WELL-BALANCED NUMERICAL APPROXIMATION FOR BAER-NUNZIATO MODEL OF TWO-PHASE FLOWS: ISENTROPIC CASE

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Abstract. We aim at constructing a steady-state capturing scheme for the Baer-Nunziato model of two-phase flows. First we transform the system into a new one with only a single source term. Second, we derive a formulation to compute steady states. This enables us to define a procedure to construct a scheme which is capable to maintain steady states. To complete the discretization, we employ the technique of the Engquist-Osher scheme to the compaction dynamics equation.

1. Introduction

We consider in this paper the Baer-Nunziato (BN) model of isentropic two-phase flows. This two-phase mixture model was developed by Baer and Nunziato to study the deflagration-to-detonation transition (DDT) in granular explosives. Precisely, the model is described by a system of four equations characterizing the conservation of mass in each phase and conservation of momentum when there is exchange of momentum between the two phases, see ([3, 10, 7]):

$$\begin{aligned}
&\partial_t(\alpha_g\rho_g) + \partial_x(\alpha_g\rho_g u_g) = 0,\\
&\partial_t(\alpha_g\rho_g u_g) + \partial_x(\alpha_g(\rho_g u_g^2 + p_g)) = p_g\partial_x\alpha_g,\\
&\partial_t(\alpha_s\rho_s) + \partial_x(\alpha_s\rho_s u_s) = 0,\\
&\partial_t(\alpha_s\rho_s u_s) + \partial_x(\alpha_s(\rho_s u_s^2 + p_s)) = -p_g\partial_x\alpha_g,
\end{aligned} \tag{1.1}$$

together with the compaction dynamics equation

$$\partial_t\alpha_g + u_s\partial_x\alpha_g = 0. \tag{1.2}$$

where $\rho_k, u_k, p_k, \alpha_k$ denote the density, the velocity, the pressure, and the volume fraction in the k-phase, $k = s, g$. Obviously, the volume

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fractions satisfy

$$\alpha_s + \alpha_g = 1. \tag{1.3}$$

In the following the g-phase is called the gas phase, the s-phase is called the solid phase. Each phase has an equation of state of the form

$$p_k = \kappa_k \rho_k^{\gamma_k}, \quad \kappa_k > 0, 1 < \gamma_k < 5/3, \quad k = s, g. \tag{1.4}$$

The system (1.1) is a system of balanced laws with *source terms*, where sources appear on the right-hand side of the equations of conservation of momentum, i.e., the second and the fourth equations of (1.1). By supplementing the system (1.1) by the compaction dynamics equation (1.2), we can rewrite the full system under the nonconservative form of system of conservation laws. Therefore, formulation of weak solutions, theoretically, can be understood in the sense of *nonconservative product*, see Dal Maso, LeFloch and Murat [8]. Construction of weak solutions of several simple systems of balanced laws with source terms has been done, see for example [26, 22, 16, 24, 25, 11]. But this is not always evident and construction of solutions remains open for a broad class of systems of balanced laws with sources. Practically, source terms cause lots of inconveniences in approximating physical solutions of the system. This has been observed even in the case of a single conservation law, shallow water equations, or in the model of fluid flows in a nozzle with variable cross-section, see [13, 20, 19, 14, 6, 12, 4, 5, 2, 18, 17, 27], etc. Thus, the discretization of source terms is important and it has been addressed by many authors, see [3, 7, 1, 9, 21, 29, 28], and the references therein. Recently, a well-balanced scheme that can capture steady states for a one-pressure model of two-phase flows was obtained in [30].

In this paper we will construct a well-balanced numerical method to the system (1.1)-(1.2) by considering individually (1.1) and (1.2). A well-balanced scheme is aimed to be built up for the system (1.1) so that it deals with the effect of the source terms. Then, a similar technique to the Engquist-Osher scheme is employed to discretize the compaction dynamics equation (1.2). Test cases show that our method can capture steady states resulted by stationary waves, and that approximate solutions converge.

2. Backgrounds

2.1. **Non-strictly hyperbolic system.** Let us investigate properties of the full supplemented system (1.1)-(1.2). For smooth solutions, the system (1.1)-(1.2) is equivalent to the following system

$$
\begin{aligned}
&\partial_t \rho_g + u_g \partial_x \rho_g + \rho_g \partial_x u_g = 0,\\
&\partial_t u_g + h_g'(\rho_g)\partial_x \rho_g + u_g \partial_x u_g = 0,\\
&\partial_t \rho_s + u_s \partial_x \rho_s + \rho_s \partial_x u_s = 0,\\
&\partial_t u_s + h_s'(\rho_s)\partial_x \rho_s + u_s \partial_x u_s + \frac{p_g - p_s}{(1-\alpha_g)\rho_s}\partial_x \alpha_g = 0,\\
&\partial_t \alpha_g + u_s \partial_x \alpha_g = 0.
\end{aligned} \tag{2.1}
$$

where

$$
h_i'(\rho) = \frac{p_i'(\rho)}{\rho}, \quad i = s, g. \tag{2.2}
$$

It is seen from (2.1) that if we choose the independent variable $V = (\rho_g, u_g, \rho_s, u_s, \alpha_g)$, we can re-write the system as a system of balanced laws in nonconservative form as

$$
V_t + A(V)V_x = 0, \tag{2.3}
$$

where

$$
A(V) = \begin{pmatrix}
u_g & \rho_g & 0 & 0 & 0\\
h_g'(\rho_g) & u_g & 0 & 0 & 0\\
0 & 0 & u_s & \rho_s & 0\\
0 & 0 & h_s'(\rho_s) & u_s & \dfrac{p_g - p_s}{(1-\alpha_g)\rho_s}\\
0 & 0 & 0 & 0 & u_s
\end{pmatrix}.
$$

