WELL-BALANCED SCHEME FOR A ONE-PRESSURE MODEL OF TWO-PHASE FLOWS

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ABSTRACT. We consider a one-pressure model of two-phase flows. The first phase if compressible and the second phase is incompressible. The system is thus closed where we have four governing equations for four unknowns. We then construct a well-balanced scheme for this system. Our scheme is capable to maintain equilibrium states.

1. INTRODUCTION

We consider in this paper a one-pressure model of two-phase flows, where the first phase, referred to as the gas phase, is compressible and the second phase, referred to as the liquid phase, is incompressible. Precisely, the model is described by a system of four equations characterizing the conservation of mass in each phase and conservation of momentum when there is exchange of momentum between the two phases, see ([10, 7]):

$$\begin{aligned}
&\partial_t(\alpha\rho) + \partial_x(\alpha\rho u) = 0,\\
&\partial_t(\alpha\rho u) + \partial_x(\alpha(\rho u^2 + p)) = p\partial_x\alpha,\\
&\partial_t(1-\alpha) + \partial_x((1-\alpha)v) = 0,\\
&\partial_t((1-\alpha)\eta v) + \partial_x((1-\alpha)(\eta v^2 + p)) = -p\partial_x\alpha,
\end{aligned} \tag{1.1}$$

Here, ρ, u, α are the density, the velocity, and the volume fraction of the gas phase and η, v are the density and the velocity in the liquid phase, respectively.

Suppose that in the compressible phase the fluid is ideal and therefore the common pressure has an equation of state of the form

$$p = \kappa\rho^\gamma, \quad \kappa > 0, 1 < \gamma < 5/3. \tag{1.2}$$

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The system (1.1) is a system of balanced laws with *source terms*, where sources appear on the right-hand side of the equations of conservation of momentum, i.e., the second and the fourth equations of (1.1). Because of the source terms, the system (1.1) has the form of nonconservative system of conservation laws and the formulation of weak solutions, theoretically, can be understood in the sense of *nonconservative product*, see Dal Maso, LeFloch and Murat [8]. Practically, source terms cause lots of inconveniences in approximating physical solutions of the system. This has been observed even in the case of a single conservation law, shallow water equations, or in the model of fluid flows in a nozzle with variable cross-section, see [12, 20, 13, 6, 11, 4, 5, 2, 17, 16, 24], etc. The work of discretizing source terms has been addressed by many authors, see [3, 7, 1, 9, 21, 26, 25], and the references therein.

In this paper we aim at deriving a well-balanced scheme that is capable to capture equilibrium states resulted by stationary waves. This is important in many applications, in particular when the process under study arrives near a steady state. We then present several test cases to show the efficiency of our method.

2. Stationary waves

As observed earlier, source terms often cause inconvenience for numerical approximations. To reduce the size of sources, we add up the two equations of balance of momentum to get the conservation of momentum of the total in place of the equation of balance of momentum for the liquid phase. So we get

$$\begin{aligned}
&\partial_t(\alpha\rho) + \partial_x(\alpha\rho u) = 0,\\
&\partial_t(\alpha\rho u) + \partial_x(\alpha(\rho u^2 + p)) = p\partial_x\alpha,\\
&\partial_t(1-\alpha) + \partial_x((1-\alpha)v) = 0,\\
&\partial_t(\alpha\rho u + (1-\alpha)\eta v) + \partial_x(\alpha(\rho u^2 + p) + (1-\alpha)(\eta v^2 + p)) = 0,
\end{aligned} \tag{2.1}$$

Setting the conservative variable

$$U = (\alpha\rho, \alpha\rho u, 1-\alpha, \alpha\rho u + (1-\alpha)\eta v)^T,$$

the flux

$$f(U) = (\alpha\rho u, \alpha(\rho^2 + p), (1-\alpha)v, \alpha(\rho u^2 + p) + (1-\alpha)(\eta v^2 + p))^T,$$

and the source

$$S(U) = (0, p\partial_x\alpha, 0, 0)^T,$$

which appears only in the second component, we can rewrite the system (2.1) in a more compact form as

$$\partial_t U(x,t) + \partial_x f(U(x,t)) = S(U(x,t)), \quad x \in \mathbb{R}, t > 0. \tag{2.2}$$

Let us find the characteristic equation the system (1.1). For smooth solutions, the system (1.1) is equivalent to the following system

$$\begin{aligned}
&\alpha_t + v\alpha_x + (\alpha - 1)v_x = 0,\\
&\rho_t + \frac{\rho(u-v)}{\alpha}\alpha_x + u\rho_x + \rho u_x + \frac{(1-\alpha)\rho}{\alpha}v_x = 0,\\
&u_t + \frac{p'(\rho)}{\rho}\rho_x + uu_x = 0,\\
&v_t + \nu p'(\rho)\rho_x + vv_x = 0.
\end{aligned} \tag{2.3}$$

Choosing the independent variable $U = (\alpha, \rho, u, v)$, we can re-write (2.3) in the nonconservative form as

$$U_t + A(U)U_x = 0, \tag{2.4}$$

where

$$A(U) = \begin{pmatrix} v & 0 & 0 & \alpha - 1\\ \dfrac{\rho(u-v)}{\alpha} & u & \rho & \dfrac{(1-\alpha)\rho}{\alpha}\\ 0 & \dfrac{p'(\rho)}{\rho} & u & 0\\ 0 & \nu p'(\rho) & 0 & v \end{pmatrix}.$$

The characteristic equation is given by

$$(v-\lambda)^2((u-\lambda)(v-\lambda) - p') + (u-\lambda)^2\frac{\nu p'(\alpha-1)\rho}{\alpha} = 0$$

which does not always admit four distinct roots. Consequently, the system may not be hyperbolic. The lack of hyperbolicity of two-phase flows models was observed in several papers, see for example [18].

