

(A2) *The conic hull* $\mathrm{cone}[\mathrm{dom}(\vartheta - \Xi)]$ *is a closed subspace of* X.

Observe again that the FM qualification condition, in contrast to the CQC one, concerns only the *constraint* convex system σ in (1.1) but *not the cost function* of the original DC program. In the rest of this section we consider relations between the FM qualification condition and some other constraint qualifications for infinite convex systems of type (3.2). The results obtained in this way apply to the CQC property of DC and convex infinite programs via the CQC–FM relations given in Proposition 4.1.

Let us next consider behavior of the FM constraint qualification (3.3) under an appropriate *linearization* of the original constraint system σ from (3.2). We construct the *linearized constraint system* as follows:

$$\sigma_1 := \begin{cases} \langle x_t^*, x\rangle \le \vartheta_t^*(x_t^*) + \alpha,\ x_t^* \in \mathrm{dom}\,\vartheta_t^*,\ t \in T,\ \alpha \ge 0, \\ \langle x^*, x\rangle \le \delta^*(x^*;\Theta) + \beta,\ x^* \in \mathrm{dom}\,\delta^*(\cdot;\Theta),\ \beta \ge 0. \end{cases} \tag{4.1}$$

Observe that the linearized constraint system σ_1 in (4.1) corresponds to the *same solution set* Ξ from (3.4) as the original convex constraint system σ defined in (3.2). The following proposition establishes the *equivalence*, in the case of $X = I\!R^n$, between the FM qualification condition for the convex constraint system σ and the *linear FM qualification condition* for the linearized system σ_1 in the sense defined at the end of Section 2.

Proposition 4.2 (FM constraint qualification under linearization in semi-infinite programming). *Let* $X = I\!R^n$. *Then the convex constraint system* (3.2) *satisfies the Farkas-Minkowski qualification condition if and only if the corresponding linearized constraint system* (4.1) *is linearly FM.*

Proof. Observe that $\vartheta_t^{**} = \vartheta_t$ for all $t \in T$, since each constraint function ϑ_t is assumed to be proper, l.s.c., and convex function. Therefore for each $t \in T$ and $x \in X$ we have the equivalent relations

$$\begin{aligned} \vartheta_t(x) \le 0 \quad &\Longleftrightarrow \vartheta_t^{**}(x) \le 0 \Longleftrightarrow \langle x_t^*, x\rangle - \vartheta_t^*(x_t^*) \le 0 \ \text{ for all } \ x_t^* \in \mathrm{dom}\,\vartheta_t^* \\ &\Longleftrightarrow \langle x_t^*, x\rangle \le \vartheta_t^*(x_t^*) \ \text{ for all } \ x_t^* \in \mathrm{dom}\,\vartheta_t^* \\ &\Longleftrightarrow \langle x_t^*, x\rangle \le \vartheta_t^*(x_t^*) + \alpha \ \text{ whenever } \ x_t^* \in \mathrm{dom}\,\vartheta_t^* \ \text{ and } \ \alpha \ge 0. \end{aligned}$$

Taking now $x \in \Theta$ and expressing this as $\delta(x;\Theta) \le 0$, via the indicator function, we get

$$\begin{aligned}\delta(x;\Theta) \le 0 \quad &\Longleftrightarrow \langle x^*, x\rangle \le \delta^*(x^*;\Theta) \ \text{ for all } \ x^* \in \operatorname{dom}\delta^*(\cdot;\Theta)\\ &\Longleftrightarrow \langle x^*, x\rangle \le \delta^*(x^*;\Theta) + \beta \ \text{ if } \ x^* \in \operatorname{dom}\delta^*(\cdot;\Theta),\ \beta \ge 0.\end{aligned}$$

The corresponding cone (2.10) for the linearized system σ_1 is computed in [9] as

$$\operatorname{cone}\Big\{\bigcup_{t\in T}\big[\operatorname{epi}\vartheta_t^* \cup \operatorname{epi}\delta^*(\cdot;\Theta)\big]\Big\} = \operatorname{cone}\Big\{\bigcup_{t\in T}\operatorname{epi}\vartheta_t^*\Big\} + \operatorname{epi}\delta^*(\cdot;\Theta) = K, \tag{4.2}$$

where the cone K is defined in (3.1). Furthermore, it is shown in [10, Proposition 1] that the linear system σ_1 is *linearly FM* in the sense discussed at the end of Section 2 *if and only if* the cone K is weak* closed in $X^* \times I\!R$. Taking finally into account the mentioned in Section 2 characterization of the linear FM property via the cone (2.10) in $I\!R^n$, we conclude that the latter property for σ_1 is *equivalent* to the FM constraint qualification for σ. This completes the proof of the proposition. $\triangle$

Now we take a closer look at several other types of constraint qualification conditions and relations between them for infinite systems of convex inequalities (3.2) in general topological vector spaces. Observe first of all that it follows from representation (4.2) of the cone K from the proof of Proposition 4.2 that the FM constraint qualification is *equivalent* to the so-called *conical epigraph hull property* recently introduced in [26] and applied therein to problems of constrained optimization.

Another dual-type qualification condition for infinite systems of convex inequalities is introduced in [10] under the name of *locally Farkas-Minkowski* (*LFM*) *constraint qualification.* The constraint system σ in (3.2) is said to be *locally FM* at $x \in \Xi$ if

$$N(x;\Xi) = N(x;\Theta) + \operatorname{cone}\Big\{\bigcup_{t\in T(x)}\partial\vartheta_t(x)\Big\}, \tag{4.3}$$

where $T(x) := \{t \in T|\ \vartheta_t(x) = 0\}$ signifies the *active index set* for σ at x. For standard (finite) convex systems, condition (4.3) is introduced in [19, p. 307] under the name of *basic constraint qualification* (BCQ); see also [15, 17] for its developments and applications in the case of convex systems of semi-infinite programming. A counterpart of the LFM condition is used in [2] in the cone-constrained setting; see Section 6 for more discussions on the latter class of convex programs.

The following relations between the FM and LFM qualification conditions can be derived from the results of [10, 26].

Proposition 4.3 (relations between FM and LFM constraint qualifications for infinite programs). *Let* $\Theta = X$ *in* (3.2) *and* (3.4)*. Then the constraint system* σ *is LFM at all* $x \in \Xi$ *if it satisfies the FM qualification condition. The converse implication holds provided that* $\operatorname{dom}\delta^*(\cdot;\Xi) \subset \operatorname{rge}\partial(\cdot;\Xi)$*, where* rge *stands as usual for the range/image of the subdifferential mapping under consideration.*

We refer the reader to [10, 25, 26] for other modifications of FM and LFM constraint qualifications for infinite systems of convex inequalities and relations between them.

Finally in this section, we compare the *dual-type* Farkas-Minkowski qualification condition for infinite systems of convex inequalities with a *primal* qualification condition of the *Slater type* in the case of finite-dimensional spaces. The latter condition for convex semi-infinite programs is introduced in [15, 27]. We prove that the FM condition is always *less restrictive* than the Slater one in finite dimensions and present a particular example showing that this implication is generally *strict*.

