GENERALIZED KKM-TYPE THEOREMS IN GFC-SPACES AND APPLICATIONS

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Abstract. We define a generalized KKM mapping, called T-KKM mapping, and the corresponding generalized KKM property, which include many counterparts existing in the literature. KKM-type theorems, coincidence theorems and geometric section theorems are established to generalize recent known results.

Keywords. GFC-spaces, T-KKM mappings, generalized KKM property, transfer compactly closed-valued (open-valued) mappings, KKM-type theorems, coincidence theorems.
2000 MR Subject Classification: 47H04, 47H10, 54H25, 55M20

1. INTRODUCTION

In 1929, three well-known Polish mathematicians established in [20] the famous classical KKM theorem in finite dimensional spaces. Fan [10] extended this theorem to infinite dimensional spaces. Since then, lots of generalizations and applications have been obtained. In early forms of this fundamental result, convexity assumptions played a crucial role and restricted the ranges of applicable areas. In [16] Horvath, replacing convex hulls by contract subsets, gave a purely topological version of the KKM theorem. Tian

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[30] proved so-called F-KKM theorems. Park and Kim [25-27] introduced G-convex spaces and developed KKM-type theorems. Ding [4,5] proposed H-KKM and L-KKM theorems for mappings from a set to an H-space and L-convex space without linear structures. Other generalizations were also proved, e.g. G-KKM theorems in [28], R-KKM theorems in [31], generalized S-KKM theorems in [22], etc.

In [3] a KKM property and a KKM class of set-valued mappings with such a property were introduced and investigated. Generalizations of such classes have been developed with many applications, especially in solution existence studies.

On the other hand, KKM-type theorems have been proved to be equivalent to many other fundamental results in nonlinear analysis. The KKM theorem was proved to be equivalent to generalization of Fan's fixed-point theorem and a minimax result in [11]. An equivalence between the KKM theorem and Brouwer's fixed-point theorem and minimax theorems was investigated in [1]. Another equivalent fixed-point theorem was shown in [29]. Later such equivalences have been extended to many kinds of coincidence theorems, matching theorems, intersection theorems, maximal-point theorems, section theorems and also several geometric results by many researchers

Applications of KKM-type theorems, especially in existence studies for variational inequalities, equilibrium problems and more general settings have been obtained by many authors, see e.g. recent papers: [8, 12-15, 17, 18, 23, 24, 32] and references therein.

To avoid in a stronger sense convexity structures in investigating KKM-type theorems, Ding [6] introduced the notion of a finitely continuous topological space (FC-space in short) to encompass many convexity structure settings. However, this notion is incomparable with that of a G-convex space. In [19] generalized FC-spaces (GFC-spaces in short) were proposed, also without any convexity structure, but include both G-convex spaces and FC-spaces as well as many recent generalized spaces. Intersection, coincidence and maximal-element theorems in GFC-spaces were then discussed in the mentioned paper.

The aim of present paper is to develop generalized KKM-type theorems along with generalized KKM classes and applications in deriving some section and coincidence theorems and geometric results, which are equivalent to these KKM-type theorems. Our results include and improve recent existing results of the same topic. The paper is organized as follows. Section 2 is devoted to recalling needed definitions and preliminary facts. In Section 3 we establish KKM type theorems. The last Section 4 deals with applications of these results in deriving section theorems and coincidence theorems. Comparisons with known recent results are always accompanied with our contributions. We leave applications in existence studies for optimization-related problems to another paper.

2. PRELIMINARIES

We recall first some notions for later uses. Let X be a topological space and Y be a set. A subset $A \subseteq X$ is said to be compactly open (compactly closed) if for each nonempty compact subset $K \subseteq X$, $A \cap K$ is open (closed, respectively) in K. The compact interior and compact closure of A are defined by

$$\mathrm{cint}A = \bigcup\{B \subseteq X : B \subseteq A \text{ and } B \text{ is compactly open in } X\},$$
$$\mathrm{ccl}A = \bigcap\{B \subseteq X : B \supseteq A \text{ and } B \text{ is compactly closed in } X\}.$$

We clearly have $K \cap \mathrm{cint}A= \mathrm{int}_K(K \cap A)$ and $K \cap \mathrm{ccl}A= \mathrm{cl}_K(K \cap A)$, where cl_K stands for the closure with respect to (wrt) the topology of K, induced by that of X. A set-valued mapping $F : Y \to 2^X$, is called transfer open-valued (transfer closed-valued) if $\forall y \in Y$, $\forall x \in F(y)$ ($\forall x \notin F(y)$), $\exists y' \in Y$ such that $x \in \mathrm{int}(F(y')$ ($x \notin \mathrm{cl}(F(y')$, respectively). $F : Y \to 2^X$ is termed transfer compactly open-valued (transfer compactly closed-valued) if $\forall y \in Y$, $\forall K \subset X$: nonempty and compact, $\forall x \in F(y) \cap K$ ($\forall x \notin F(y) \cap K$), $\exists y' \in Y$ such that $x \in \mathrm{int}_K(F(y') \cap K)$ ($x \notin \mathrm{cl}_K(F(y') \cap K)$, respectively). Of course transfer open-valuedness (transfer closed-valuedness) implies transfer compact open-valuedness(transfer compact closed-valuedness, respectively).