Let us now investigate the stationary contact waves of the system (2.1). Motivated by our earlier works ([23, 20]), we look for stationary waves resulted by source terms. Thus, they are concerned only on the first two equations of the compressible phase. Stationary waves are just the limit of stationary smooth solutions of (2.1). *A stationary smooth solution U* of (2.1) is a time-independent smooth solution. Therefore, stationary solutions of (2.1) satisfy the following ordinary differential equations

$$
\begin{aligned}
&(\alpha\rho u)' = 0,\\
&(\alpha(\rho u^2 + p))' = p\alpha',\\
&((1-\alpha)v)' = 0,\\
&(\alpha(\rho u^2 + p) + (1-\alpha)(\eta v^2 + p))' = 0,
\end{aligned}
\tag{2.5}
$$

where $(.)'$ stands for d/dx. We have to make sure that if α is constant, then no stationary waves are formed. From (2.5) we have

$$
\begin{aligned}
&(\alpha\rho u)' = 0,\\
&\Big(\frac{u^2}{2} + h(\rho)\Big)' = 0,\\
&((1-\alpha)v)' = 0,\\
&(\alpha(\rho u^2 + p) + (1-\alpha)(\eta v^2 + p))' = 0,
\end{aligned}
\tag{2.6}
$$

where $h'(\rho) = p'(\rho)/\rho$, or

$$
h(\rho) = \frac{\kappa\gamma}{\gamma - 1}\rho^{\gamma-1}.
$$

We look for stationary jumps which are limit of smooth solutions of (2.6). Then (2.6) yields the following result which gives us the way to compute stationary waves.

Lemma 2.1. *The left-hand and right-hand states of a stationary contact satisfy*

$$
\begin{aligned}
&[\alpha\rho u] = 0,\\
&\Big[\frac{u^2}{2} + h(\rho)\Big] = 0,\\
&[(1-\alpha)v] = 0,\\
&[\alpha(\rho u^2 + p) + (1-\alpha)(\eta v^2 + p)] = 0,
\end{aligned}
\tag{2.7}
$$

where $[\alpha\rho u] := \alpha^+\rho^+u^+ - \alpha^-\rho^-u^-$*, and so on, denotes the difference of the corresponding value* $\alpha\rho u$ *between the right-hand and left-hand states of the stationary contact.*

From Lemma 2.1, we deduce that a stationary wave from a given state $U_0 = (\alpha_0, \rho_0, u_0, v_0)$ to some state $U = (\alpha, \rho, u, v)$ must satisfy the relations on the gas phase

$$\begin{aligned} &\alpha\rho u = \alpha_0\rho_0 u_0,\\ &\frac{u^2}{2} + h(\rho) = \frac{u_0^2}{2} + h(\rho_0). \end{aligned} \tag{2.8}$$

This leads us to finding roots of the equation

$$F(U_0,\rho,\alpha) := \operatorname{sgn}(u_0)\Big(u_0^2 - \frac{2\kappa\gamma}{\gamma-1}(\rho^{\gamma-1}-\rho_0^{\gamma-1})\Big)^{1/2}\rho - \frac{\alpha_0 u_0\rho_0}{\alpha} = 0. \tag{2.9}$$

To find zeros of the function $F(U_0,\rho,\alpha)$, we observe that it is well-defined whenever

$$u_0^2 - \frac{2\kappa\gamma}{\gamma-1}(\rho^{\gamma-1}-\rho_0^{\gamma-1}) \geq 0,$$

or

$$\rho \leq \bar\rho(U_0) := \Big(\frac{\gamma-1}{2\kappa\gamma}u_0^2 + \rho_0^{\gamma-1}\Big)^{\frac{1}{\gamma-1}}.$$

We have

$$\frac{\partial F(U_0,\rho;\alpha)}{\partial\rho} = \frac{u_0^2 - \frac{2\kappa\gamma}{\gamma-1}(\rho^{\gamma-1}-\rho_0^{\gamma-1}) - \kappa\gamma\rho^{\gamma-1}}{\Big(u_0^2 - \frac{2\kappa\gamma}{\gamma-1}(\rho^{\gamma-1}-\rho_0^{\gamma-1})\Big)^{1/2}}.$$

Assume, for definitiveness, that $u_0 > 0$. The last expression yields

$$\begin{aligned} &\frac{\partial F(U_0,\rho;\alpha)}{\partial\rho} > 0, \quad \rho < \rho_{\max}(\rho_0,u_0),\\ &\frac{\partial F(U_0,\rho;\alpha)}{\partial\rho} < 0, \quad \rho > \rho_{\max}(\rho_0,u_0), \end{aligned} \tag{2.10}$$

where

$$\rho_{\max}(\rho_0,u_0) := \Big(\frac{\gamma-1}{\kappa\gamma(\gamma+1)}u_0^2 + \frac{2}{\gamma+1}\rho_0^{\gamma-1}\Big)^{\frac{1}{\gamma-1}}.$$

Since

$$F(U_0,\rho=0,a) = F(U_0,\rho=\bar\rho,a) = -\frac{\alpha_0 u_0\rho_0}{\alpha} < 0,$$

the function $\rho \mapsto F(U_0,\rho;\alpha)$ admits a root if and only if the maximum value is non-negative:

$$F(U_0,\rho=\rho_{\max},\alpha) \geq 0,$$

or, equivalently,

$$\alpha \geq \alpha_{\min}(U_0) := \frac{\alpha_0\rho_0|u_0|}{\sqrt{\kappa\gamma}\,\rho_{\max}^{\frac{\gamma+1}{2}}(\rho_0,u_0)}. \tag{2.11}$$

Similar argument can be made for $u_0 < 0$.