Definition 4.4 (Slater constraint qualification for semi-infinite programs). *Let* $\Theta = X = I\!R^n$ *for the constraint system* σ *in* (3.2). *We say that* σ *satisfies the* SLATER QUALIFICATION CONDITION *if the following properties hold:*

(a) *the index set* T *is a compact subset of a finite-dimensional space;*

(b) *the constraint family* $\vartheta_t(x)$ *is continuous with respect to both variables* (t,x) *on the set* $T \times I\!R^n$*;*

(c) *there is a point* $x_0 \in I\!R^n$ *such that* $\vartheta_t(x_0) < 0$ *for all* $t \in T$*.*

The next theorem is the main result of this section establishing the afore-mentioned relation between the FM and Slater qualification conditions in the setting of Definition 4.4.

Theorem 4.5 (relation between Farkas-Minkowski and Slater qualification conditions for semi-infinite programs). *Let* σ *be the infinite system of convex inequalities* (3.2) *in the framework of Definition* 4.4. *Assume in addition that the set of feasible solutions* Ξ *in* (3.4) *with* $\Theta = I\!R^n$ *is bounded. Then the Slater qualification condition for* σ *implies the Farkas-Minkowski constraint qualification in the sense of* (3.3).

Proof. Along with the linearization σ_1 from (4.1) of the original convex system σ in (3.2), consider another linearization of σ defined by

$$\sigma_2 := \big\{\langle u,x\rangle \le \langle u,y\rangle - \vartheta_t(y),\ (t,y) \in T \times I\!R^n,\ u \in \partial\vartheta_t(y)\big\}. \tag{4.4}$$

Due to assumed continuity of the constraint functions $\vartheta_t(\cdot)$ as $t \in T$, their are subdifferentiable and satisfy the relation

$$\vartheta_t^*(u) = \langle u,y\rangle - \vartheta_t(y) \ \text{ whenever } \ u \in \partial\vartheta_t(y), \quad (t,y) \in T \times I\!R^n.$$

The latter yields that $u \in \partial\vartheta_t(y)$, and thus the inequality $\langle u,x\rangle \le \langle u,y\rangle - \vartheta_t(y)$ reduces to $\langle u,x\rangle \le \vartheta_t^*(u)$. Therefore we have $\sigma_2 \subset \sigma_1$.

Furthermore, it follows from [27, Theorem 4.5] that the Slater condition for σ implies under the assumptions made that the linear system σ_2 built in (4.4) is *linearly FM* in the sense defined at the end of Section 2. By $\sigma_2 \subset \sigma_1$ this ensures the linear FM property of the linearization σ_1 from (4.1). The latter implies the Farkas-Minkowski qualification condition for σ by Proposition 4.2, which completes the proof of the theorem. △

The following example shows that the Farkas-Minkowski qualification condition for σ is *strictly better* than the Slater one.

Example 4.6 (the Farkas-Minkowski constraint qualification strictly improves the Slater one). Let the constraint system σ in (3.2) is given by

$$\vartheta_t(x_1,x_2) := \max\big\{-x_1, 0\big\} - tx_2 \ \text{ with } \ t \in T = [0,1) \ \text{ and } \ (x_1,x_2) \in I\!R^2.$$

It is clear that the Slater qualification condition from Definition 4.4 does not hold in this example, since first T is not compact and then there is no $x_0 = (x_{01}, x_{02}) \in I\!R^2$ such that $\vartheta_t(x_{01}, x_{02}) < 0$ for all $t \in [0, 1)$. On the other hand, it is easy to verify that the Farkas-Minkowski constraint qualification is satisfied. Indeed,

$$\operatorname{epi}\vartheta_t^* = [-1,0] \times \{-t\} \times I\!R_+ \ \text{ and hence } \ \operatorname{cone}\Big(\bigcup_{t\in T}\operatorname{epi}\vartheta_t^*\Big) = I\!R_-^2 \times I\!R_+,$$

which is a closed subset of $I\!R^3$.

5 Optimality Conditions for DC Infinite Programs

In this section we employ the CQC property to derive *qualified* optimality conditions in DC and convex problems of infinite programming in topological vector spaces. We obtain various results in this direction: *necessary optimality conditions* as well as *necessary and sufficient optimality conditions* for both *local* and *global* minimizers. Let us start with the following necessary and sufficient conditions for global minimizers in DC infinite programs expressed in terms of *approximate subgradients* for convex functions and *approximate normals* to convex sets defined in Section 2.

Theorem 5.1 (qualified necessary and sufficient optimality conditions for global minimizers in DC infinite programs). *Let the DC infinite program* (1.1) *satisfy the CQC qualification condition. Then $\bar{x} \in \Xi \cap \operatorname{dom}\vartheta$ is a global minimizer for* (1.1) *if and only if for each $\varepsilon \ge 0$ and $x^* \in \partial_\varepsilon \theta(\bar{x})$ there are $\lambda \in \widetilde{I\!R}_+^T$, $\eta, \nu, \varepsilon_t \ge 0$ as $t \in \operatorname{supp}\lambda$ such that*

$$\eta + \sum_{t\in\operatorname{supp}\lambda} \lambda_t\big[\varepsilon_t - \vartheta_t(\bar{x})\big] + \nu = \varepsilon \ \textit{and} \tag{5.1}$$

$$x^* \in \partial_\varepsilon\vartheta(\bar{x}) + \sum_{t\in\operatorname{supp}\lambda} \lambda_t \partial_{\varepsilon_t}\vartheta_t(\bar{x}) + N_\nu(\bar{x};\Theta). \tag{5.2}$$

Proof. Observe that the original DC infinite program (1.1) can be equivalently rewritten in the following *unconstrained form*:

$$\text{minimize } \ \big(\vartheta + \delta(\cdot;\Xi)\big)(x) - \theta(x), \quad x \in X, \tag{5.3}$$

where Ξ is the feasible solution set defined in (3.4). The well-known characterization of optimal solutions to unconstrained DC problems presented in [18, 19] allows us to conclude that $\bar{x}$ is a *global minimizer* for (5.3) *if and only if* for each $\varepsilon \ge 0$ we have the inclusion

$$\partial_\varepsilon\theta(\bar{x}) \subset \partial_\varepsilon\big(\vartheta + \delta(\cdot;\Xi)\big)(\bar{x}). \tag{5.4}$$

By the assumed CQC property in (1.1) we get from the calculus rule of Theorem 3.2(iii) that representation (3.7) for $\partial_\varepsilon(\vartheta + \delta(\cdot;\Xi))(\bar{x})$ holds whenever $\varepsilon \ge 0$. Thus the characterization (5.4) of global solutions to the unconstrained problem (5.3) and the above calculus rule yield that $\bar{x}$ is a *global minimizer* for the original DC problem (1.1) with infinite constraints *if and only if* the inclusion

$$\partial_\varepsilon\theta(\bar{x}) \subset \bigcup_{\lambda\in\widetilde{I\!R}^T_+} \bigcup_{\substack{\eta,\nu,\varepsilon_t\ge 0,\ t\in T\\ \eta+\sum_{t\in\operatorname{supp}\lambda}\lambda_t[\varepsilon_t-\vartheta_t(\bar{x})]+\nu=\varepsilon}} \Big\{\partial_\eta\vartheta(\bar{x}+\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial_{\varepsilon_t}\vartheta_t(\bar{x})+N_\nu(\bar{x};\Theta)\Big\}$$

holds for each $\varepsilon \ge 0$. The latter means that for every subgradient $x^* \in \partial_\varepsilon\theta(\bar{x})$ there exist multipliers $\lambda \in \widetilde{I\!R}^T_+$ and numbers $\eta, \nu, \varepsilon_t \ge 0$ as $t \in \operatorname{supp}\lambda$ satisfying both conditions (5.1) and (5.2). This completes the proof of the theorem. △

The next theorem establishes *necessary* optimality conditions for *local* solutions to the DC infinite program (1.1) in general topological vector spaces. It extends our recent result in [13, Theorem 3.2] derived by a different method in the Banach space setting.