Lemma 2.1(e.g.[7]) *Let Y be a set, X be a topological space and $F : Y \to 2^X$. The following statements are equivalent*

(i) *F is transfer compactly closed-valued (transfer compactly open-valued, respectively).*

(ii) *for each compact subset $K \subset X$,*

$$\bigcap_{y\in Y}(K \cap F(y)) = \bigcap_{y\in Y}(K \cap \mathrm{ccl}F(y)) = \bigcap_{y\in Y}(K \cap \mathrm{cl}_K F(y))$$

$$(\bigcup_{y\in Y}(K \cap F(y)) = \bigcup_{y\in Y}(K \cap \mathrm{cint}F(y)) = \bigcup_{y\in Y}(K \cap \mathrm{int}_K F(y))).$$

For a nonempty set Y, $\langle Y \rangle$ stands for the set of all finite subsets of Y. For $n \in \mathbb{N}$, the set of the natural numbers, Δ_n stands for the n-simplex with the vertices being the unit vectors e_1, e_2, ..., e_{n+1} of a basis of R^{n+1}. For a topological space X, a subset $A \subset X$ and a mapping $H : X \to 2^Y$, $\overline{A}$, A^c and $H|_A$ denote the closure and complement of A and the mapping H restricted on A, respectively.

Definition 2.1([19])

(i) Let X be a topological space, Y be a nonempty set and Φ be a family of continuous mappings $\varphi : \Delta_n \to X, n \in \mathbb{N}$. Then a triple (X, Y, Φ) is said to be a generalized finitely continuous topological space (GFC-space in short) if for each finite subset $N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$, there is $\varphi_N : \Delta_n \to X$ of the family Φ. Later we also use $(X, Y, \{\varphi_N\})$ to denote (X, Y, Φ).

(ii) Let $D, C \subseteq Y$ and set-valued $S : Y \to 2^X$ be given. D is called an S-subset of Y (S-subset of Y wrt C) if $\forall N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$, $\forall \{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N \cap D$ ($\subseteq N \cap C$, respectively), $\varphi_N(\Delta_k) \subseteq S(D)$, where Δ_k is the face of Δ_n corresponding to $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\}$.

If $Y = X$, we simply write (X, Φ), instead of (X, X, Φ) which collaps to an FC-space, introduced in [6]. If in addition, S is the identity map then an S-subset of Y coincides with an FC-subspace of Y [6]. If $Y \subseteq X$ and (X, Y, Γ)

is a G-convex space (with Γ being a generalized convex hull operator) introduced in [26], then taking Φ as the family of the continuous mappings $\varphi_N : \Delta_n \to \Gamma(N)$ as defined in [26], we obtain a special case (X, Y, Φ) of GFC-spaces. Both G-convex spaces and FC-spaces are general and include many spaces with general convexity structures, but are incomparable.

Definition 2.2

(i) Let (X, Y, Φ) be a GFC-space and Z be a topological space. Let $F : Y \to 2^Z$ and $T : X \to 2^Z$ be set-valued mappings. F is said to be a generalized KKM mapping (wrt) T (T-KKM mapping in short) if, for each $N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$ and each $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$, one has $T(\varphi_N(\Delta_k)) \subseteq \bigcup_{j=0}^{k} F(y_{i_j})$, where $\varphi_N \in \Phi$ is corresponding to N and Δ_k is the face of Δ_n corresponding to $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\}$.

(ii) We say that a set-valued mapping $T : X \to 2^Z$ has the generalized KKM property if, for each T-KKM mapping $F : Y \to 2^Z$, the family $\{\overline{F(y)} : y \in Y\}$ has the finite intersection property, i.e. all finite intersections of sets of this family are nonempty. By KKM(X,Y,Z) we denote the class of all the mappings $T : X \to 2^Z$ which enjoy the generalized KKM property.

Later we will see that Definition 2.2 encompasses most of existing classes of KKM-type mappings. The following simple lemma is needed for our arguments in the sequel.

Lemma 2.2. *Let (X, Y, Φ) be a GFC-space, Z be a topological space, $S : Y \to 2^X$ be a set-valued mapping, D be a S-subset of Y and $T \in KKM(X, Y, Z)$. Then $T|_{S(D)} \in KKM(S(D), D, \overline{T(S(D))})$.*

Proof. Assume that $R : D \to 2^{\overline{T(S(D))}}$ is a $T|_{S(D)}$-KKM mapping. Then, for each $N = \{y_0, y_1, ..., y_n\} \in \langle D \rangle \subseteq \langle Y \rangle$ and each $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$,

$$T(\varphi_N(\Delta_k)) = T|_{S(D)}(\varphi_N(\Delta_k)) \subseteq \bigcup_{j=0}^{k} R(y_{i_j}).$$

Define a set-valued mapping $F : Y \to 2^Z$ by

$$F(y) = \begin{cases} R(y) & \text{if} \quad y \in D, \\ Z & \text{if otherwise.} \end{cases}$$

Clearly F is a T-KKM mapping. Since $T \in \text{KKM}(X, Y, Z)$, the family $\{\overline{F(y)} : y \in Y\}$ has the finite intersection property. It follows that the family $\{\overline{R(y)} : y \in D\}$ has this property too. Thus, the lemma is proved. □

3. KKM-TYPE THEOREMS

Throughout this section, if not otherwise specified, let (X, Y, Φ) be a GFC-space, Z be a topological space, $S : Y \to 2^X$ and $F : Y \to 2^Z$ be set-valued mappings and $T \in \text{KKM}(X, Y, Z)$.