The characteristic equation of $A(U)$ is given by

$$
(u_s - \lambda)((u_g - \lambda)^2 - p_g')((u_s - \lambda)^2 - p_s') = 0, \tag{2.4}
$$

which admits five roots as

$$
\begin{aligned}
&\lambda_1 = u_g - \sqrt{p_g'} < \lambda_2 = u_g + \sqrt{p_g'},\\
&\lambda_3 = u_s - \sqrt{p_s'} < \lambda_4 = u_s < \lambda_5 = u_s + \sqrt{p_s'}.
\end{aligned} \tag{2.5}
$$

It is easy to see from (2.5) that the characteristic fields may coincide and thus the system is *not strictly hyperbolic.*

2.2. **System of a single source term.** As observed earlier, source terms often cause inconvenience for numerical approximations. To reduce the size of sources, we add up the two equations of balance of momentum to get the conservation of momentum of the total in place of the equation of balance of momentum for the liquid phase. So we get three sets of equations:

- Governing equations in the gas phase

$$\begin{aligned} &\partial_t(\alpha_g\rho_g) + \partial_x(\alpha_g\rho_g u_g) = 0,\\ &\partial_t(\alpha_g\rho_g u_g) + \partial_x(\alpha_g(\rho_g u_g^2 + p_g)) = p_g\partial_x\alpha_g, \end{aligned} \tag{2.6}$$

- "composite" conservation laws

$$\begin{aligned} &\partial_t(\alpha_s\rho_s) + \partial_x(\alpha_s\rho_s u_s) = 0,\\ &\partial_t(\alpha_s\rho_s u_s + \alpha_g\rho_g u_g) + \partial_x\big(\alpha_s(\rho_s u_s^2 + p_s) + \alpha_g(\rho_g u_g^2 + p_g)\big) = 0, \end{aligned} \tag{2.7}$$

- compaction dynamics equation

$$\partial_t\alpha_g + u_s\partial_x\alpha_g = 0. \tag{2.8}$$

Set the conservative variable

$$U = (\alpha_g\rho_g, \alpha_g\rho_g u_g, \alpha_s\rho_s, \alpha_g\rho_g u_g + \alpha_s\rho_s u_s)^T,$$

the flux

$$f(U) = (\alpha_g\rho_g u_g, \alpha_g(\rho_g u_g^2 + p_g), \alpha_s\rho_s u_s, \alpha_s(\rho_s u_s^2 + p_s) + \alpha_g(\rho_g u_g^2 + p_g))^T,$$

and the source

$$S(U) = (0, p_g\partial_x\alpha_g, 0, 0)^T.$$

We can see that a unique source appears only in the second component. Thus, we can rewrite the system (2.6)-(2.7) as a system of conservation laws *with a single source term*

$$\partial_t U(x,t) + \partial_x f(U(x,t)) = S(U(x,t)), \quad x \in \mathbb{R}, t > 0. \tag{2.9}$$

In the next section, we will see that the numerical approximation for the full system (1.1)-(1.2) can be performed as follows: first, we study and obtain the values of steady states of (2.6), then the values of steady states of (2.7), second we discretize the system with source (2.9) and finally we discretize the compaction dynamics equation (2.8).

3. Stationary waves

Let us now investigate the stationary contact waves of the system (2.9). Motivated by our earlier works ([24, 20]), we look for stationary waves resulted by source terms. Thus, they are concerned only on the first two equations of the compressible phase. Stationary waves are just the limit of stationary smooth solutions of (2.9). *A stationary smooth solution* U of (2.9) is a time-independent smooth solution. Therefore, stationary solutions of (2.9) satisfy the following ordinary differential equations

$$
\begin{aligned}
&(\alpha_g\rho_g u_g)' = 0,\\
&\Big(\frac{u_g^2}{2} + h_g\Big)' = 0,\\
&(\alpha_s\rho_s u_s)' = 0,\\
&\big(\alpha_s(\rho_s u_s^2 + p_s) + \alpha_g(\rho_g u_g^2 + p_g)\big)' = 0,
\end{aligned}
\tag{3.1}
$$

where $(.)' = d/dx$ and $h_i'(\rho) = p_i'(\rho)/\rho, \quad i = s, g$, or

$$h_i(\rho) = \frac{\kappa_i\gamma_i}{\gamma_i - 1}\rho^{\gamma_i - 1}.$$

We look for stationary jumps which are limit of smooth solutions of (3.1). Then (3.1) yields the following result which gives us the way to compute stationary waves.

Lemma 3.1. *The left-hand and right-hand states of a stationary contact satisfy*

$$
\begin{aligned}
&[\alpha_g\rho_g u_g] = 0,\\
&\Big[\frac{u_g^2}{2} + h_g\Big] = 0,\\
&[\alpha_s\rho_s u_s] = 0,\\
&\big[\alpha_s(\rho_s u_s^2 + p_s) + \alpha_g(\rho_g u_g^2 + p_g)\big] = 0,
\end{aligned}
\tag{3.2}
$$

where $[\alpha\rho u] := \alpha^+\rho^+u^+ - \alpha^-\rho^-u^-$, and so on, denotes the difference of the corresponding value $\alpha\rho u$ between the right-hand and left-hand states of the stationary contact.

To simplify the expressions, we omit the subindex $\rho_g, u_g, \ldots$ in the gas phase. From Lemma 3.1, we deduce that a stationary wave from a given state $U_0 = (\alpha_0, \rho_0, u_0, v_0)$ to some state $U = (\alpha, \rho, u, v)$ must satisfy the relations on the gas phase

$$
\begin{aligned}
&\alpha\rho u = \alpha_0\rho_0 u_0,\\
&\frac{u^2}{2} + h(\rho) = \frac{u_0^2}{2} + h(\rho_0).
\end{aligned}
\tag{3.3}
$$

This leads us to finding roots of the equation

$$F(U_0, \rho, \alpha) := \mathrm{sgn}(u_0)\Big(u_0^2 - \frac{2\kappa\gamma}{\gamma - 1}(\rho^{\gamma-1} - \rho_0^{\gamma-1})\Big)^{1/2}\rho - \frac{\alpha_0 u_0 \rho_0}{\alpha} = 0. \tag{3.4}$$