Theorem 5.2 (necessary optimality conditions for local solutions to DC infinite programs). *Let $\bar{x} \in \Xi \cap \operatorname{dom}\vartheta$ be a local minimizer for the DC program* (1.1) *under the CQC qualification condition. Then we have the inclusion*

$$\partial\theta(\bar{x}) \subset \partial\vartheta(\bar{x}) + \bigcup_{\lambda\in A(\bar{x})}\Big[\sum_{t\in\operatorname{supp}\,\lambda}\lambda_t\partial\vartheta_t(\bar{x})\Big] + N(\bar{x};\Theta), \tag{5.5}$$

where the set of active constraint multipliers $A(\bar{x})$ is defined in (3.12).

Proof. Since $\bar{x}$ is a local minimizer for (1.1), it is also a local minimizer for the *unconstrained* DC problem (5.3). Employing a standard necessary optimality condition for unconstrained DC programs (see, e.g., [8, 19]), we get the inclusion

$$\partial\theta(\bar{x}) \subset \partial\big(\vartheta + \delta(\cdot;\Xi)\big)(\bar{x}). \tag{5.6}$$

The subdifferential sum rule (3.13) from Corollary 3.3 ensures, under the CQC qualification condition, the equality representation for $\partial(\vartheta+\delta(\cdot;\Xi))(\bar{x})$. Combining the latter with (5.6), we arrive at (5.5) and complete the proof of the theorem. △

Finally in this section, consider a particular case of the DC problem (1.1) with $\theta(x) \equiv 0$ when (1.1) reduces to the *convex program* involving *infinite constraints*:

$$\begin{cases} \text{minimize } \ \vartheta(x) \ \text{ subject to} \\ \vartheta_t(x) \le 0, \ \ t \in T, \ \text{ and } \ x \in \Theta. \end{cases} \tag{5.7}$$

In this case, the necessary condition (5.5) for local minimizers in Theorem 5.2 reads as

$$0 \in \partial\vartheta(\bar{x}) + \sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial\vartheta_t(\bar{x}) + N(\bar{x};\Theta). \tag{5.8}$$

The next result, which extends the recent one from [13, Theorem 3.5] obtained in Banach spaces, shows that condition (5.8) is *necessary and sufficient* for *global* solutions to the convex infinite program (5.7).

Theorem 5.3 (necessary and sufficient optimality conditions for convex infinite programs). *Let the qualification condition CQC hold for the convex program* (5.7). *Then* $\bar{x} \in \Xi \cap \operatorname{dom} \vartheta$ *is a (global) solution to* (5.7) *if and only if there is* $\lambda \in A(\bar{x})$ *such that inclusion* (5.8) *is satisfied.*

Proof. The *necessity* of (5.8) for the global optimality of $\bar{x}$ in (5.7) follows immediately from Theorem 5.2 with $\theta(x) \equiv 0$. Let us prove the *sufficiency* of (5.8) for the global optimality of $\bar{x}$ in the convex program (5.7). To proceed, we suppose that (5.8) holds with some $\lambda \in A(\bar{x})$ and so find $u \in X^*$ such that $-u \in N(\bar{x}; \Theta)$ and

$$u \in \partial\vartheta(\bar{x}) + \sum_{t \in \operatorname{supp} \lambda} \lambda_t \partial\vartheta_t(\bar{x}) \subset \partial\big(\vartheta + \sum_{t \in \operatorname{supp} \lambda} \lambda_t \vartheta_t\big)(\bar{x}).$$

By the subgradient definition (2.5) as $\varepsilon = 0$ the latter implies that

$$\vartheta(x) + \sum_{t \in \operatorname{supp} \lambda} \lambda_t \vartheta_t(x) \ge \vartheta(\bar{x}) + \sum_{t \in \operatorname{supp} \lambda} \lambda_t \vartheta_t(\bar{x}) + \langle u, x - \bar{x} \rangle, \quad x \in X.$$

Taking into account that $\vartheta_t(\bar{x}) = 0$ for all $t \in \operatorname{supp} \lambda$ and that $-u \in N(\bar{x}; \Theta)$, we get

$$\vartheta(x) + \sum_{t \in \operatorname{supp} \lambda} \lambda_t \vartheta_t(x) - \vartheta(\bar{x}) \ge \langle u, x - \bar{x} \rangle \ge 0 \;\text{ whenever }\; x \in \Theta. \tag{5.9}$$

Furthermore, if $x \in \Xi$ satisfies all the constraints in (5.7), then (5.9) gives

$$\vartheta(x) \ge \vartheta(x) + \sum_{t \in \operatorname{supp} \lambda} \lambda_t \vartheta_t(x) \ge \vartheta(\bar{x}),$$

which means that $\bar{x}$ is an optimal solution to (5.7). △

6 DC Programs with Cone Constraints

In this section we discuss a remarkable class of *cone-constrained DC programs*, which are of significant interest for optimization theory and its applications and can be reduced to the general class of DC *infinite* programs of type (1.1). In this way we present certain specifications for the case of cone-constrained programs of some results for general DC infinite problems obtained in the previous sections, which will be applied in Sections 7 and 8 to DC programs with more particular forms of cone constraints.

The *cone-constrained DC programs* under consideration are written generally as:

$$\begin{cases} \text{minimize } \; \vartheta(x) - \theta(x) \; \text{ subject to} \\ f(x) \in -S \subset Y \; \text{ and } \; x \in \Theta \subset X, \end{cases} \tag{6.1}$$

where $\vartheta, \theta, \Theta$ satisfy the standing assumptions of Section 1, X and Y are locally convex Hausdorff topological vector spaces, S is a closed *convex cone* in Y, and $f\colon X \to Y$ is a continuous S-*convex mapping* in the sense that

$$f\big(\nu x_1 + (1-\nu)x_2\big) - \nu f(x_1) - (1-\nu) f(x_2) \in -S \;\text{ for all }\; x_1, x_2 \in X \;\text{ and }\; \nu \in [0,1].$$

The *feasible set* in the cone-constrained DC problem (6.1) is denoted by

$$\widehat{\Xi} := \big\{x \in X \big|\ f(x) \in -S,\ x \in \Theta\big\}. \tag{6.2}$$

Observe that $f(x) \in -S$ *if and only if*

$$\vartheta_\lambda(x) := \lambda f(x) \le 0 \ \text{ whenever } \ \lambda \in S^+, \tag{6.3}$$

where S^+ stands for the *positive dual cone* to S defined by

$$S^+ := \big\{\lambda \in Y^* \big|\ \lambda s \ge 0 \ \text{ for all } \ s \in S\big\} \tag{6.4}$$

with the simplified notation for the *inner product* $\lambda s := \langle \lambda, s\rangle$ in (6.3), (6.4), and in what follows. Thus the cone-constrained problem (6.1) can be *equivalently* rewritten as:

$$\begin{cases} \text{minimize } \ \vartheta(x) - \theta(x) \ \text{ subject to} \\ \vartheta_\lambda(x) \le 0,\ \lambda \in S^+, \ \text{ and } \ x \in \Theta, \end{cases} \tag{6.5}$$

which is obviously a *special case* of the underlying DC infinite program (1.1).