Theorem 3.1. *Assume that Y is an S-subset of itself. Let the following conditions hold*

(i) $\overline{T(S(Y))}$ *is a compact subset (of Z);*

(ii) *F is T-KKM and transfer compactly closed-valued.*

Then

$$\overline{T(S(Y))} \cap \bigcap_{y \in Y} F(y) \neq \emptyset.$$

Proof. We define a new set-valued mapping $\widehat{F} : Y \to 2^{\overline{T(S(Y))}}$ by, for $y \in Y$, $\widehat{F}(y) = \overline{T(S(Y))} \cap \text{ccl}F(y)$. Then $\widehat{F}$ has clearly closed values in $\overline{T(S(Y))}$. We claim that $\widehat{F}$ is T-KKM. Indeed, for $N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$

and $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$, as F is T-KKM one has $T(\varphi_N(\Delta_k)) \subseteq \bigcup_{j=0}^{k} F(y_{i_j})$. Since Y is an S-subset of Y, $T(\varphi_N(\Delta_k)) \subseteq T(S(Y))$. Therefore

$$\begin{aligned} T(\varphi_N(\Delta_k)) &\subseteq \overline{T(S(Y))} \cap \textstyle\bigcup_{j=0}^{k} F(y_{i_j}) \\ &= \textstyle\bigcup_{j=0}^{k} [\overline{T(S(Y))} \cap F(y_{i_j})] \\ &\subseteq \textstyle\bigcup_{j=0}^{k} \widehat{F}(y_{i_j}). \end{aligned}$$

As $T \in$ KKM (X, Y, Z), the family $\{\overline{\widehat{F}(y)} : y \in Y\} = \{\widehat{F}(y) : y \in Y\}$ has the finite intersection property. Since this is a family of closed subsets of compact set $\overline{T(S(Y))}$, by Lemma 2.1 one has

$$\begin{aligned} \overline{T(S(Y))} \cap \textstyle\bigcap_{y \in Y} F(y) &= \textstyle\bigcap_{y \in Y} (\overline{T(S(Y))} \cap F(y)) \\ &= \textstyle\bigcap_{y \in Y} (\overline{T(S(Y))} \cap \mathrm{ccl} F(y)) \\ &= \textstyle\bigcap_{y \in Y} \widehat{F}(y) \neq \emptyset. \ \square \end{aligned}$$

Now we discuss particular cases of Theorem 3.1. Let (X, Φ) be an FC-space, Z be a topological space, Y be a set, $F : Y \to 2^Z$, $s : Y \to X$ and $T \in \mathrm{KKM}(X, Y, Z)$ be given. Following [7], F is called an s-KKM wrt T if for each $N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$ and each $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$, $T(\varphi_{s(N)}(\Delta_k)) \subseteq \bigcup_{j=0}^{k} F(y_{i_j})$, where $\varphi_{s(N)} : \Delta_n \to X$ is the mapping of the family Φ, corresponding to $\{s(y_0), s(y_1), ..., s(y_n)\}$ in the definition of an FC-space [6]. We define a GFC-space (X, Y, Φ) as follows: for $N \in \langle Y \rangle$ and $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$, as the corresponding mapping from Φ we take $\varphi_{s(N)}$. Then an s-KKM mapping wrt T acting on the FC-space (X, Φ) becomes a T-KKM mapping acting on the GFC-space (X, Y, Φ), according to Definition 2.2(i). Therefore, Theorem 3.2 of [7] is special case of Theorem 3.1, where $S \equiv s$. If in addition $Y = X$, $S \equiv s \equiv I$ (the identity map) and T is a compact mapping, our Theorem 3.1 collaps to Theorem 3.3 of [7] and Theorem 3.2 of [9]. If a set-valued mapping is compactly closed-valued then it is also transfer compactly closed-valued. So Theorem 3.1 includes properly Theorem 3.1 of [7] and Theorem 3.1 of [9]. When $(X, Y, \Phi) = (X, Y, \Gamma)$ is a G-convex space [25] and $S \equiv I$, our Theorem 3.1 implies Theorem 1 of [21], where the assumption corresponding to our condition (ii) is more stringent.