To find zeros of the function $F(U_0, \rho, \alpha)$, we observe that it is well-defined whenever

$$u_0^2 - \frac{2\kappa\gamma}{\gamma - 1}(\rho^{\gamma-1} - \rho_0^{\gamma-1}) \ge 0,$$

or

$$\rho \le \bar{\rho}(U_0) := \Big(\frac{\gamma - 1}{2\kappa\gamma} u_0^2 + \rho_0^{\gamma-1}\Big)^{\frac{1}{\gamma-1}}.$$

We have

$$\frac{\partial F(U_0, \rho; \alpha)}{\partial \rho} = \frac{u_0^2 - \frac{2\kappa\gamma}{\gamma-1}(\rho^{\gamma-1} - \rho_0^{\gamma-1}) - \kappa\gamma\rho^{\gamma-1}}{\Big(u_0^2 - \frac{2\kappa\gamma}{\gamma-1}(\rho^{\gamma-1} - \rho_0^{\gamma-1})\Big)^{1/2}}.$$

Assume, for definitiveness, that $u_0 > 0$. The last expression yields

$$\begin{aligned} \frac{\partial F(U_0, \rho; \alpha)}{\partial \rho} &> 0, \quad \rho < \rho_{\max}(\rho_0, u_0), \\ \frac{\partial F(U_0, \rho; \alpha)}{\partial \rho} &< 0, \quad \rho > \rho_{\max}(\rho_0, u_0), \end{aligned} \tag{3.5}$$

where

$$\rho_{\max}(\rho_0, u_0) := \Big(\frac{\gamma - 1}{\kappa\gamma(\gamma + 1)} u_0^2 + \frac{2}{\gamma + 1}\rho_0^{\gamma-1}\Big)^{\frac{1}{\gamma-1}}.$$

Since

$$F(U_0, \rho = 0, a) = F(U_0, \rho = \bar{\rho}, a) = -\frac{\alpha_0 u_0 \rho_0}{\alpha} < 0,$$

the function $\rho \mapsto F(U_0, \rho; \alpha)$ admits a root if and only if the maximum value is non-negative:

$$F(U_0, \rho = \rho_{\max}, \alpha) \ge 0,$$

or, equivalently,

$$\alpha \ge \alpha_{\min}(U_0) := \frac{\alpha_0 \rho_0 |u_0|}{\sqrt{\kappa\gamma}\rho_{\max}^{\frac{\gamma+1}{2}}(\rho_0, u_0)}. \tag{3.6}$$

Similar argument can be made for $u_0 < 0$.

It will be convenient to set in the (ρ, u)-plan the following sets, referred to as the "lower region" G_1, the "middle region" G_2, and the "upper region" G_3, and the "boundary" $\mathcal{C}$, as

$$\begin{aligned} G_1 &:= \{(\rho, u) \,:\, u < -\sqrt{p'(\rho)}\}, \\ G_2 &:= \{(\rho, u) \,:\, |u| < \sqrt{p'(\rho)}\}, \\ G_3 &:= \{(\rho, u) \,:\, u > \sqrt{p'(\rho)}\}. \end{aligned} \tag{3.7}$$

The existence of the zeros are followed immediately from (3.5) and (3.6). We are now at a position to say about the the existence as well as properties of zeros of the function $F(U_0, \rho, \alpha)$.

Lemma 3.2. *Given $U_0 = (\alpha_0, \rho_0, u_0)$ and $0 \le \alpha \le 1$. The function $F(U_0, \rho, \alpha)$ in (3.5) admits a zero if if and only if $a \ge \alpha_{\min}(U_0)$. In this case, $F(U_0, \rho, \alpha)$ admits two distinct zeros, denoted by $\rho = \varphi_1(U_0, \alpha), \rho = \varphi_2(U_0, \alpha)$ such that*

$$\varphi_1(U_0, \alpha) \le \rho_{\max}(U_0) \le \varphi_2(U_0, \alpha) \tag{3.8}$$

the equality in (3.8) holds only if $\alpha = \alpha_{\min}(U_0)$.

Lemma 3.3. *(a) We have*

$$\begin{aligned} &\rho_{\max}(\rho_0, u_0) < \rho_0, \quad (\rho_0, u_0) \in G_2, \\ &\rho_{\max}(\rho_0, u_0) > \rho_0, \quad (\rho_0, u_0) \in G_3 \cup G_1, \\ &\rho_{\max}(\rho_0, u_0) = \rho_0, \quad (\rho_0, u_0) \in \mathcal{C}_\pm. \end{aligned} \tag{3.9}$$

(b) $(\varphi_1(U_0, \alpha), u) \in G_1$ if $u_0 < 0$, and $(\varphi_1(U_0, \alpha), u) \in G_3$ if $u_0 > 0$; $(\varphi_2(U_0, \alpha), u) \in G_2$, where u is defined by (3.5). Moreover,

$$(\rho_{\max}(U_0, \alpha), u) \in \mathcal{C}. \tag{3.10}$$

In addition, we have

(i) *If $\alpha > \alpha_0$, then*

$$\varphi_1(U_0, \alpha) < \rho_0 < \varphi_2(U_0, \alpha). \tag{3.11}$$

(ii) *If $\alpha < \alpha_0$, then*

$$\begin{aligned} &\rho_0 < \varphi_1(U_0, \alpha) \quad \text{for} \quad U_0 \in G_1 \cup G_3, \\ &\rho_0 > \varphi_2(U_0, \alpha) \quad \text{for} \quad U_0 \in G_3. \end{aligned} \tag{3.12}$$

(c)

$$\begin{aligned} &\alpha_{\min}(U, \alpha) < \alpha, \quad (\rho, u) \in G_i,\ i = 1, 2, 3, \\ &\alpha_{\min}(U, \alpha) = \alpha, \quad (\rho, u) \in \mathcal{C}, \\ &\alpha_{\min}(U, \alpha) = 0, \quad \rho = 0 \quad \text{or} \quad u = 0. \end{aligned} \tag{3.13}$$