It will be convenient to set in the (ρ, u)-plan the following sets, referred to as the "lower region" G_1, the "middle region" G_2, and the "upper region" G_3, and the "boundary" $\mathcal{C}$, as

$$\begin{aligned} G_1 &:= \{(\rho, u) \,:\, u < -\sqrt{p'(\rho)}\}, \\ G_2 &:= \{(\rho, u) \,:\, |u| < \sqrt{p'(\rho)}\}, \\ G_3 &:= \{(\rho, u) \,:\, u > \sqrt{p'(\rho)}\}. \end{aligned} \tag{2.12}$$

The existence of the zeros are followed immediately from (2.10) and (2.11). We are now at a position to say about the the existence as well as properties of zeros of the function $F(U_0, \rho, \alpha)$.

Lemma 2.2. *Given $U_0 = (\alpha_0, \rho_0, u_0)$ and $0 \le \alpha \le 1$. The function $F(U_0, \rho, \alpha)$ in (2.10) admits a zero if if and only if $a \ge \alpha_{\min}(U_0)$. In this case, $F(U_0, \rho, \alpha)$ admits two distinct zeros, denoted by $\rho = \varphi_1(U_0, \alpha), \rho = \varphi_2(U_0, \alpha)$ such that*

$$\varphi_1(U_0, \alpha) \le \rho_{\max}(U_0) \le \varphi_2(U_0, \alpha) \tag{2.13}$$

the equality in (2.13) holds only if $\alpha = \alpha_{\min}(U_0)$.

Lemma 2.3. *(a) We have*

$$\begin{aligned} &\rho_{\max}(\rho_0, u_0) < \rho_0, \quad (\rho_0, u_0) \in G_2, \\ &\rho_{\max}(\rho_0, u_0) > \rho_0, \quad (\rho_0, u_0) \in G_3 \cup G_1, \\ &\rho_{\max}(\rho_0, u_0) = \rho_0, \quad (\rho_0, u_0) \in \mathcal{C}_\pm. \end{aligned} \tag{2.14}$$

(b) $(\varphi_1(U_0, \alpha), u) \in G_1$ if $u_0 < 0$, and $(\varphi_1(U_0, \alpha), u) \in G_3$ if $u_0 > 0$; $(\varphi_2(U_0, \alpha), u) \in G_2$, where u is defined by (2.10). Moreover,

$$(\rho_{\max}(U_0, \alpha), u) \in \mathcal{C}. \tag{2.15}$$

In addition, we have

(i) *If $\alpha > \alpha_0$, then*

$$\varphi_1(U_0, \alpha) < \rho_0 < \varphi_2(U_0, \alpha). \tag{2.16}$$

(ii) *If $\alpha < \alpha_0$, then*

$$\begin{aligned} &\rho_0 < \varphi_1(U_0, \alpha) \quad \textit{for} \quad U_0 \in G_1 \cup G_3, \\ &\rho_0 > \varphi_2(U_0, \alpha) \quad \textit{for} \quad U_0 \in G_2. \end{aligned} \tag{2.17}$$

(c)

$$\begin{aligned}
&\alpha_{\min}(U,\alpha) < \alpha, \quad (\rho,u) \in G_i,\ i = 1,2,3,\\
&\alpha_{\min}(U,\alpha) = \alpha, \quad (\rho,u) \in \mathcal{C},\\
&\alpha_{\min}(U,\alpha) = 0, \quad \rho = 0 \quad or \quad u = 0.
\end{aligned} \tag{2.18}$$

Proof. Most of the proof was available in [23]. However, for completeness, we will show the steps. Assume for simplicity that $u_0 > 0$. Define

$$g(U_0,\rho) = u_0^2 - \frac{2\kappa\gamma}{\gamma-1}(\rho^{\gamma-1} - \rho_0^{\gamma-1}) - \kappa\gamma\rho^{\gamma-1}. \tag{2.19}$$

Then, a straightforward calculation gives

$$g(U_0, \rho_{\max}(U_0)) = 0,$$

which proves (2.15). On the other hand, since

$$\frac{dg(U_0,\rho)}{d\rho} = -(\gamma+1)\kappa\gamma\rho^{\gamma-2} < 0,$$

and that $\varphi_1(U_0,\alpha) < \rho_{\max}(U_0,\alpha) < \varphi_2(U_0,\alpha)$ it holds that

$$g(U_0,\varphi_1(U_0,\alpha)) > g(U_0,\rho_{\max}(U_0)) = 0 > g(U_0,\varphi_1(U_0,\alpha)).$$

The last two inequalities justify the statement in (b). Moreover,

$$F(U_0,\rho_0;\alpha) = \rho_0 u_0(1-\alpha_0/\alpha) > 0 \quad \text{iff} \quad a > \alpha_0,$$

which proves (2.16), and shows that ρ_0 is located outside of the interval $[\varphi_1(U_0,\alpha), \varphi_2(U_0,\alpha)]$ in the opposite case. Since

$$\frac{\partial F(U_0,\rho_0;\alpha)}{\partial\rho} = \frac{u_0^2 - \kappa\gamma\rho_0^{\gamma-1}}{u_0} < 0 \quad \text{iff} \quad U_0 \in G_2,$$

which, together with the earlier observation, implies (2.17).