Assume that X is a convex space, $\mathrm{co}(\cdot)$ is the usual convex hull operator in this convex space, Y, Z, S and T are as at the beginning of this section. In [2] $F : Y \to 2^Z$ is called a generalized S-KKM mapping wrt T if, for any $N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$,

$$T(\mathrm{co}S(N)) \subseteq F(N).$$

We define a GFC-space $(X, Y, \{\varphi_N\})$ as follows. Take $s : Y \to X$ which is any fixed selection of S and, for any $N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$, take $\varphi_N : \Delta_n \to X$ by the definition $\varphi_N(e) = \sum_{i=0}^{n} \lambda_i s(y_i)$ for all $e = \sum_{i=0}^{n} \lambda_i e_i \in \Delta_n$. Then $(X, Y, \{\varphi_N\})$ is clearly a GFC-space. It is equally obvious that a set-valued mapping F is S-KKM wrt T only if F is T-KKM by our Definition 2.2. Consequently Theorem 4.3 of that paper is a true special case of our Theorem 3.1 with $S(\cdot)$ replaced by $\mathrm{co}(S(\cdot))$.

The compactness condition (i) in Theorem 3.1 can be replaced by a coercivity condition as follows.

Theorem 3.2. *Assume that*

(i_1) *for each compact subset $D \subseteq X$, $\overline{T(D)}$ is compact* ;

(i_2) *there is a compact subset K of Z such that for each $N \in \langle Y \rangle$, there is an S-subset L_N of Y, containing N with either $S(L_N)$ or $\overline{S(L_N)}$ being compact and*

$$\overline{T(S(L_N))} \cap \bigcap_{y \in L_N} \mathrm{ccl}F(y) \subseteq K;$$

(ii) *F is T-KKM and transfer compactly closed-valued.*

Then

$$\overline{T(S(Y))} \cap \bigcap_{y \in Y} F(y) \neq \emptyset.$$

Proof. (a) First, assume that F is compactly closed-valued. Suppose ab absurdo that

$$Z = [\overline{T(S(Y))} \cap \bigcap_{y \in Y} F(y)]^c = \overline{T(S(Y))}^c \cup \bigcup_{y \in Y} F(y)^c.$$

Then the compact set K (in (i_2)) has an open covering $K \cap \overline{T(S(Y))}^c$, $\{K \cap F(y)^c\}_{y\in Y}$ and hence there is $N \in \langle Y \rangle$ such that

$$K \subseteq \overline{T(S(Y))}^c \cup \bigcup_{y\in N} F(y)^c$$

$$\subseteq \overline{T(S(L_N))}^c \cup \bigcup_{y\in N} F(y)^c.$$

By assumption (i_2),

$$K^c \subseteq \overline{T(S(L_N))}^c \cup \bigcup_{y\in L_N} F(y)^c.$$

Therefore,

$$Z = K \cup K^c \subseteq \overline{T(S(L_N))}^c \cup \bigcup_{y\in L_N} F(y)^c.$$

So

$$\overline{T(S(L_N))} \cap \bigcap_{y\in L_N} F(y) = \emptyset.$$

We consider now the first case of assumption (i_2). We apply Theorem 3.1 to GFC-space $(S(L_N), L_N, \Phi)$ with Z, S, T, F replaced by $\overline{T(S(L_N))}$, $S|_{L_N}$, $T|_{S(L_N)}$ and $F_1 : L_N \to 2^{\overline{T(S(L_N))}}$, where $F_1(y) = \overline{T(S(L_N))} \cap F(y)$. We see that L_N is an $S|_{L_N}$-subset of itself. Moreover, $\mathrm{cl}_{\overline{T(S(L_N))}} T|_{S(L_N)}(S|_{L_N}(L_N))$ is compact by (i_1) and (i_2), where $\mathrm{cl}_{\overline{T(S(L_N))}}$ stands for the closure in $\overline{T(S(L_N))}$. To check the remaining assumption (ii) of Theorem 3.1, we observe that F_1 has compactly closed-values in $\overline{T(S(L_N))}$ and $T|_{S(L_N)} \in \mathrm{KKM}(S(L_N), L_N, \overline{T(S(L_N))})$ (by Lemma 2.2). To see that F_1 is $T|_{S(L_N)}$-KKM let $N^* \in \langle L_N \rangle$ and $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N^*$. Then, since L_N is an $S|_{L_N}$-subset of L_N,

$$\begin{aligned} T|_{S(L_N)}(\varphi_{N^*}(\Delta_k)) &\subseteq T(\varphi_{N^*}(\Delta_k)) \\ &\subseteq \overline{T(S(L_N))} \cap \bigcup_{j=0}^{k} F(y_{i_j}) \\ &= \bigcup_{j=0}^{k} F_1(y_{i_j}), \end{aligned}$$

i.e. F_1 is $T|_{S(L_N)}$-KKM. Making use of Theorem 3.1 yields

$$\emptyset \neq \mathrm{cl}_{\overline{T(S(L_N))}}(T|_{S(L_N)}(S|_{L_N}(L_N))) \cap \bigcap_{y\in L_N} F_1(y)$$

$$\subseteq \overline{T(S(L_N))} \cap \bigcap_{y \in L_N} F(y),$$

a contradiction.