Proof. Most of the proof was available in [30]. However, for completeness, we will show the steps. Assume for simplicity that $u_0 > 0$. Define

$$g(U_0, \rho) = u_0^2 - \frac{2\kappa\gamma}{\gamma - 1}(\rho^{\gamma-1} - \rho_0^{\gamma-1}) - \kappa\gamma\rho^{\gamma-1}. \tag{3.14}$$

Then, a straightforward calculation gives

$$g(U_0, \rho_{\max}(U_0)) = 0,$$

which proves (3.10). On the other hand, since

$$\frac{dg(U_0, \rho)}{d\rho} = -(\gamma + 1)\kappa\gamma\rho^{\gamma-2} < 0,$$

and that $\varphi_1(U_0, \alpha) < \rho_{\max}(U_0, \alpha) < \varphi_2(U_0, \alpha)$ it holds that

$$g(U_0, \varphi_1(U_0, \alpha)) > g(U_0, \rho_{\max}(U_0)) = 0 > g(U_0, \varphi_1(U_0, \alpha)).$$

The last two inequalities justify the statement in (b). Moreover,

$$F(U_0, \rho_0; \alpha) = \rho_0 u_0 (1 - \alpha_0/\alpha) > 0 \quad \text{iff} \quad a > \alpha_0,$$

which proves (3.11), and shows that ρ_0 is located outside of the interval $[\varphi_1(U_0, \alpha), \varphi_2(U_0, \alpha)]$ in the opposite case. Since

$$\frac{\partial F(U_0, \rho_0; \alpha)}{\partial \rho} = \frac{u_0^2 - \kappa\gamma\rho_0^{\gamma-1}}{u_0} < 0 \quad \text{iff} \quad U_0 \in G_2,$$

which, together with the earlier observation, implies (3.12).

We next check (3.13) for $a = \alpha_0$. It comes from the definition of $\alpha_{\min}(U_0)$ that $\alpha_{\min}(U_0) < \alpha_0$ if and only if

$$\sqrt{\kappa\gamma}\rho^{*\frac{\gamma+1}{2}} > \rho_0|u_0|,$$

that can be equivalently written as

$$Q(m) := \frac{2}{\gamma + 1} m - (\kappa\gamma)^{\frac{1-\gamma}{\gamma+1}} m^{\frac{2}{\gamma+1}} + \frac{\gamma - 1}{\kappa\gamma(\gamma + 1)} > 0,$$

where $m := \rho_0^{\gamma-1}/u_0^2$. Then, we can see that

$$Q(1/\kappa\gamma) = 0, \tag{3.15}$$

which, in particular shows that the second equation in (3.13) holds, since $(\rho_0, u_0) \in \mathcal{C}_\pm$ for $m = 1/\kappa\gamma$. Moreover,

$$\frac{dQ(m)}{dm} = \frac{2}{\gamma + 1}(1 - (\kappa\gamma m)^{\frac{1-\gamma}{\gamma+1}}),$$

which is positive for $m > 1/\kappa\gamma$ and negative for $m < \kappa\gamma$. This together with (3.15) establish the first statement in (3.13). The third statement in (3.13) is straightforward. This completes the proof of Lemma 3.3.

□

To select a unique stationary wave, we need the following so-called Monotonicity criterion. The relationships (3.2) also defines a curve $\rho \mapsto \alpha = \alpha(U_0, \rho)$. So we require that

MONOTONICITY CRITERION. *The volume fraction* $\alpha = \alpha(U_0, \rho)$ *must vary monotonically between the two values* ρ_0 *and* ρ_1*, where* ρ_1 *is the* ρ*-value of the corresponding state of a stationary wave having* U_0 *as one state.*

A similar criterion was used by Kröner, LeLoch, and Thanh [23, 20, 19], Isaacson and Temple [15, 16].

Geometrically, we can choose either φ_1 or φ_2 in the domains G_1, G_2, G_3 using the following lemma.

Lemma 3.4. *The Monotonicity Criterion is equivalent to saying that any stationary shock does not cross the boundary* $\mathcal{C}$*. In other words:*

(i) *If* $U_0 \in G_1 \cup G_3$*, then only the zero* $\varphi(U_0, \alpha) = \varphi_1(U_0, \alpha)$ *is selected.*

(ii) *If* $U_0 \in G_2$*, then only the zeros* $\varphi(U_0, \alpha) = \varphi_2(U_0, \alpha)$ *is selected.*

Proof. The second equation of (3.2) determines the u-value as $u = u(\rho)$. Taking the derivative with respect to ρ in the equation

$$\alpha^2(u(\rho)\rho)^2 = (\alpha_0 u_0 \rho_0)^2,$$

we get

$$\alpha(\rho)\alpha'(\rho)(u\rho)^2 + 2\alpha^2(u\rho)(u(\rho)'\rho + u(\rho)) = 0. \tag{3.16}$$

Thus, to prove the lemma, it is sufficient to show that the factor $(u(\rho)'\rho + u(\rho))$ remains of a constant sign whenever (ρ, u) remains in the same domain. Indeed, assume for simplicity that $u_0 > 0$, then

$$\begin{aligned} u'(\rho)\rho + u(\rho) &= \frac{-\kappa\gamma\rho^{\gamma-1}}{u} + u \\ &= \frac{u^2 - \kappa\gamma\rho^{\gamma-1}}{u}, \end{aligned}$$

which remains of a constant sign as long as (ρ, u) remain in the same domain. This completes the proof of Lemma **??**. □

For the solid phase, we set

$$G(\rho) = \kappa_s\alpha_s^2\rho^{\gamma+1} - \big(\alpha_{s0}(\rho_{s0}u_{s0}^2 + p_{s0}) - [\alpha_g(\rho_g u_g^2 + p_g)]\big)\alpha_s\rho + (\alpha_{s0}\rho_{s0}u_{s0})^2. \tag{3.17}$$