The *CQC qualification condition* from Definition 3.1 reads in case (6.5) as

$$\text{epi}\,\vartheta^* + \bigcup_{\lambda \in S^+} \text{epi}\,(\lambda f)^* + \text{epi}\,\delta^*(\cdot;\Theta) \text{ is weak}^* \text{ closed in } X^* \times I\!R \tag{6.6}$$

while the *Farkas-Minkowski constraint qualification* reduces to

$$\bigcup_{\lambda \in S^+} \text{epi}\,(\lambda f)^* + \text{epi}\,\delta^*(\cdot;\Theta) \ \text{ is weak}^* \text{ closed } \ X^* \times I\!R, \tag{6.7}$$

where the set $\bigcup_{\lambda \in S^+} \text{epi}\,(\lambda f)^*$ in (6.6) and (6.7) is a closed and convex cone [20].

The FM condition (6.7) for the cone-constrained systems in (6.1) has been first introduced in [21] under the name of the CCCQ condition and then was rediscovered in [1]. It was further used in many publications; see, e.g., [2, 10, 11, 12] and the references therein. The CQC qualification condition (6.6) seems to be new for the DC and convex cone-constrained problems under consideration.

The next proposition taken from [21] gives sufficient conditions of the *primal type* ensuring the fulfillment of the FM qualification condition (3.3) and therefore our major CQC property for the cone-constrained programs (6.1) due to Proposition 4.1.

Proposition 6.1 (primal sufficient conditions for the FM constraint qualification in cone-constrained programming). *The FM constraint qualification* (6.7) *holds for the cone-constrained program* (6.1) *provided that one of the following primal qualification conditions is satisfied:*

(CQ1) $0 \in \text{icr}\big(f(\Theta) + S\big)$ *and the affine hull* $\text{aff}\big(f(\Theta) + S\big)$ *is a closed subspace of* X.

(CQ2) $0 \in \text{sqri}\big(f(\Theta) + S\big)$.

(CQ3) $0 \in \text{core}\big(f(\Theta) + S\big)$.

(CQ4) *There is* $x_0 \in \Theta$ *such that* $-f(x_0) \in \text{int}\, S$.

We can see that the primal qualification conditions (CQ1)—(CQ4) are of *Slater*, or of (*generalized*) *interiority*, type; cf. Definition 4.4 in a different setting of semi-infinite programming. In [21], the reader can find detailed discussions on these qualification conditions and their relations with dual-type constraint qualifications. In particular, it is shown therein that the FM constraint qualification (6.7) is *strictly weaker* (less restrictive) than the primal-type conditions (CQ1)–(CQ4) for the cone-constrained problems.

As an immediate consequence of Proposition 4.1 and Proposition 6.1 we get the following sufficient conditions ensuring the CQC property in the *DC cone-constrained* programs (6.1).

Corollary 6.2 (sufficient conditions for the CQC property in cone-constrained programs). *Suppose in the framework of* (6.1) *that one of the qualification conditions* (CQ1)–(CQ4) *is satisfied and that either assumption* (A1) *or assumption* (A2) *from Proposition* 4.1 *holds with the replacement of* Ξ *by the feasible set* $\widehat{\Xi}$ *to* (6.1) *defined in* (6.2). *Then we have the CQC property* (6.6) *in the cone-constrained problem* (6.1).

Note that for the cone-constrained DC program (6.1) the characterizations of the CQC property given in Theorem 3.2 reduce to the ones recently obtained in [11, Theorem 3.1]. Let us now establish a counterpart of the sum rule (3.13) from Corollary 3.3 in the case of cone constraints in problem (6.1).

Proposition 6.3 (subdifferential sum rule for cone-constrained systems). *Let the CQC qualification condition* (6.6) *hold for the cone-constrained problem* (6.1). *Then the subdifferential sum rule* (3.13) *in the specified setting of* (6.5) *is equivalent to*

$$\partial\big(\vartheta+\delta(\cdot;\widehat{\Xi})\big)(\bar{x})=\partial\vartheta(\bar{x})+\bigcup_{\lambda\in\Lambda(\bar{x})}\partial(\lambda f)(\bar{x})+N(\bar{x};\Theta), \tag{6.8}$$

where $\Lambda(\bar{x}):=\{\lambda\in S^+|\ \lambda f(\bar{x})=0\}$.

Proof. First observe that in the cone-constrained setting of (6.5) the right-hand side of the subdifferential rule (3.13) becomes

$$\partial\vartheta(\bar{x})+\bigcup_{\gamma\in\widehat{A}(\bar{x})}\Big\{\sum_{\lambda\in\operatorname{supp}\gamma}\gamma_\lambda\partial\vartheta_\lambda(\bar{x})\Big\}+N(\bar{x};\Theta), \tag{6.9}$$

where the set of active constraint multipliers (3.12) equivalently reduces to

$$\widehat{A}(\bar{x}):=\big\{\gamma=(\gamma_\lambda)\in\widetilde{I\!R}^S_+\big|\ \gamma_\lambda\vartheta_\lambda(\bar{x})=0\ \text{ for all }\ \lambda\in\operatorname{supp}\gamma\big\}. \tag{6.10}$$

For each $\gamma\in\widehat{A}(\bar{x})$ in (6.10) denote

$$\widetilde{\lambda}:=\sum_{\lambda\in\operatorname{supp}\gamma}\gamma_\lambda\lambda$$

and notice that $\widetilde{\lambda}\in S^+$, since S^+ is a cone. Taking the notation in (6.3) into account, we have therefore the equalities

$$\widetilde{\lambda}f(\bar{x})=\sum_{\lambda\in\operatorname{supp}\gamma}\gamma_\lambda\big(\lambda f\big)(\bar{x})=\sum_{\lambda\in\operatorname{supp}\gamma}\gamma_\lambda\vartheta_\lambda(\bar{x})=0, \tag{6.11}$$

which imply that $\widetilde{\lambda} \in \Lambda(\bar{x})$ for the multiplier set defined in the proposition. It follows from the continuity of the function $(\lambda f)(x)$ for all $\lambda \in \operatorname{supp}\gamma$ that

$$\sum_{\lambda \in \operatorname{supp}\gamma} \gamma_\lambda \partial \vartheta_\lambda(\bar{x}) = \sum_{\lambda \in \operatorname{supp}\gamma} \gamma_\lambda \partial\big(\lambda f\big)(\bar{x}) = \partial\big(\widetilde{\lambda} f\big)(\bar{x}). \tag{6.12}$$

Combining finally (6.11) and (6.12), we get

$$\partial\vartheta(\bar{x}) + \bigcup_{\gamma \in \widehat{A}(\bar{x})} \Big\{ \sum_{\lambda \in \operatorname{supp}\gamma} \gamma_\lambda \partial\vartheta_\lambda(\bar{x}) \Big\} + N(\bar{x};\Theta) \subset \partial\vartheta(\bar{x}) + \bigcup_{\lambda \in \Lambda(\bar{x})} \partial\big(\lambda f\big)(\bar{x}) + N(\bar{x};\Theta),$$

while the opposite inclusion is obvious. This completes the proof of the proposition. △

Let us finally present specified versions of *optimality conditions* from Section 5 in the case of cone-constrained programs (6.1). Using the above arguments, we similarly justify that the results of Theorem 5.1 and Theorem 5.2 can be combined in the following proposition established recently in [11, Theorem 6.1].