Now consider the second case of (i_2). By (i_1), $\overline{T(\overline{S(L_N)})}$ is compact. Then its closed subset $\overline{T(S(L_N))}$ is compact as well. Therefore we can apply Theorem 3.1 as for the first case of (i_2).

(b) If F is transfer compactly closed-valued, we consider $\widehat{F}$ defined by $\widehat{F}(y) = \mathrm{ccl}F(y)$ and apply part (a) together with Lemma 2.1. □

Since our Definition 2.2 of a T-KKM mapping includes many definitions of KKM type mappings as discussed after Theorem 3.1, it is easy to see that Theorem 3.2 has true special cases as follows. When applied to the particular case, where $X = Y$ and $S \equiv I$, Theorem 3.2 improves Theorem 3.3 of [9]. If $(X, Y, \Phi) = (X, Y, \Gamma)$ is a G-convex space and $S \equiv I$ then Theorem 3.2 implies Theorem 3 of [21].

For S-KKM mappings with respect to T and the class S-KKM(X, Y, Z) defined in [2] we have the following consequence which is Theorem 5.1 of [2].

Corollary 3.3 *Let X, Y be convex spaces and Z be a Hausdorff topological space. Let $S : Y \to 2^X$, $F : Y \to 2^Z$ and $T \in S$-$KKM(X, Y, Z)$. Let the following conditions hold*

(a) $\overline{S(C)}$ *is compact and convex for each compact convex subset C* of Y;

(b) *for each compact subset $D \subseteq X$, $\overline{T(D)}$ is compact*;

(c) *F is an S-KKM mapping wrt T and compactly closed-valued*;

(d) *there is a nonempty compact convex subset L of Y and a compact subset K of Z such that $\bigcap_{y \in L} F(y)) \subseteq K$.*

Then

$$\overline{T(\mathrm{co}S(Y))} \cap \bigcap_{y \in Y} F(y) \neq \emptyset.$$

Proof. We apply Theorem 3.2 with the second case of assumption (i_2). For L_N required in this assumption we simply take $L_N = \mathrm{co}(L \cup N)$, where L is given in assumption (d). □

The condition that F is T-KKM, imposed in Theorems 3.1 and 3.2, can be replaced by other assumptions which may be easier to check in some cases as follows.

Theorem 3.4 *Let $G, M : Y \to 2^Z$ be additionally given. Assume that Y is an S-subset of itself. Assume further that*

(i) $\overline{T(S(Y))}$ *is compact*;

(ii_1) *F is transfer compactly closed-valued*;

(ii_2) *for each $y \in Y$, $G(y) \subseteq F(y)$ and $T(S(y)) \subseteq M(y)$*;

(ii_3) *for each $z \in Z$, $Y \backslash M^{-1}(z)$ an S-subset of Y wrt $Y \backslash G^{-1}(z)$.*

Then

$$\overline{T(S(Y)} \cap \bigcap_{y \in Y} F(y) \neq \emptyset.$$

Proof. It suffices to check that F is T-KKM. Suppose the existence of $N \in \langle Y \rangle$ and $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$ such that $T(\varphi_N(\Delta_k)) \nsubseteq \bigcup_{j=0}^{k} F(y_{i_j})$, i.e. $x \in \varphi_N(\Delta_k))$ and $z \in T(x)$ exist such that $z \notin F(y_{i_j})$ for each $j = 0, 1, ..., k$. Then

$$\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N \cap (Y \backslash F^{-1}(z)) \subseteq N \cap (Y \backslash G^{-1}(z)).$$

By (ii_3), $\varphi_N(\Delta_k) \subseteq S(Y \backslash M^{-1}(z))$ and hence there exists $y \in Y \backslash M^{-1}(z)$ such that $x \in S(y)$. Consequently, $z \in T(S(y)) \subseteq M(y)$, contradicting the fact that $y \in Y \backslash M^{-1}(z)$. □

Note that Theorem 3.4 of [7] is a consequence of Theorem 3.4 for the case where $Y = X$ and $S \equiv I$.

Finally, the compactness in condition (i) can be weakened to a coercivity assumption as follows.

Theorem 3.5 *Let* $G, M : Y \to 2^Z$ *be additionally given. Let* (ii_1), (ii_2), (ii_3) *and the following conditions hold*

(i_1) *for each compact subset* D *of* X, $\overline{T(D)}$ *is compact*;

(i_2) *there is compact subset* K *of* Z *such that for each* $N \in \langle Y \rangle$, *there is an* S*-subset* L_N *of* Y *containing* N *so that* $S(L_N)$ *is compact and*

$$\overline{T(S(L_N))} \cap \bigcap_{y \in L_N} \mathrm{ccl} F(y) \subseteq K.$$

Then

$$\overline{T(S(Y)} \cap \bigcap_{y \in Y} F(y) \neq \emptyset.$$

The proof is similar to that of Theorems 3.2 and 3.4 and hence omitted.

Remark 3.1. Applying Lemma 2.5 of [7] it is easy to see that even for the special case where $Y = X$ and $S \equiv I$, Theorem 3.5 improves Theorem 3.2 of [8].