The value ρ_s satisfies

$$G(\rho_s) = 0. \tag{3.18}$$

The u_s value is then given by

$$u_s = \frac{\alpha_{s0}\rho_{s0}u_{s0}}{\alpha_s\rho_s}. \tag{3.19}$$

The function G is convex so finding the root $\rho = \rho_{s,\pm}$ from a given value of the ρ_s value in the other side $\rho_{s,\mp}$ of stationary waves can be done using Newton-Raphson method. Since

$$G(0) = (\alpha_{s0}\rho_{s0}u_{s0})^2 \geq 0, \quad \lim_{\rho\to+\infty} G(\rho) = +\infty,$$

the convexity implies that the functions $G(\rho)$ when it has a zero it will have two zeros. The Newton-Raphson method starting at $\rho_{s,\pm}$ will converge to the zero $\rho_{s,\mp}$, where G is monotone between these two values. The fact that there are two zeros of $G(\rho)$ means there would be different choices for stationary waves and that would lead to multiple approximating solutions. This coincides with what has been known in systems of balanced laws with source terms when there are probably multiple solutions, see [24, 25, 11]. We observe that the uniqueness of solutions and/or criteria to select a unique solution of systems of balanced laws with sources is still an open question.

4. Construction of the well-balanced scheme

Let us consider the system when sources are reduced to a single term

$$\partial_t U(x,t) + \partial_x f(U(x,t)) = S(U(x,t)), \quad x \in \mathbb{R}, t > 0, \tag{4.1}$$

where the independent variable is given by

$$U = (\alpha_g\rho_g, \alpha_g\rho_g u_g, \alpha_s\rho_s, \alpha_g\rho_g u_g + \alpha_s\rho_s u_s)^T,$$

the flux functions are given by

$$f(U) = (\alpha_g\rho_g u_g, \alpha_g(\rho_g u_g^2 + p_g), \alpha_s\rho_s u_s, \alpha_s(\rho_s u_s^2 + p_s) + \alpha_g(\rho_g u_g^2 + p_g))^T,$$

and the source is given by

$$S(U) = (0, p_g\partial_x\alpha_g, 0, 0)^T.$$

Given a uniform time step Δt, and a spacial mesh size Δx, setting $x_j = j\Delta x, j \in \mathbf{Z}$, and $t_n = n\Delta t, n \in \mathbf{N}$, we denote U_j^n to be an approximation of the exact value $U(x_j, t_n)$.

A C.F.L condition is also required on the mesh sizes:

$$\lambda \max_U\{|u_g| + \sqrt{p_g'(\rho_g)}, |u_s| + \sqrt{p_s'(\rho_s)}\} < 1, \quad \lambda := \frac{\Delta t}{\Delta x}. \tag{4.2}$$

The well-balanced scheme is defined by

$$U_j^{n+1} = U_j^n - \lambda(g(U_j^n, U_{j+1,-}^n) - g(U_{j-1,+}^n, U_j^n)). \tag{4.3}$$

for some numerical flux g. The states $U_{j+1,-}^n, U_{j-1,+}^n$ are defined as followed. Set

$$\begin{aligned} U_0 = U_{j+1}^n =& (\alpha_{g,j+1}^n \rho_{g,j+1}^n, \alpha_{g,j+1}^n \rho_{g,j+1}^n u_{g,j+1}^n, \alpha_{s,j+1}^n \rho_{s,j+1}^n, \\ & \alpha_{g,j+1}^n \rho_{g,j+1}^n u_{g,j+1}^n + \alpha_{s,j+1}^n \rho_{s,j+1}^n u_{s,j+1}^n)^T \end{aligned}.$$

Then, we take $\alpha = \alpha_{g,j}^n$ and first compute the corresponding $\rho = \rho_{g,j+1,-}^n := \varphi(U_0, \alpha)$ as seen by Lemma 3.4. Second, the value $u = u_{g,j+1,-}^n$ is computed using (3.3). Third, observe that $\alpha_{s,j}^n = 1 - \alpha_{g,j}^n$, the value $\rho_{s,j+1,-}^n$ is computed using (3.18). Then $u_{s,j+1,-}^n$ is given by (3.19).

Similarly, set

$$\begin{aligned} U_0 = U_{j-1}^n =& (\alpha_{g,j-1}^n \rho_{g,j-1}^n, \alpha_{g,j-1}^n \rho_{g,j-1}^n u_{g,j-1}^n, \alpha_{s,j-1}^n \rho_{s,j-1}^n, \\ & \alpha_{g,j-1}^n \rho_{g,j-1}^n u_{g,j-1}^n + \alpha_{s,j-1}^n \rho_{s,j-1}^n u_{s,j-1}^n)^T. \end{aligned}$$

Then, we take $\alpha = \alpha_j^n$ and first compute the corresponding $\rho = \rho_{g,j-1+}^n := \varphi(U_0, \alpha)$ as seen by Lemma 3.4. Second, the value $u = u_{g,j-1,+}^n$ is computed using (3.3). Third, $\rho_{s,j-1,+}^n$ is computed using (3.18). Then $u_{s,j-1,+}^n$ is given by (3.19).

To complete the discretization of the whole model, we employ the technique in the Engquist-Osher scheme to discretize the compaction dynamics equation (1.2). We first write

$$u = \max\{u, 0\} + \min\{u, 0\} = u^+ + u^-. \tag{4.4}$$

and then we apply the backward difference scheme for u^+ and forward difference scheme for u^-. This can be done as arrive at

$$\alpha_j^{n+1} = \alpha_j^n - \lambda\Big(u_j^{+,n}(\alpha_j^n - \alpha_{j-1}^n) + u_j^{-,n}(\alpha_{j+1}^n - \alpha_j^n)\Big). \tag{4.5}$$