We next check (2.18) for $a = \alpha_0$. It comes from the definition of $\alpha_{\min}(U_0)$ that $\alpha_{\min}(U_0) < \alpha_0$ if and only if

$$\sqrt{\kappa\gamma}\rho^{*\frac{\gamma+1}{2}} > \rho_0|u_0|,$$

that can be equivalently written as

$$Q(m) := \frac{2}{\gamma+1}m - (\kappa\gamma)^{\frac{1-\gamma}{\gamma+1}} m^{\frac{2}{\gamma+1}} + \frac{\gamma-1}{\kappa\gamma(\gamma+1)} > 0,$$

where $m := \rho_0^{\gamma-1}/u_0^2$. Then, we can see that

$$Q(1/\kappa\gamma) = 0, \tag{2.20}$$

which, in particular shows that the second equation in (2.18) holds, since $(\rho_0, u_0) \in \mathcal{C}_\pm$ for $m = 1/\kappa\gamma$. Moreover,

$$\frac{dQ(m)}{dm} = \frac{2}{\gamma+1}(1 - (\kappa\gamma m)^{\frac{1-\gamma}{\gamma+1}}),$$

which is positive for $m > 1/\kappa\gamma$ and negative for $m < \kappa\gamma$. This together with (2.20) establish the first statement in (2.18). The third statement in (2.18) is straightforward. This completes the proof of Lemma 2.3. □

To select a unique stationary wave, we need the following so-called Monotonicity criterion. The relationships (2.7) also defines a curve $\rho \mapsto \alpha = \alpha(U_0, \rho)$. So we require that

MONOTONICITY CRITERION. *The volume fraction $\alpha = \alpha(U_0, \rho)$ must vary monotonically between the two values ρ_0 and ρ_1, where ρ_1 is the ρ-value of the corresponding state of a stationary wave having U_0 as one state.*

A similar criterion was used by Kröner, LeLoch, and Thanh [22, 20, 19], Isaacson and Temple [14, 15].

Geometrically, we can choose either φ_1 or φ_2 in the domains G_1, G_2, G_3 using the following lemma.

Lemma 2.4. *The Monotonicity Criterion is equivalent to saying that any stationary shock does not cross the boundary $\mathcal{C}$. In other words:*

(i) *If $U_0 \in G_1 \cup G_3$, then only the zero $\varphi(U_0, \alpha) = \varphi_1(U_0, \alpha)$ is selected.*
(ii) *If $U_0 \in G_2$, then only the zeros $\varphi(U_0, \alpha) = \varphi_2(U_0, \alpha)$ is selected.*

Proof. The second equation of (2.7) determines the u-value as $u = u(\rho)$. Taking the derivative with respect to ρ in the equation

$$\alpha^2(u(\rho)\rho)^2 = (\alpha_0 u_0 \rho_0)^2,$$

we get

$$\alpha(\rho)\alpha'(\rho)(u\rho)^2 + 2\alpha^2(u\rho)(u(\rho)'\rho + u(\rho)) = 0. \tag{2.21}$$

Thus, to prove the lemma, it is sufficient to show that the factor $(u(\rho)'\rho + u(\rho))$ remains of a constant sign whenever (ρ, u) remains in the same domain. Indeed, assume for simplicity that $u_0 > 0$, then

$$\begin{aligned} u'(\rho)\rho + u(\rho) &= \frac{-\kappa\gamma\rho^{\gamma-1}}{u} + u \\ &= \frac{u^2 - \kappa\gamma\rho^{\gamma-1}}{u}, \end{aligned}$$

which remains of a constant sign as long as (ρ, u) remain in the same domain. This completes the proof of Lemma 2.4. □

3. Construction of the well-balanced scheme

Given a uniform time step Δt, and a spacial mesh size Δx, setting $x_j = j\Delta x, j \in \mathbf{Z}$, and $t_n = n\Delta t, n \in \mathbf{N}$, we denote U_j^n to be an approximation of the exact value $U(x_j, t_n)$.

A C.F.L condition is also required on the mesh sizes:

$$\lambda \max_U ||A(U)|| < 1, \quad \lambda := \frac{\Delta t}{\Delta x}. \tag{3.1}$$

The well-balanced scheme is defined by

$$U_j^{n+1} = U_j^n - \lambda(g(U_j^n, U_{j+1,-}^n) - g(U_{j-1,+}^n, U_j^n)). \tag{3.2}$$

where $U = (\alpha, \alpha v, \alpha\rho, \alpha\rho u)^T$, for some numerical flux g. The states $U_{j+1,-}^n, U_{j-1,+}^n$ are defined as followed. Set

$$U_0 = U_{j+1}^n = (\alpha_{j+1}^n, \alpha_{j+1}^n v_{j+1}^n, \alpha_{j+1}^n \rho_{j+1}^n, \alpha_{j+1}^n \rho_{j+1}^n u_{j+1}^n)^T.$$

Then, we take $\alpha = \alpha_j^n$ and compute the corresponding $\rho = \rho_{n+1,-}^n := \varphi(U_0, \alpha)$, and then $u = u_{n+1,-}^n$, and $v = v_{n+1,-}^n$, from (2.8).