4. COINCIDENCE THEOREMS AND GEOMETRIC VERSIONS

In this section we establish coincidence theorems and some their geometric versions. These results are either equivalent to or consequences of KKM-type theorems obtained in the preceding section. So generalized KKM properties are essentially employed.

Let (X, Y, Φ) be a GFC-space, Z be a topological space, $S : Y \to 2^X$ be such that Y is an S-subset of itself and $T \in \mathrm{KKM}(X, Y, Z)$ be such that $\overline{T(S(Y))}$ is compact. Our first purpose is to demonstrate the equivalence between each of Theorems 4.1 - 4.3 below and Theorem 3.4.

Theorem 4.1 *Let A, B and C be subsets of $Y \times Z$ with $B \subseteq A$. Let the following conditions hold*

(i) $\widetilde{F}$ *is transfer compactly open-valued, where* $\widetilde{F} : Y \to 2^Z$ *is defined by* $\widetilde{F}(y) = \{z \in Z : (y, z) \notin A\}$ *for all* $y \in Y$;

(ii) *for each* $y \in Y$, $\{y\} \times T(S(y)) \subseteq C$;

(iii) *for each* $z \in Z$, *the set* $\{y \in Y : (y, z) \notin C\}$ *is an S-subset of Y wrt the set* $\{y \in Y : (y, z) \notin B\}$.

Then, there exist $\widehat{z} \in \overline{T(S(Y)}$ *such that* $Y \times \{\widehat{z}\} \subseteq A$.

Theorem 4.2 *Let* $H, P, Q : Z \to 2^Y$ *be given. Let the following conditions hold*

(i) H^{-1} *has transfer compactly open values and* $H(z) \neq \emptyset$ *for each* $z \in \overline{T(S(Y))}$;

(ii) *for each* $z \in Z$, $H(z) \subseteq P(z)$;

(iii) *for each* $z \in Z$, $Q(z)$ *is an S-subset of Y wrt* $P(z)$.

Then, there exists a coincidence point $(\widehat{x}, \widehat{y}, \widehat{z})$ *for S, Q, T, i.e.* $\widehat{x} \in S(\widehat{y})$, $\widehat{y} \in Q(\widehat{z})$ *and* $\widehat{z} \in T(\widehat{x})$.

Theorem 4.3 *Let* $P, Q : Z \to 2^Y$ *and* $F : Y \to 2^Z$ *be given. Let the following conditions hold*

(i) F *is transfer compactly closed-valued and* $F^{-1}(z) \neq Y$ *for each* $z \in \overline{T(S(Y))}$;

(ii) *for each* $y \in Y$, $Z \setminus P^{-1}(y) \subseteq F(y)$*;*

(iii) *for each* $z \in Z$, $Q(z)$ *is an S-subset of* Y *wrt* $P(z)$*.*

Then, there is a coincidence point $(\widehat{x}, \widehat{y}, \widehat{z})$ *for* S, Q, T *.*

Proof. Theorem 3.4 $\Rightarrow$ *Theorem* 4.1. Define set-valued mappings $F, G, M : Y \to 2^Z$ by, for $y \in Y$,

$$\begin{aligned} F(y) &= \{z \in Z : (y, z) \in A\}, \\ G(y) &= \{z \in Z : (y, z) \in B\}, \\ M(y) &= \{z \in Z : (y, z) \in C\}. \end{aligned}$$

By (i), $F(y)$ is transfer compactly closed-valued. $G(y) \subseteq F(y)$ as $B \subseteq A$ and $T(S(y)) \subseteq M(y)$ for each y due to (ii). The last assumption (ii$_3$) of Theorem 3.4 is nothing else than (iii). now that all the conditions hold, this theorem confirms the existence of $\widehat{z} \in \overline{T(S(Y))}$ such that $\widehat{z} \in F(y)$ for each $y \in Y$, i.e. $Y \times \{\widehat{z}\} \subseteq A$.

Theorem 4.1 $\Rightarrow$ *Theorem* 4.2. Set $A = (Y \times Z) \setminus \mathrm{graph} H$, $B = (Y \times Z) \setminus \mathrm{graph} P$ and $C = (Y \times Z) \setminus \mathrm{graph} Q$. We check the assumptions of Theorem 4.1. For $y \in Y$, $\widetilde{F}(y) = \{z \in Z : (y, z) \notin A\} = H^{-1}(y)$ and hence $\widetilde{F}$ is transfer compactly open-valued by (i) of Theorem 4.2. (ii) implies that $B \subseteq A$. The set $\{y \in Y : (y, z) \notin C\} = Q(z)$ is an S-subset of Y wrt the set $\{y \in Y : (y, z) \notin C\} = P(z)$. Now suppose, to the contrary of the conclusion, that for each $y \in Y$ and $z \in T(S(y))$, $y \notin Q(z)$, i.e. $(y, z) \in C$. This means $\{y\} \times T(S(y)) \subseteq C$ and consequently all the assumptions of Theorem 4.1 are fulfilled. So, $\widehat{z} \in \overline{T(S(Y))}$ exists such that $Y \times \{\widehat{z}\} \subseteq A$, i.e. $H(\widehat{z}) = \emptyset$ contradicting assumption (i).