Remark. Observe that we have for stationary solutions

$$\begin{aligned} & \alpha_{g,j+1}^n \rho_{g,j+1}^n u_{g,j+1}^n = \alpha_{g,j}^n \rho_{g,j}^n u_{g,j}^n, \\ & \frac{(u_{g,j+1}^n)^2}{2} + h_g(\rho_{g,j+1}^n) = \frac{(u_{g,j}^n)^2}{2} + h_g(\rho_{g,j}^n), \\ & \alpha_{s,j+1}^n \rho_{s,j+1}^n u_{s,j+1}^n = \alpha_{s,j}^n \rho_{s,j}^n u_{s,j}^n, \\ & \alpha_{s,j+1}^n(\rho_{s,j+1}^n (u_{s,j+1}^n)^2 + p_{s,j+1}^n) + \alpha_{g,j+1}^n(\rho_{g,j+1}^n (u_{g,j+1}^n)^2 + p_{g,j+1}^n) \\ & \qquad = \alpha_{s,j}^n(\rho_{s,j}^n (u_{s,j}^n)^2 + p_{s,j}^n) + \alpha_{g,j}^n(\rho_{g,j}^n (u_{g,j}^n)^2 + p_{g,j}^n), \end{aligned} \tag{4.6}$$

This implies that in the stationary case it holds that

$$\begin{aligned} \rho^n_{g,j+1,-} &= \rho^n_{g,j}, \quad u^n_{g,j+1,-} = u^n_{g,j}, \\ \rho^n_{g,j-1,+} &= \rho^n_{g,j}, \quad u^n_{g,j-1,+} = u^n_{g,j}, \\ \rho^n_{s,j+1,-} &= \rho^n_{s,j}, \quad u^n_{s,j+1,-} = u^n_{s,j}, \\ \rho^n_{s,j-1,+} &= \rho^n_{s,j}, \quad u^n_{s,j-1,+} = u^n_{s,j}, \end{aligned} \tag{4.7}$$

so that

$$U^n_{j+1,-} = U^n_j, \quad U^n_{j-1,+} = U^n_j.$$

This yields

$$U^{n+1}_j = U^n_j. \tag{4.8}$$

The equation (4.8) means that *our scheme captures exactly stationary waves.*

We will present several tests, using the first-order Lax-Friedrich scheme. The $n+1$th time step values of the first variable group U of five components is calculated by the Lax-Friedrichs scheme:

$$U^{n+1}_j = \frac{1}{2}(U^n_{j+1,-} + U^n_{j-1,+}) - \frac{\lambda}{2}(f(U^n_{j+1,-}) - f(U^n_{j-1,+})). \tag{4.9}$$

Test 1. Let us consider the Riemann problem for the system (1.1) with the Riemann data

$$(\rho_{g,0}(x), u_{g,0}(x), \rho_{s,0}, u_{s,0}(x), \alpha_{g,0}(x)) = \begin{cases} U_L, & \text{if} \quad x < 0 \\ U_R, & \text{if} \quad x > 0, \end{cases}$$

where

$$\begin{aligned} U_L &= (0.5, 0.3, 1, 0, 0.7), \\ U_R &= (0.514866336524944, 0.254920531192354, 1.246397691948884, 0, 0.8). \end{aligned} \tag{4.10}$$

It is not difficult to check that the solution is a stationary wave of the form

$$U(x,t) = \begin{cases} U_L, & \text{if} \quad x < 0 \\ U_R, & \text{if} \quad x > 0. \end{cases}$$

The classical (modified) Lax-Friedrichs gives unsatisfactory result, see Figure 1; our scheme captures exactly the stationary wave, see Figure 2.

Test 2. Let us consider the Riemann problem for the system (1.1) with the Riemann data

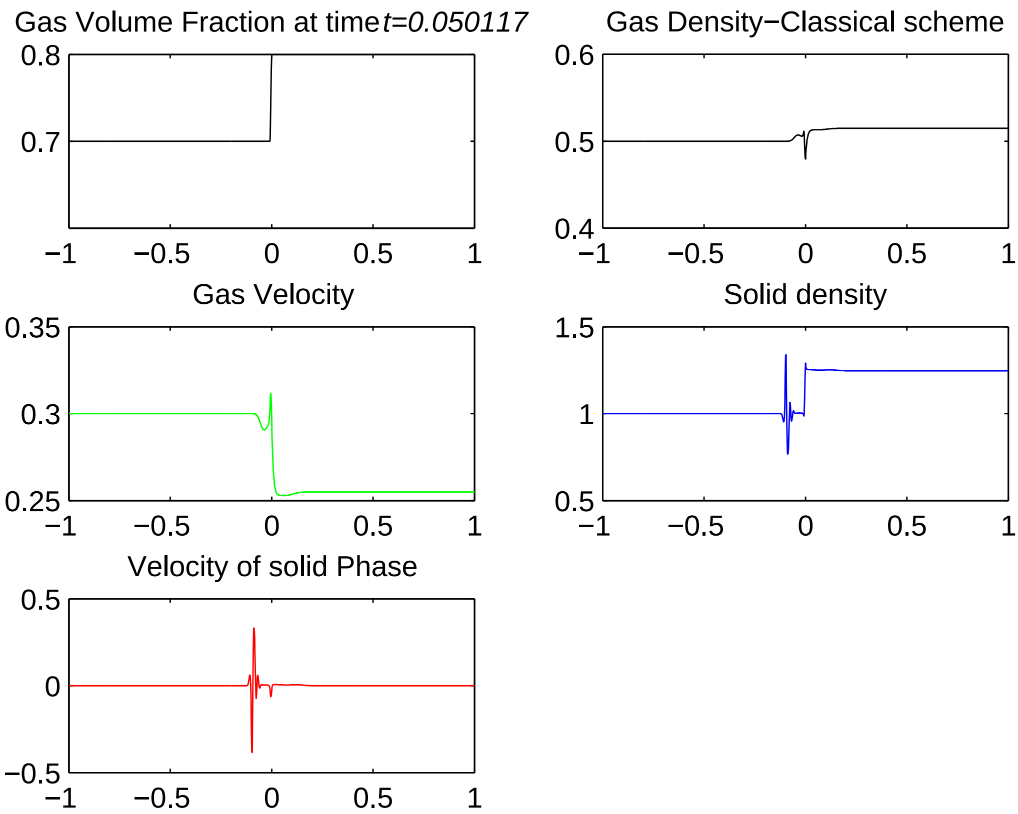

FIGURE 1. Steady states (4.10) by classical scheme

$$(\rho_{g,0}(x), u_{g,0}(x), \rho_{s,0}, u_{s,0}(x), \alpha_{g,0}(x)) = \begin{cases} U_L & \text{if} \quad x < 0 \\ U_R & \text{if} \quad x > 0, \end{cases} \tag{4.11}$$

where

$$U_L = (0.2, 0.8, 1, 0.5, 0.5), \quad U_R = (0.5, 0.9, 1, 1, 0.8, 0.6). \tag{4.12}$$

Our results show a monotone sequence of solutions corresponding to the discretization the interval $[-1, 1]$ into 1000, 2000, 3000 and 4000 points, see Figures 3, 4, 5, 6, and 7.