Similarly, Set

$$U_0 = U_{j-1}^n = (\alpha_{j-1}^n, \alpha_{j-1}^n v_{j-1}^n, \alpha_{j-1}^n \rho_{j-1}^n, \alpha_{j-1}^n \rho_{j-1}^n u_{j-1}^n)^T.$$

Then, we take $\alpha = \alpha_j^n$ and compute the corresponding $\rho = \rho_{n-1,+}^n := \varphi(U_0, \alpha)$, and then $u = u_{n-1,+}^n$, and $v = v_{n-1,+}^n$, from (2.8).

Remark. Observe that we have for stationary solutions

$$\begin{aligned} &\alpha_{j+1}^n \rho_{j+1}^n u_{j+1}^n = \alpha_j^n \rho_j^n u_j^n, \\ &\frac{(u_{j+1}^n)^2}{2} + h(\rho_{j+1}^n) = \frac{(u_j^n)^2}{2} + h(\rho_j^n). \end{aligned} \tag{3.3}$$

This implies that in the stationary case it holds that

$$\begin{aligned} \alpha_{j+1,-}^n &= \alpha_j^n, \quad \rho_{j+1,-}^n = \rho_j^n, \quad u_{j+1,-}^n = u_j^n, \\ \alpha_{j-1,+}^n &= \alpha_j^n, \quad \rho_{j-1,+}^n = \rho_j^n, \quad u_{j-1,+}^n = u_j^n, \end{aligned} \tag{3.4}$$

so that

$$U_{j+1,-}^n = U_j^n, \quad U_{j-1,+}^n = U_j^n.$$

This yields

$$U_j^{n+1} = U_j^n. \tag{3.5}$$

The equation (3.5) means that *our scheme captures exactly stationary waves.*

We will present several tests, using the first-order Lax-Friedrich scheme. The $n+1$th time step values of the first variable group U of five components is calculated by the Lax-Friedrichs scheme:

$$U_j^{n+1} = \frac{1}{2}(U_{j+1,-}^n + U_{j-1,+}^n) - \frac{\lambda}{2}(f(U_{j+1,-}^n) - f(U_{j-1,+}^n)). \tag{3.6}$$

Test 1. Take $\eta = 0.5$. Let us consider the Riemann problem for the system (1.1) with the Riemann data

$$(\alpha_0(x), \rho_0(x), u_0(x), v_0(x)) = \begin{cases} U_L, & \text{if} \quad x < 0 \\ U_R, & \text{if} \quad x > 0, \end{cases}$$

where

$$\begin{aligned} U_L = (0.3, 0.5, 1, 1), \quad U_R =& (0.362272850517834, 0.372463472934059, \\ & 1.111659496416635, 1.097648109490711). \end{aligned} \tag{3.7}$$

It is not difficult to check that the solution is a stationary wave of the form

$$U(x,t) = \begin{cases} U_L, & \text{if} \quad x < 0 \\ U_R, & \text{if} \quad x > 0. \end{cases}$$

The modified Lax-Friedrichs gives unsatisfactory result, see Figure 1; our scheme captures exactly the stationary wave, see Figure 2.

Test 2. Take $\eta = 0.5$. Let us consider the Riemann problem for the system (1.1) with the Riemann data

$$(\alpha_0(x), \rho_0(x), u_0(x), v_0(x)) = \begin{cases} (0.4, 0.1, 1, 1) & \text{if} \quad x < 0 \\ (0.5, 0.2, 1.5, 1.2) & \text{if} \quad x > 0. \end{cases} \tag{3.8}$$

Our results show a monotone sequence of solutions corresponding to the discretization the interval $[-1, 1]$ into 500, 1000, and 2000 points, see Figures 3, 4, 5, and 6.