Proposition 6.4 (optimality conditions for DC cone-constrained programs). *Assume that the CQC property* (6.6) *holds for the cone-constrained program* (6.1). *Then* $\bar{x} \in \widehat{\Xi} \cap \operatorname{dom}\vartheta$ *is a global solution to* (6.1) *if and only if for each* $\varepsilon \geq 0$ *and* $x^* \in \partial_\varepsilon\theta(\bar{x})$ *there exist* $\lambda \in S^+$ *and* $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$ *satisfying the conditions*

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon + \lambda f(\bar{x}) \quad \textit{and}$$

$$x^* \in \partial_{\varepsilon_1}\vartheta(\bar{x}) + \partial_{\varepsilon_2}(\lambda f)(\bar{x}) + N_{\varepsilon_3}(\bar{x};\Theta).$$

If furthermore $\bar{x} \in \widehat{\Xi} \cap \operatorname{dom}\vartheta$ *is a local solution to* (6.1), *then for each* $x^* \in \partial\theta(\bar{x})$ *there exists* $\lambda \in S^+$ *such that*

$$x^* \in \partial\vartheta(\bar{x}) + \partial\big(\lambda f\big)(\bar{x}) + N(\bar{x};\Theta) \quad \textit{and} \quad \lambda f(\bar{x}) = 0.$$

Observe that for *convex* programs with cone constraints, i.e., for problems in form (6.1) with $\theta(x) \equiv 0$, the last condition of Proposition 6.4 reduces to the familiar *Karush-Kuhn-Tucker form*: there is $\lambda \in S^+$ such that

$$0 \in \partial\vartheta(\bar{x}) + \partial\big(\lambda f\big)(\bar{x}) + N(\bar{x};\Theta), \quad \lambda f(\bar{x}) = 0,$$

which is *necessary and sufficient* for the global optimality in this problem; cf. Theorem 5.3.

7 DC Infinite Programs under Polyhedrality

This section is devoted to studying a special class of DC infinite programs, where the "minus" term θ in the objective of (1.1) is described by a function of the *polyhedral convex* type. For simplicity we restrict ourselves to the case of *cone-constrained polyhedral* DC problems, i.e., DC programs of the afore-mentioned polyhedral type with constraints written in the form of (6.1). The results obtained can be easily extended to polyhedral DC programs with general *infinite* convex constraints as in (1.1). Let us first recall the definition of polyhedral convex functions; see, e.g., [18, 28].

A real-valued function $\theta\colon X \to I\!R$ is *polyhedral convex*, or *piecewise affine convex*, if it can be represented in the form

$$\theta(x) = \max_{i\in I}\big\{\langle a_i^*, x\rangle + b_i\big\}, \quad x \in X, \tag{7.1}$$

where $I = \{1, 2, \ldots, n\}$, $n \in I\!N$ with $a_1^*, a_2^*, \ldots, a_n^* \in X^*$ and $b_1, \ldots, b_n \in I\!R$. Every polyhedral convex function is obviously convex and continuous on X.

Thus the *polyhedral DC cone-constrained program* under consideration in this section is formulated as follows:

$$\begin{cases} \text{minimize } \ \vartheta(x) - \max\limits_{i\in I}\big\{\langle a_i^*, x\rangle + b_i\big\} \ \text{ subject to} \\ f(x) \in -S \subset Y \ \text{ and } \ x \in \Theta \subset X, \end{cases} \tag{7.2}$$

where ϑ, f, S, X, Y are the same as in Section 6, while the "minus" function in the objective of (7.2) is given in the *polyhedral form* (7.1). We keep the notation $\widehat{\Xi} = f^{-1}(-S) \cap \Theta$ for the feasible set (6.2) in (7.2) and denote further

$$\theta_i(x) := \langle a_i^*, x\rangle + b_i \ \text{ and } \ I(x) := \big\{i \in I \big|\ \theta_i(x) = \theta(x)\big\}, \quad x \in X. \tag{7.3}$$

The following main result of this section gives *necessary and sufficient* conditions for *global solutions* to the polyhedral DC programs (7.2).

Theorem 7.1 (necessary and sufficient conditions for global optimality in polyhedral DC programs problem). *Let the CQC qualification condition* (6.6) *hold in problem* (7.2). *Then* $\bar{x} \in \widehat{\Xi} \cap \operatorname{dom}\vartheta$ *is a global solution to* (7.2) *if and only if for each* $i \in I$ *there exist multiplier* $\lambda_i \in S^+$ *and numbers* $\mu_i, \nu_i, \rho_i \geq 0$ *such that*

$$\mu_i + \nu_i + \rho_i = \lambda_i f(\bar{x}) + \theta(\bar{x}) - \theta_i(\bar{x}) \ \ \textit{and} \tag{7.4}$$

$$a_i^* \in \partial_{\mu_i} f(\bar{x}) + \partial_{\nu_i}\big(\lambda_i f\big)(\bar{x}) + N_{\rho_i}(\bar{x}; \Theta), \tag{7.5}$$

where the polyhedral functions $\theta(x)$ *and* $\theta_i(x)$ *are defined in* (7.1) *and* (7.3)*, respectively.*

Proof. Let us first prove the *necessity* part. If $\bar{x} \in \widehat{\Xi} \cap \operatorname{dom}\vartheta$ is a global solution to (7.2), then for each $i \in I$ we have

$$\begin{aligned} \theta(x) - \theta(\bar{x}) \ &\geq \theta_i(x) - \theta(\bar{x}) \geq \langle a_i^*, x\rangle + b_i - \theta(\bar{x}) \\ &\geq \langle a_i^*, x - \bar{x}\rangle + \langle a_i^*, \bar{x}\rangle + b_i - \theta(\bar{x}) \\ &\geq \langle a_i^*, x - \bar{x}\rangle + \theta_i(\bar{x}) - \theta(\bar{x}), \end{aligned}$$

which imply that $a_i^* \in \partial_{\varepsilon_i}\theta(\bar{x})$ with $\varepsilon_i := \theta(\bar{x}) - \theta_i(\bar{x}) \geq 0$. It follows from Proposition 6.4 by the assumed CQC qualification condition in (7.2) that there are $\lambda_i \in S^+$ and $\mu_i, \eta_i, \rho_i \geq 0$ satisfying the necessary optimality conditions (7.4) and (7.5). .

To justify next the *sufficiency* part of the theorem, for each $i \in I$ we take $\lambda_i \in S^+$ and $\mu_i, \nu_i, \rho_i \geq 0$ satisfying conditions (7.4) and (7.5). Denoting

$$\alpha_i := \mu_i + \nu_i + \rho_i = \lambda_i f(\bar{x}) + \theta(\bar{x}) - \theta_i(\bar{x}), \quad i \in I, \tag{7.6}$$

we thus have the inclusion

$$\partial_{\mu_i}\vartheta(\bar{x})+\partial_{\nu_i}\big(\lambda_i f\big)(\bar{x})+N_{\rho_i}(\bar{x};\Theta)\subset\partial_{\alpha_i}\big(\vartheta+\lambda_i f+\delta(\cdot;\Theta)\big)(\bar{x}),$$

which yields by definition (2.3) of the approximate subdifferentials that

$$\vartheta(x)+\lambda_i f(x)-\vartheta(\bar{x})-\lambda_i f(\bar{x})\ge\langle a_i^*,x-\bar{x}\rangle-\alpha_i\ \text{ whenever }\ x\in\Theta.$$