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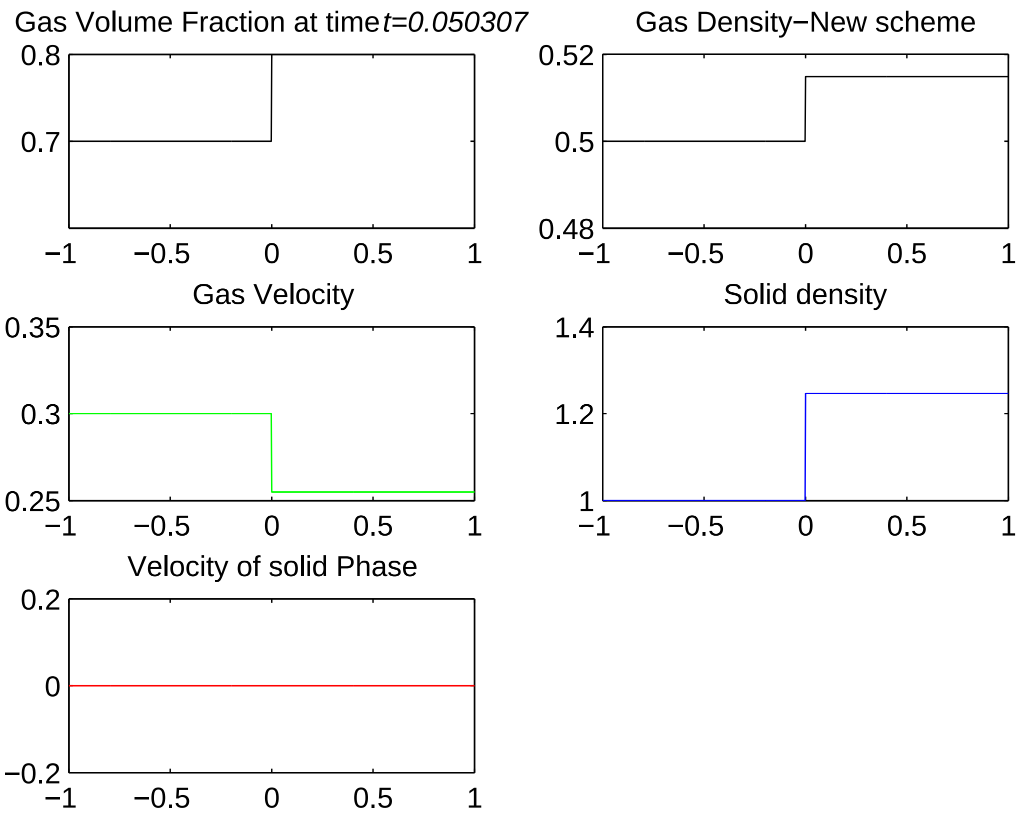

FIGURE 2. Steady states (4.10) are captured by our well-balanced scheme

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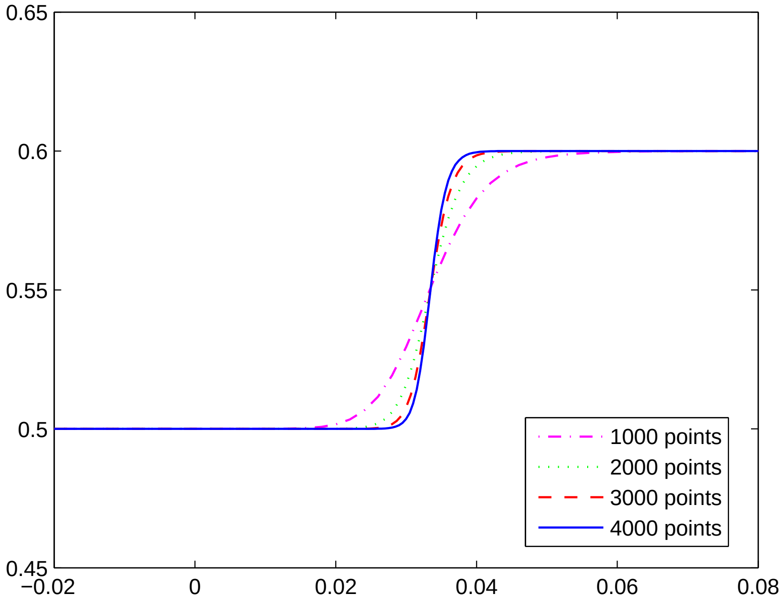

FIGURE 3. Volume fraction of the gas phase of the problem (1.1), (1.2) and (4.12) by our scheme with increasing numbers of discretization