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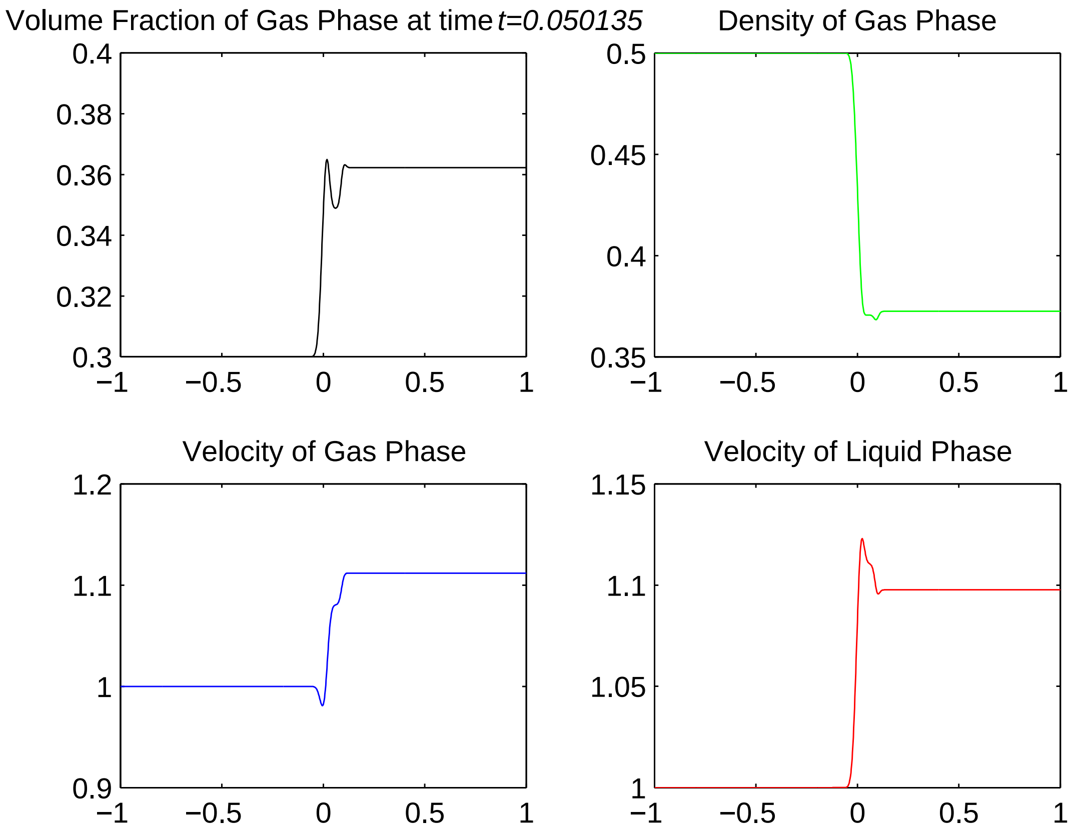

FIGURE 1. A stationary wave by classical scheme

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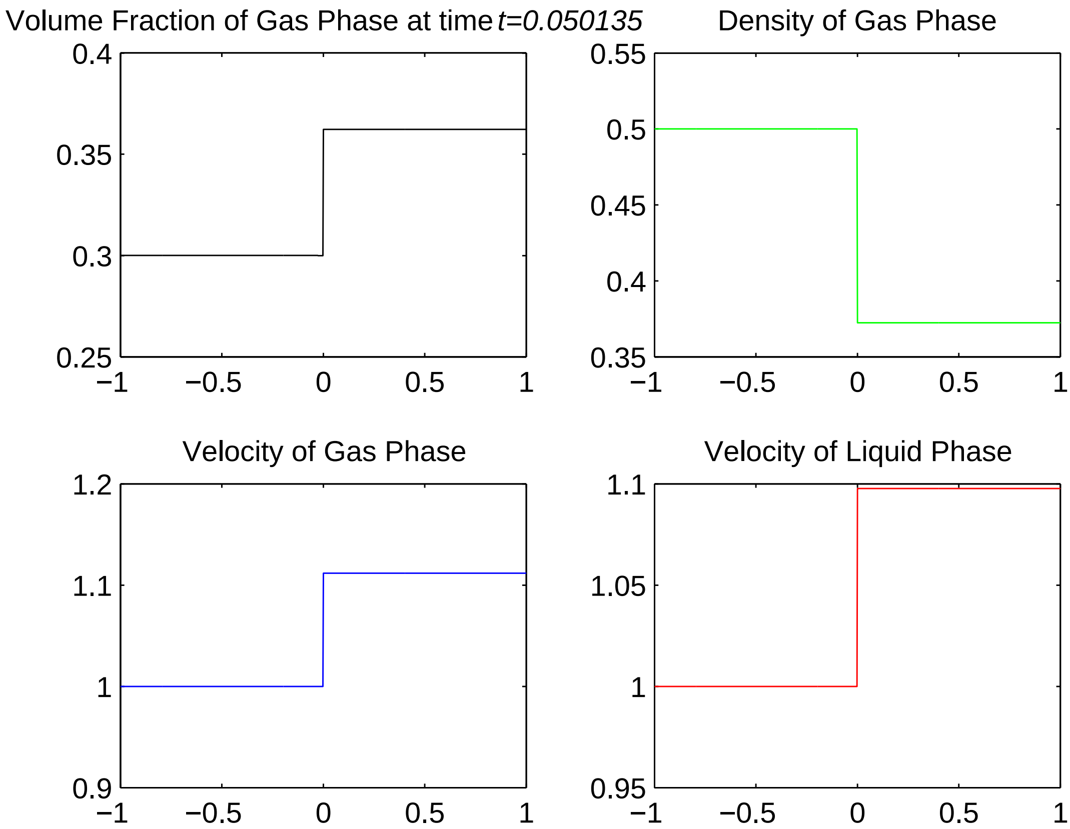