Theorem 4.2 $\Rightarrow$ *Theorem* 4.3. Choose $H : Z \to 2^Y$ by setting $H(z) = Y \setminus F^{-1}(z)$ to see that all the assumptions of Theorem 4.2 are satisfied.

Theorem 4.3 $\Rightarrow$ *Theorem* 3.4. Define $P, Q : Z \to 2^Y$ by, for $z \in Z$,

$$P(z) = Y \setminus G^{-1}(z), \quad Q(z) = Y \setminus M^{-1}(z).$$

By (ii$_2$), $Z \setminus P^{-1}(y) = G(y) \subseteq F(y)$. Assumption (ii$_3$) implies (iii) of Theorem 4.3. Now suppose to the contrary of the conclusion of Theorem 3.4 that, for each $z \in \overline{T(S(Y))}$, $z \notin \bigcap_{y\in Y} F(y)$, i.e. $F^{-1}(z) \neq Y$ and all the assumptions of Theorem 4.3 are satisfied. For a coincidence point $(\widehat{x}, \widehat{y}, \widehat{z})$ existing by this theorem we see that $\widehat{z} \in T(S(\widehat{y}))$ and $\widehat{y} \in Q(\widehat{z}) = Y \setminus M^{-1}(\widehat{z})$, i.e. $\widehat{z} \notin M(\widehat{y})$, contradicting (ii$_2$). □

Note that when applied to the particular case where $Y = X$ (and we have an FC-space) and $S \equiv I$, Theorems 4.1 - 4.3 slightly improve Theorems 4.1 - 4.3 of [7].

Similarly as for KKM-type theorems the compactness of $\overline{T(S(Y)}$ can be relaxed to a coercivity assumption as follows.

Theorem 4.4 (X, Y, Φ), *Z, S and T be as specified at the beginning of Section 4 except the compactness of* $\overline{T(S(Y))}$). *Let* (ii), (iii) *of Theorem 4.2 and the following conditions hold*

(i) H^{-1} *has transfer compactly open values and* $H(z) \neq \emptyset$ *for each* $z \in Z$;

(iv$_1$) *for each compact subset D of X,* $\overline{T(D)}$ *is compact;*

(iv$_2$) *there is a compact subset K of Z such that for each* $N \in \langle Y \rangle$, *there is an S-subset* L_N *of Y containing N such that* $S(L_N)$ *is compact and* $T(S(L_N)) \setminus K \subseteq \bigcup_{y\in L_N} \text{cint} H^{-1}(y)$.

Then, coincidence points exist for S, Q and T .

Proof. By (i) and the compactness of K we have $N \in \langle Y \rangle$ such that

$$K = \bigcup_{y\in Y} (K \cap \text{cint} H^{-1}(y)) \subseteq \bigcup_{\text{y}\in\text{N}} \text{cint} H^{-1}(y).$$

Hence, assumption (iv$_2$) implies that

$$T(S(L_N)) \subseteq K \cup (T(S(L_N)) \setminus K) \subseteq \bigcup_{y\in L_N} \text{cint} H^{-1}(y).$$

Then

$$T(S(L_N)) = \bigcup_{y \in L_N} (\mathrm{cint} H^{-1}(y) \cap T(S(L_N))) = \bigcup_{\mathrm{y} \in \mathrm{L_N}} \mathrm{int}_{T(S(L_N))}(H^{-1}(y) \cap T(S(L_N))).$$

Define new set-valued mapping $H_1, P_1, Q_1 : T(S(L_N)) \to 2^{L_N}$ by

$$H_1(z) = H(z) \cap L_N, \;\; P_1(z) = P(z) \cap L_N, \;\; Q_1(z) = Q(z) \cap L_N.$$

It is easy to see that all the assumptions of Theorem 4.2 are satisfied with $(S(L_N), L_N, \Phi), T(S(L_N)), S|_{L_N}, T|_{S(L_N)}, H_1, P_1, Q_1$ in the places of (X, Y, Φ), Z, S, T, H, P and Q, respectively. By this theorem a point $(\widehat{x}, \widehat{y}, \widehat{z}) \in S(L_N) \times L_N \times T(S(L_N))$ exists such that $\widehat{x} \in S|_{L_N}(\widehat{y})$, $\widehat{y} \in Q_1(\widehat{z})$ and $\widehat{z} \in T|_{S(L_N)}(\widehat{x})$. This point is also a required point of Theorem 4.4. □

Note that, for the case $Y = X$ (and we have an FC-space) and $S \equiv I$, Theorems 4.4 becomes Theorems 4.4 of [7].

The theorems established in this paper can be employed to develop sufficient conditions for the solution existence in various general optimization - related problems. We leave this development to our forthcoming paper.

REFERENCES

1. K. C. Border, *Fixed-point theorems with applications to economics and game theory*, Cambridge University Press, Cambridge-London-New York-Sydney (1985).

2. T. H. Chang, Y. Y. Huang, J. C. Jeng, K. W. Kuo, On S-KKM property and related topics, *J. Math. Anal. Appl.*, **229**(1999), 212-227.