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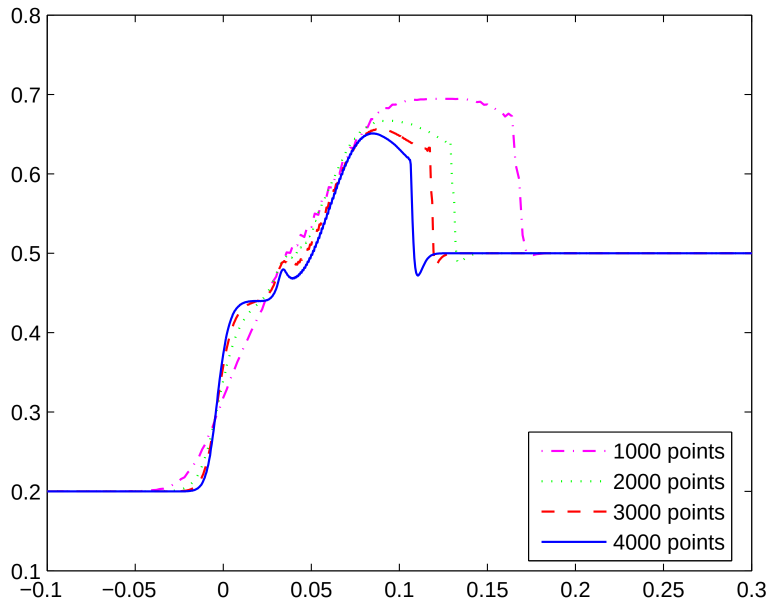

FIGURE 4. Density of the gas phase of the problem (1.1), (1.2) and (4.12) by our scheme with increasing numbers of discretization

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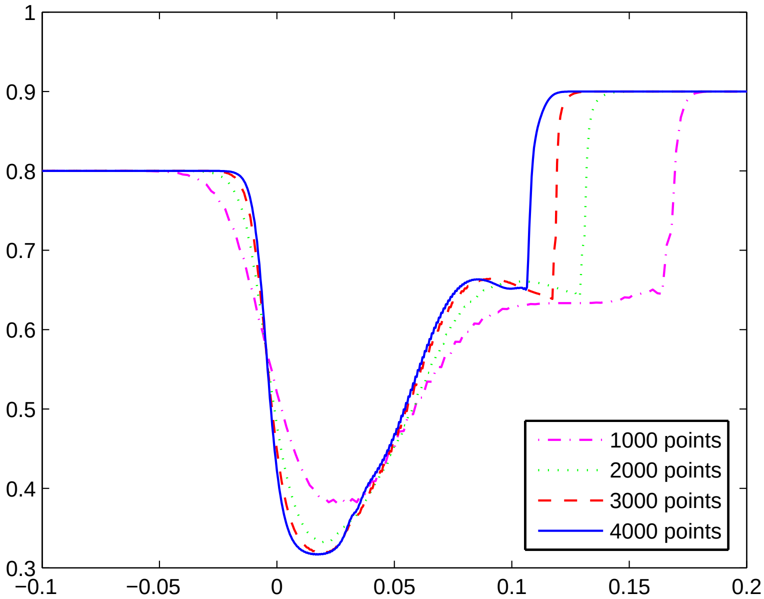

FIGURE 5. Velocity of the gas phase of the problem (1.1), (1.2) and (4.12) by our scheme with increasing numbers of discretization

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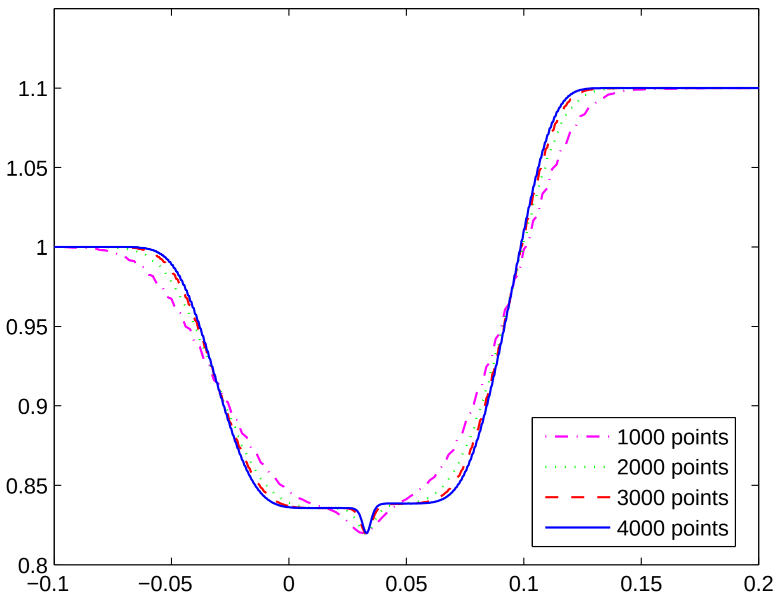

FIGURE 6. Density of the solid phase of the problem (1.1), (1.2) and (4.12) by our scheme with increasing numbers of discretization

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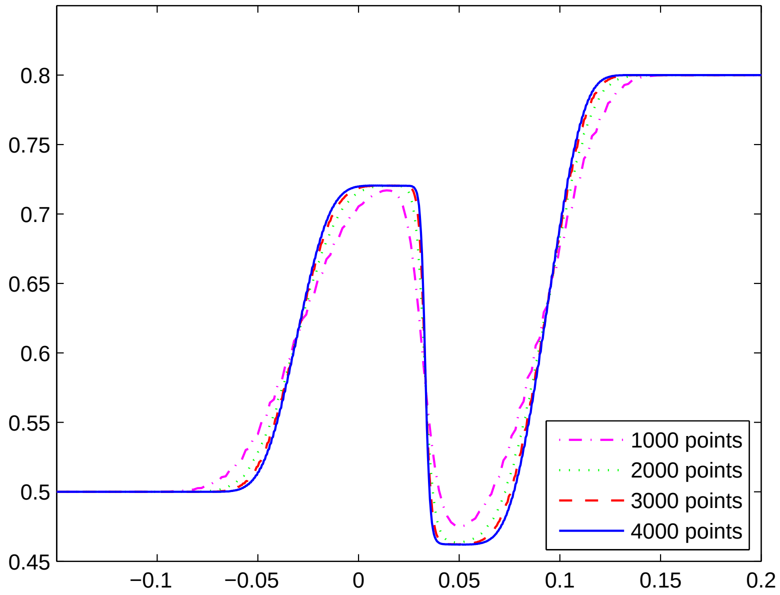

FIGURE 7. Velocity of the solid phase of the problem (1.1), (1.2) and (4.12) by our scheme with increasing numbers of discretization