FIGURE 2. The stationary wave by well-balanced scheme

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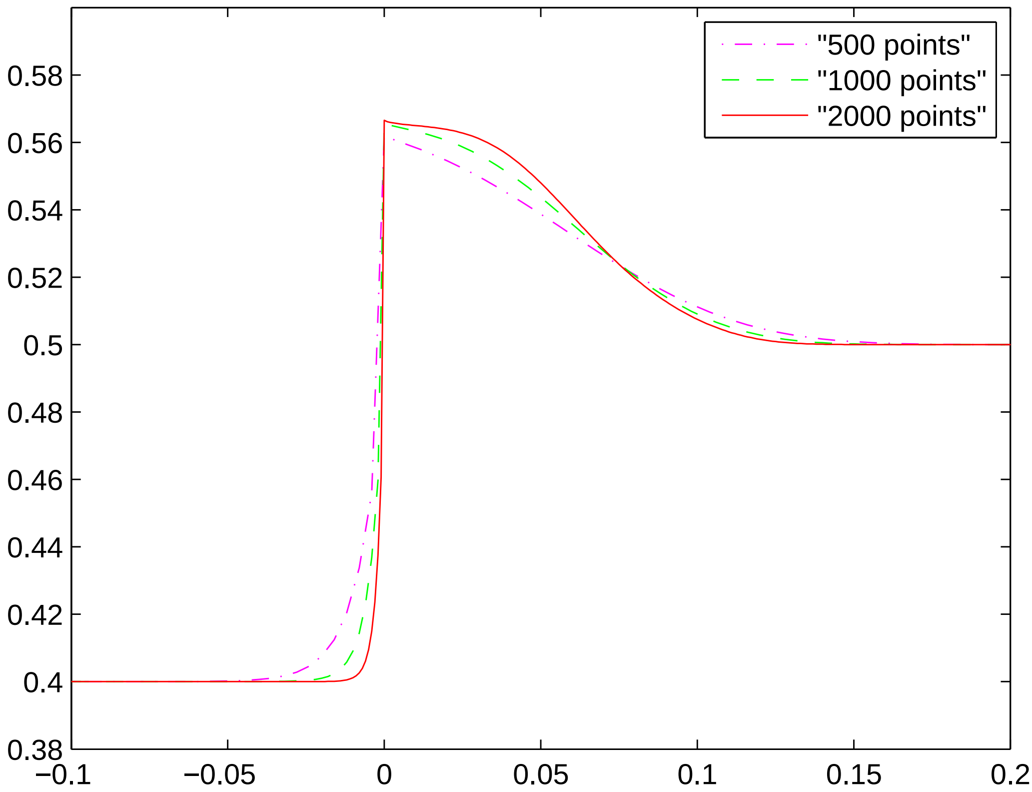

FIGURE 3. Volume fraction of the gas phase by our scheme with increasing numbers of discretization

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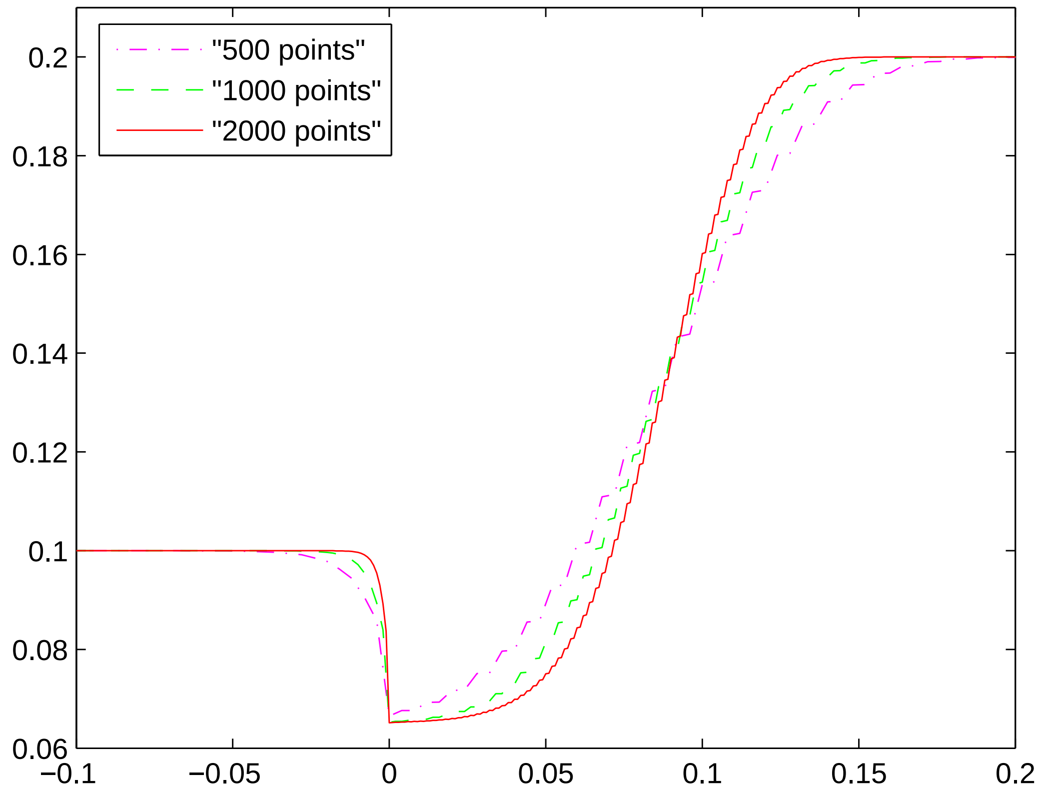

FIGURE 4. Density of the gas phase by our scheme with increasing numbers of discretization