Since $\lambda_i f(x)\le 0$ for each $x\in\widehat{\Xi}$, the latter inequality implies that

$$\begin{aligned}\vartheta(x)\ &\ge\langle a_i^*,x-\bar{x}\rangle+\vartheta(\bar{x})-\alpha_i+\lambda_i f(\bar{x})=\langle a_i^*,x-\bar{x}\rangle+\vartheta(\bar{x})-\theta(\bar{x})+\theta_i(\bar{x})\\&\ge\langle a_i^*,x-\bar{x}\rangle+\langle a_i^*,\bar{x}\rangle+b_i+\vartheta(\bar{x})-\theta(\bar{x})\\&\ge\langle a_i^*,x\rangle+b_i+\vartheta(\bar{x})-\theta(\bar{x})=\theta_i(x)+\vartheta(\bar{x})-\theta(\bar{x})\end{aligned}$$

and therefore $\vartheta(x)-\theta_i(x)\ge\vartheta(\bar{x})-\theta(\bar{x})$ for all $i\in I$ and $x\in\widehat{\Xi}$. Hence we conclude that

$$\vartheta(x)-\theta(x)=\vartheta(x)-\max_{i\in I}\theta_i(x)\ge\vartheta(\bar{x})-\theta(\bar{x})\ \text{ for all }\ x\in\widehat{\Xi},$$

which justifies the optimality of $\bar{x}$ in (7.2) and completes the proof of the theorem. △

It is not hard to observe that the necessary optimality conditions of Theorem 7.1 can be equivalently formulated in somewhat different and more convenient form involving the active index set $I(\bar{x})$ from (7.3).

Theorem 7.1' (equivalent necessary and sufficient conditions for global optimality in polyhedral DC problems). *Let the CQC qualification condition* (6.6) *be satisfied in* (7.2)*. Then $\bar{x}$ is a global solution to* (7.2) *if and only if the following statements hold:*

(i) *For any $i\in I(\bar{x})$ there is $\lambda_i\in S^+$ such that*

$$a_i^*\in\partial\vartheta(\bar{x})+\partial\big(\lambda_i f\big)(\bar{x})+N(\bar{x};\Theta)\ \text{ and }\ \lambda_i f(\bar{x})=0. \tag{7.7}$$

(ii) *For any $i\in I\setminus I(\bar{x})$ there is $\lambda_i\in S^+$ such that*

$$a_i^*\in\bigcup_{\mu_i,\nu_i,\rho_i\ge 0,\ \mu_i+\nu_i+\rho_i=\alpha_i}\Big\{\partial_{\mu_i}\vartheta(\bar{x})+\partial_{\nu_i}\big(\lambda_i f\big)(\bar{x})+N_{\rho_i}(\bar{x};\Theta)\Big\}, \tag{7.8}$$

where the numbers $\alpha_i\ge 0$ are defined in (7.6).

Proof. Observe that for $i\in I(\bar{x})$ conditions (7.4) and (7.5) of Theorem 7.1 reduce to (7.7). Indeed, for such indices i we get from (7.3) and (7.4) that

$$0\le\mu_i+\nu_i+\rho_i=\lambda_i f(\bar{x})+\theta(\bar{x})-\theta_i(\bar{x})=\lambda_i f(\bar{x})\le 0,$$

and thus $\lambda_i f(\bar{x})=0$ with $\mu_i=\nu_i=\rho_i=0$. Condition (7.8) in (ii) clearly reduces to (7.5) by the definition of α_i in (7.6). △

The next theorem provides a *necessary and sufficient* condition for *local optimality* in the polyhedral DC program (7.2). This result is a polyhedral counterpart of Theorem 5.2 obtained in the general infinite DC setting. In contrast to Theorem 5.2, we now get the condition that is not only necessary but also sufficient for local optimality in polyhedral DC (generally nonconvex) cone-constrained programs.

Theorem 7.2 (qualified necessary and sufficient condition for local optimality in polyhedral DC cone-constrained programming). *Let the CQC qualification condition* (6.6) *hold in the polyhedral DC program* (7.2). *Then* $\bar{x} \in \widehat{\Xi} \cap \operatorname{dom} \vartheta$ *is a local minimizer for this problem if and only if we have the inclusion*

$$\operatorname{co}\left\{a_i^* \,\middle|\, i \in I(\bar{x})\right\} \subset \partial\vartheta(\bar{x}) + \bigcup_{\lambda \in \Lambda(\bar{x})} \partial\big(\lambda f\big)(\bar{x}) + N(\bar{x}; \Theta), \tag{7.9}$$

where the multiplier set $\Lambda(\bar{x})$ *is defined in Proposition* 6.3.

Proof. As usual we rewrite the constrained problem (7.2) in the equivalent *unconstrained form* involving the *infinite penalty*:

$$\text{minimize} \;\; \big(\vartheta + \delta(\cdot; \widehat{\Xi})\big)(x) - \theta(x), \quad x \in X, \tag{7.10}$$

where $\theta(x)$ is a polyhedral convex function of type (7.1). Applying now a delicate polyhedral result obtain in [29] (see also [18, Theorem 4.1]), we conclude that $\bar{x}$ is a *local minimizer* for problem (7.10) *if and only if* the inclusion

$$\partial\theta(\bar{x}) \subset \partial\big(\vartheta + \delta(\cdot; \widehat{\Xi})\big)(\bar{x}) \tag{7.11}$$

is satisfied. It is easy to compute the classical subdifferential of the polyhedral convex function $\theta(x)$ given in (7.1) by

$$\partial\theta(\bar{x}) = \operatorname{co}\left\{a_i^* \,\middle|\, i \in I(\bar{x})\right\}. \tag{7.12}$$

Furthermore, by the calculus rule of Proposition 6.3 we have

$$\partial\big(\vartheta + \delta(\cdot; \widehat{\Xi})\big)(\bar{x}) = \partial\vartheta(\bar{x}) + \bigcup_{\lambda \in \Lambda(\bar{x})} \partial\big(\lambda f\big)(\bar{x}) + N(\bar{x}; \Theta). \tag{7.13}$$

Substituting finally equalities (7.12) and (7.13) into the unconstrained optimality criterion (7.11), we conclude that inclusion (7.9) provides a necessary and sufficient condition for local optimality in the constrained polyhedral DC program (7.2). △

8 DC Programs with Positive Semi-Definite Constraints

The final section of the paper is devoted to some specifications of the major optimality conditions obtained above in the general case of DC infinite programs for a remarkable class of the so-called *DC programs with positive semi-definite constraints*, which are highly important in optimization theory and its applications. We refer the reader to the fundamental books [3, 4], where *convex* models of this type are studied in detailed and applied to various practical problems. To the best of our knowledge, *DC programs* with positive semi-definite constraints have nor received much attention in the literature. The results obtained below in finite-dimensional spaces extend those established in [22, 23] for convex programs with positive semi-definite constraints. Note also that, in addition to semi-definite constraints as in [3, 4, 22, 23], we impose convex *set/geometric constraints* on the decision variables.