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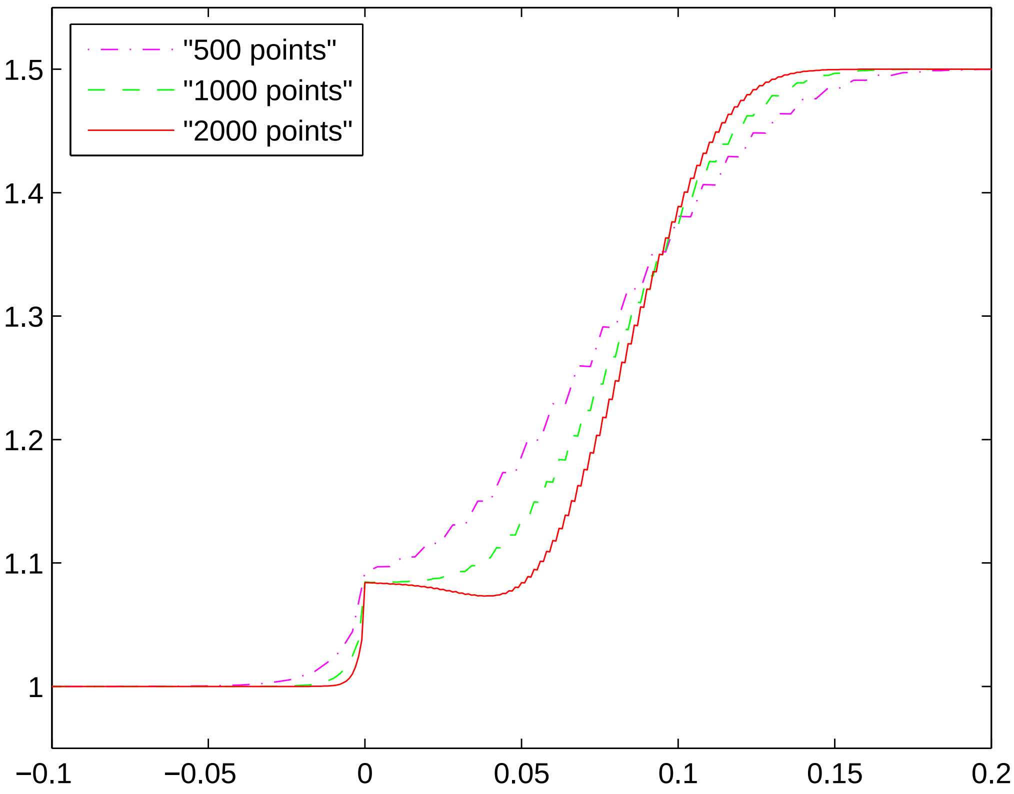

FIGURE 5. Velocity of the gas phase by our scheme with increasing numbers of discretization

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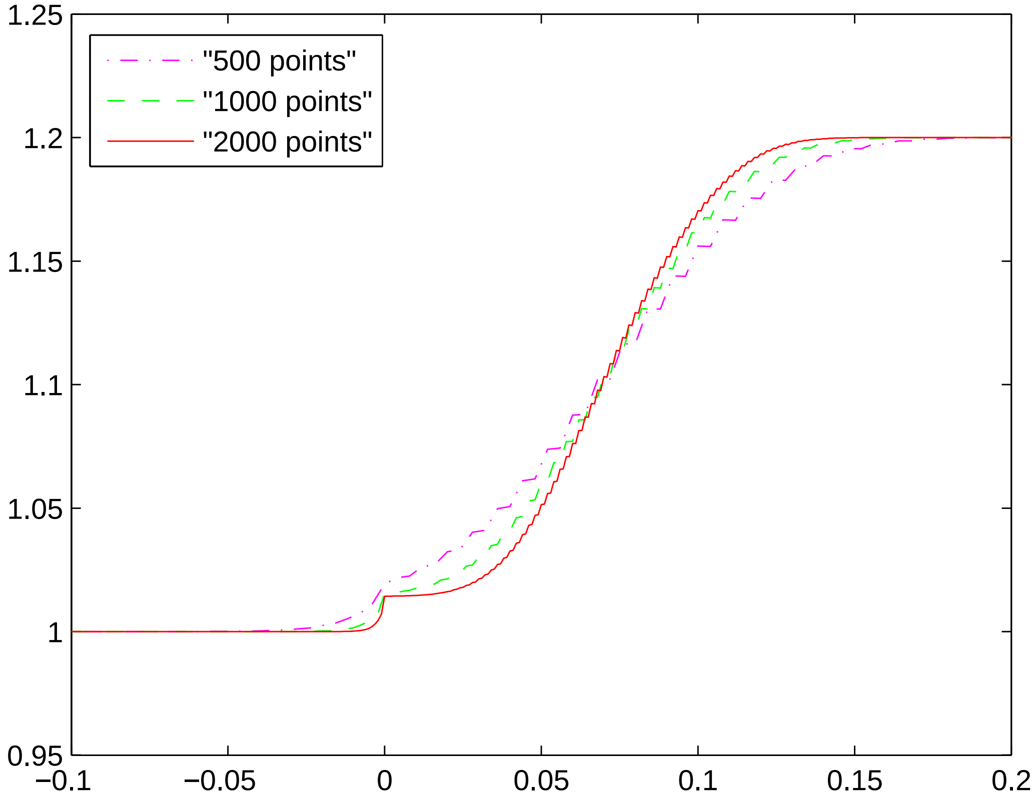

FIGURE 6. Velocity of the liquid phase by our scheme with increasing numbers of discretization