The basic problem under consideration in this section is formulated as follows:

$$\begin{cases} \text{minimize } \vartheta(x) - \theta(x) \text{ subject to} \\ x \in \Theta \subset I\!R^n, \quad F_0 + \displaystyle\sum_{i=1}^m x_i F_i \succeq 0, \end{cases} \tag{8.1}$$

where ϑ, θ, and Θ satisfy the standing assumptions of Section 1 while F_i as $i = 0, \ldots, m$ belong to the space S_n of *symmetric* $(n \times n)$*-matrices*. The symbol $\succeq$ signifies in this section the *Löwer partial order* on S_n defined as follows:

$$M \succeq P \text{ for } M, P \in S_n \Longleftrightarrow M - P \text{ is positive semi-definite.}$$

Recall that S_n is considered as a vector space with the *trace inner product* defined by

$$\langle M, P \rangle := \text{Tr}[MP], \quad \text{where } \text{Tr}[\cdot] \text{ is the trace operation.}$$

Let $S = \{M \in S_n |\ M \succeq 0\}$ be the closed convex cone of all positive semi-definite $(n \times n)$-matrices. Then $S^+ = S$ and $M \in S$ *if and only if* $\text{Tr}[ZM] \geq 0$ for all $Z \in S$. Further, for each $x \in I\!R^m$ and $Z \in S_n$ denote

$$F(x) := F_0 + \sum_{i=1}^m x_i F_i, \ \widehat{F}(x) := \sum_{i=1}^m x_i F_i, \text{ and } \widehat{F}^*(Z) := \big(\text{Tr}[ZF_1], \ldots, Tr[ZF_m]\big). \tag{8.2}$$

Then the DC problem with positive semi-definite constraints (8.1) can be equivalently rewritten as the DC cone-constrained program of type (6.1):

$$\begin{cases} \text{minimize } \vartheta(x) - \theta(x) \text{ subject to} \\ x \in \Theta \subset I\!R^n \text{ and } -F(x) \in -S. \end{cases} \tag{8.3}$$

Observe that for all $Z \in S$ and $u \in I\!R^m$ we have

$$\begin{aligned} (-ZF)^*(u) &= \sup_{x \in I\!R^m} \big\{\langle u, x\rangle + \langle Z, F(x)\rangle\big\} = \sup_{x \in I\!R^m} \Big\{\langle u, x\rangle + \sum_{i=1}^m x_i \text{Tr}[ZF_i] + \text{Tr}[ZF_0]\Big\} \\ &= \text{Tr}[ZF_0] + \sup_{x \in I\!R^m} \langle u + \widehat{F}^*(Z), x\rangle. \end{aligned}$$

Hence the adjoint operator to $(-ZF)$ is represented as

$$(-ZF)^*(u) = \begin{cases} \text{Tr}[ZF_0] & \text{if } u = -\widehat{F}^*(Z), \\ \infty & \text{otherwise.} \end{cases}$$

Consequently, for all $Z \in S$ and $x \in I\!R^m$ we get the representations

$$\partial(-ZF)(x) = -\widehat{F}^*(Z), \quad \text{epi}\,(-ZF)^* = \big(-\widehat{F}^*(Z), \text{Tr}[ZF_0]\big) + \{0\} \times I\!R_+. \tag{8.4}$$

It follows from (8.4) that the cone K in (3.1) corresponding to the constraint system

$$\widehat{\sigma} := \big\{x \in \Theta, -F(x) \in -S\big\} \tag{8.5}$$

in problem (8.3) is represented via the initial data of this problem as

$$K=\bigcup_{Z\in S,\, r\geq 0}\Big(-\widehat{F}^*(Z),\mathrm{Tr}[ZF_0]+r\Big)+\mathrm{epi}\,\delta^*(\cdot;\Theta).$$

Thus the *Farkas-Minkowski* (*FM*) *qualification condition* (3.3) for the constraint system $\widehat{\sigma}$ in (8.5) reads as follows:

$$\text{the set}\quad\bigcup_{Z\in S,\, r\geq 0}\Big(-\widehat{F}^*(Z),\mathrm{Tr}[ZF_0]+r\Big)+\mathrm{epi}\,\delta^*(\cdot;\Theta)\;\text{ is closed in }\;I\!\!R^{m+1}. \tag{8.6}$$

For the case of $\Theta=I\!\!R^m$ it reduces to the one introduced in [31] and then used in [22]. The set-constrained extension (8.6) first appeared and employed in [23].

The following theorem establishes *necessary and sufficient* conditions for *global optimality* in the DC problem with positive semi-definite constraints (8.1). It can be considered as a specification of Theorem 5.1 and Propositions 6.4 in the case of program (8.1) written in the equivalent cone-constrained form (8.3). As in Proposition 6.4, denote by $\widehat{\Xi}:=\{x\in\Theta|\;-F(x)\in -S\}$ the set of feasible solutions to problem (8.3).

Theorem 8.1 (necessary and sufficient conditions for global optimality in DC programs with positive semi-definite constraints). *Assume that the Farkas-Minkowski qualification condition* (8.6) *holds for the constraint system* (8.5) *in problem* (8.1) *and the "plus" function ϑ in the objective is continuous at some point from $\widehat{\Xi}$. Then $\bar{x}\in\widehat{\Xi}\cap\mathrm{dom}\,\vartheta$ is a global solution to* (8.1) *if and only if for each $\varepsilon\geq 0$ and each $x^*\in\partial_\varepsilon\theta(\bar{x})$ there exist $Z\in S$ and $\varepsilon_1,\varepsilon_2\geq 0$ such that*

$$\varepsilon_1+\varepsilon_2=\varepsilon-\mathrm{Tr}[ZF(\bar{x})]\quad\textit{and} \tag{8.7}$$

$$x^*+\widehat{F}^*(Z)\in\partial_{\varepsilon_1}\vartheta(\bar{x})+N_{\varepsilon_2}(\bar{x};\Theta). \tag{8.8}$$

In particular, if $\bar{x}\in\widehat{\Xi}\cap\mathrm{dom}\,\vartheta$ is a local solution to (8.1)*, then for each subgradient $x^*\in\partial\theta(\bar{x})$ there exists $Z\in S$ such that*

$$x^*+\widehat{F}^*(Z)\in\partial\vartheta(\bar{x})+N(\bar{x};\Theta)\quad\textit{and}\quad\mathrm{Tr}[ZF(\bar{x})]=0. \tag{8.9}$$

Proof. Since the Farkas-Minkowski constraint qualification (8.6) is satisfied and ϑ is assumed to be continuous at some feasible point, the CQC condition (3.1) holds in (8.3) by Proposition 4.1. Applying now Proposition 6.4 to (8.3) and taking into account the specific structure of this problem, we get the following necessary and sufficient conditions for the global minimizer $\bar{x}$ under consideration: for each $\varepsilon\geq 0$ and each $x^*\in\partial_\varepsilon\theta(\bar{x})$ there are $Z\in S^+=S$ and $\varepsilon_1',\varepsilon_2',\varepsilon_3'\geq 0$ such that

$$\varepsilon_1'+\varepsilon_2'+\varepsilon_3'=\varepsilon+\langle Z,-F(\bar{x})\rangle=\varepsilon-\mathrm{Tr}[ZF(\bar{x})]\;\text{ and}$$

$$x^*\in\partial_{\varepsilon_1'}\vartheta(\bar{x})+N_{\varepsilon_2'}(\bar{x};\Theta)+\partial_{\varepsilon_3'}(-ZF)(\bar{x}).$$

Note further that $\partial_{\varepsilon'_3}(-ZF)(\bar{x}) = \partial(-ZF)(\bar{x}) = -\widehat{F}^*(Z)$ and $\partial_{\varepsilon'_1}\vartheta(\bar{x}) \subset \partial_{\varepsilon'_1+\varepsilon'_3}\vartheta(\bar{x})$. Thus setting $\varepsilon_1 := \varepsilon'_1 + \varepsilon'_3$ and $\varepsilon_2 := \varepsilon'_2$, we get

$$x^* \in \partial_{\varepsilon_1}\vartheta(\bar{x}) + N_{\varepsilon_2}(\bar{x};\Theta) - \widehat{F}^*(Z) \text{ and } \varepsilon_1 + \varepsilon_2 = \varepsilon - \mathrm{Tr}[ZF(\bar{x})].$$

This justifies that the fulfillment of both conditions (8.7) and (8.8) is necessary and sufficient for the global optimality in (8.1). The necessary conditions in (8.9) follow immediately from (8.7) and (8.8) as $\varepsilon = 0$. △

When $\theta(x) \equiv 0$, problem (8.1) reduces to a *convex semi-definite program* of the form

$$\begin{cases} \text{minimize } \vartheta(x) \text{ subject to} \\ x \in \Theta \subset I\!R^n, \quad F_0 + \displaystyle\sum_{i=1}^m x_i F_i \succeq 0. \end{cases} \tag{8.10}$$

Thus we arrive at the following consequence of Theorem 8.1, which was previously derived in [22, Corollary 4.1] for convex problems of type (8.10) with *no* set constraints $x \in \Theta$.

Corollary 8.2 (necessary and sufficient optimality conditions for convex programs with positive semi-definite constraints). *Assume that the Farkas-Minkowski qualification condition* (8.6) *holds for the constraint system* (8.5) *in problem* (8.10) *and that the cost function* ϑ *in* (8.10) *is continuous at some point of the feasible set* $\widehat{\Xi}$. *Then* $\bar{x} \in \Xi$ *is an optimal solution to* (8.10) *if and only if there exists* $Z \in S$ *such that*

$$\widehat{F}^*(Z) \in \partial\vartheta(\bar{x}) + N(\bar{x};\Theta), \quad \mathrm{Tr}[ZF(\bar{x})] = 0. \tag{8.11}$$

Proof. The *necessity* of the conditions in (8.11) for the (global) optimality of $\bar{x}$ in problem (8.10) follows immediately from relations (8.9) of Theorem 8.1. The *sufficiency* part can be derived from Theorem 5.3 due to the fully convex structure of (8.10). △

Finally in this section, we present an *example* of using the necessary and sufficient optimality conditions obtained in Theorem 8.1 to solve a DC program with positive semi-definite and geometric constraints of type (8.1) in the case of $n = 2$ and $m = 3$.

Example 8.3 (solving a DC program with positive semi-definite and geometric constraints). Consider the following DC program of type (8.1):

$$\begin{cases} \text{minimize } (x_1^2 + x_2) - x_2^2 \\ \text{subject to } (x_1, x_2) \in \Theta, \begin{pmatrix} 0 & x_1 & 0 \\ x_1 & x_2 & 0 \\ 0 & 0 & x_1 \end{pmatrix} \succeq 0, \end{cases} \tag{8.12}$$

where $\Theta := \{(x_1, x_2) |\, x_1 \le 0, x_2 \le 1\}$ is a closed convex subset of $I\!R^2$.

Let us show, based on the optimality conditions of Theorem 8.1, that $\bar{x} = (0, 0)$ is an optimal solution to problem (8.12). To write down problem (8.12) in the framework of (8.1), put $\vartheta(x_1, x_2) := x_1^2 + x_2$, $\theta(x_1, x_2) := x_2^2$, and

$$F_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ F_1 = \begin{pmatrix} 0 & 1 & 0, \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Based on the constructions in (8.2), we can prove that the set

$$\bigcup_{Z\in S,\ r\geq 0}\Big(-\widehat{F}^*(Z),\mathrm{Tr}[ZF_0]+r\Big)+\mathrm{epi}\,\delta^*(\cdot;\Theta)$$

is closed in $I\!R^3$; see [23, Example 6.1] for more details. Therefore, the Farkas-Minkowski qualification condition (8.6) holds for the corresponding constraint system $\widehat{\sigma}$ in (8.5). Furthermore, it is easy to check that for each $\varepsilon,\varepsilon_1,\varepsilon_2\geq 0$ and each $Z\in S$ we have

$$\begin{aligned}
&\partial_\varepsilon\theta(0,0)=\{0\}\times[-2\sqrt{\varepsilon},2\sqrt{\varepsilon}],\\
&\partial_{\varepsilon_1}\vartheta(0,0)=[-2\sqrt{\varepsilon_1},2\sqrt{\varepsilon_1}]\times\{1\},\\
&N_{\varepsilon_2}((0,0);\Theta)=[0,\infty)\times[0,\varepsilon_2],\\
&\widehat{F}^*(Z)=(2\lambda_2+\lambda_6,\lambda_4),
\end{aligned}$$

where Z is a positive semi-definite matrix given by

$$Z=\begin{pmatrix}\lambda_1&\lambda_2&\lambda_3\\ \lambda_2&\lambda_4&\lambda_5\\ \lambda_3&\lambda_5&\lambda_6\end{pmatrix}\succeq 0.$$

To proceed with checking the optimality conditions of Theorem 8.1, fix $\varepsilon\geq 0$ and take

$$(0,a)\in\partial_\varepsilon\theta(0,0)=\{0\}\times[-2\sqrt{\varepsilon},2\sqrt{\varepsilon}].$$

Observe that for $a\leq 1$ we can choose $\varepsilon_1=0,\varepsilon_2=\varepsilon$, and

$$Z=\begin{pmatrix}0&0&0\\ 0&\lambda_4&0\\ 0&0&0\end{pmatrix}\succeq 0$$

with $\lambda_4:=1-a\geq 0$. Then $\varepsilon_1+\varepsilon_2=\varepsilon-\mathrm{Tr}[ZF(0,0)]=\varepsilon$ and

$$(0,a)+\widehat{F}^*(Z)\in\partial_{\varepsilon_1}\vartheta(0,0)+N_{\varepsilon_2}\big((0,0);\Theta\big).$$

Otherwise, for $a>1$ choose $\varepsilon_1=0$, $\varepsilon_2=\varepsilon$, $Z=O_{3\times 3}$. Then $\varepsilon_1+\varepsilon_2=\varepsilon-\mathrm{Tr}[ZF(\bar{x})]$ and

$$a\in[1,2\sqrt{\varepsilon}]\subset[1,1+\varepsilon],$$

since $1+\varepsilon\geq 2\sqrt{\varepsilon}$ and $a\in[-2\sqrt{\varepsilon},2\sqrt{\varepsilon}]$. The latter implies that

$$(0,a)+\widehat{F}^*(Z)=(0,a)\in(0,1)+[0,\infty)\times[0,\varepsilon]=\partial_{\varepsilon_1}\vartheta(0,0)+N_{\varepsilon_2}\big((0,0);\Theta\big).$$

Therefore for each $\varepsilon\geq 0$ and each $(0,a)\in\partial_\varepsilon\theta(0,0)$ there exist $Z\in S$ and $\varepsilon_1,\varepsilon_2\geq 0$ satisfying the conditions $\varepsilon_1+\varepsilon_2=\varepsilon-\mathrm{Tr}[ZF(\bar{x})]$ and

$$(0,a)+\widehat{F}^*(Z)\in\partial_{\varepsilon_1}\vartheta(\bar{x})+N_{\varepsilon_2}(\bar{x};\Theta),$$

which justify the global optimality of $\bar{x}=(0,0)$ in the DC problem (8.12) with positive semi-definite and geometric constraints under consideration.

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