3. T. H. Chang, C. L. Yen, KKM property and fixed-point theorems, *J. Math. Anal. Appl.*, **203**(2003), 224-235.

4. X. P. Ding, New H-KKM theorems and their applications to geometric properties, coincidence theorems, minimax inequalitities and maximal elements, *Indian J. Pure Appl. Math.*, **26**(1995), 1-19.

5. X. P. Ding, Generalized L-KKM type theorems in L-convex spaces with applications, *Comput. Math. Appl.*, **43**(2002), 1249-1256.

6. X. P. Ding, Maximal-element theorems in product FC-spaces and generalized games, *J. Math. Anal. Appl.*, **305**(2005), 29-42.

7. X. P. Ding, Generalized KKM type theorems in FC-spaces with applications (I), *J. Glob. Optim.*, **36**(2006), 581-596.

8. X. P. Ding, Generalized KKM type theorems in FC-spaces with applications (II). *J. Glob. Optim.*, **38**(2007), 367-385.

9. X. P. Ding, T. M. Ding, KKM type theorems and generalized vector equilibrium problems in noncompact FC-spaces, *J. Math. Anal. Appl.*, **331**(2007), 1230-1245.

10. K. Fan, A generalization of Tichonoff's fixed-point theorem *Math. Anal.*, **142**(1961), 305-310.

11. C. W. Ha, Minimax and fixed-point theorems, *Math. Anal.*, **248**(1980), 73-77.

12. N. X. Hai, P. Q. Khanh, Systems of multivalued quasiequilibrium problems, *Adv. Nonlin. Var. Ineq.*, **9**(2006), 97-108.

13. N. X. Hai, P. Q. Khanh, The solution existence of general variational inclusion problems, *J. Math. Anal. Appl.*, **328**(2007), 1268-1277.

14. N. X. Hai, P. Q. Khanh, Existence of solutions to general quasiequilibrium problems and applications, *J. Optim. Theory Appl.*, **133**(2007), 317-327.

15. N. X. Hai, P. Q. Khanh, Systems of set-valued quasivariational inclusion problems, *J. Optim. Theory Appl.*, **135**(2007), 55-67.

16. C. Horvath, Some results on multi-valued mappings and inequalities without convexity, in: *Nonlinear and Convex Analysis* (B. L. Lin and S. Simons, Eds.), (1987), pp. 99-106.

17. P. Q. Khanh, L. M. Luu, The existence of solutions to vector quasivariational inequalities and quasicomplementarity problems with applications to traffic network equilibria, *J. Optim. Theory Appl.*, **123**(2004), 533-548.

18. P. Q. Khanh, L. M. Luu, Some existence results vector quasivariational inequalities involving multifunctions and applications to traffic equilibrium problems, *J. Glob. Optim.*, **32**(2005), 551-568.

19. P. Q. Khanh, N. H. Quan, Intersection theorems, coincidence theorems and maximal-element theorems in GFC-spaces, submitted.

20. B. Knaster, K. Kuratowski, S. Mazurkiewski, Ein Beweis des Fixpunktsatzes fur n-dimensional Simplexe, *Fund. Math.*, **14**(1929), 132-137.

21. L. J. Lin, A KKM type theorem and it's applications, *Bull. Autral. Math. Soc..*, **59**(1999), 481-493.

22. L. J. Lin, T. H. Chang, S-KKM theorems, saddle points and minimax inequalities, *Nonlin. Anal.*, **34**(1998), 73-86.

23. L. J. Lin, H. L. Chen, The study of KKM theorems with applications to vector equilibrium problems and implicit variational inequalities problems, *J. Glob. Optim.*, **32**(2005), 135-157.

24. L. J. Lin, W. P. Wan, KKM theorems and coincidence theorems with applications to the existence of equilibria, *J. Optim*, *Theory Appl.*, **123 (1)**(2004), 105-122.

25. S. Park, H. Kim, Admissible classes of multifunction on generalized convex spaces, *Proc. Coll. Natur. Sci. Seoul Nation. Univ..*, **18**(1993), 1-21.

26. S. Park, H. Kim, Coincidence theorems for admissible multifunction on generalized convex spaces, *J. Math. Anal. Appl.*, **197**(1996), 173-187.

27. S. Park, H. Kim, Foundations of the KKM theory on generalized convex spaces, *J. Math. Anal. Appl.*, **209(2)**(1997), 551-571.

28. K. K. Tan, G-KKM theorem, minimax inequalities and saddle points, *Nonlin. Anal.*, **30**(1997), 4151-4160.

29. E. Tarafdar, A fixed-point theorem equivalent to the Fan-Knaster-Kuratowski-Mazurkiewicz theorem, *J. Math. Anal. Appl.*, **128**(1987), 475-479.

30. G. Tian, Generalization of F-KKM theorem and the Ky Fan minimax inequality with applications to maximal elements, price equilibrium and complementarity, *J. Math. Anal. Appl.*, **170**(1992), 457-471.

31. R. U. Verma, R-KKM mapping theorems and variational inequalities, *Mat. Proc. Roy. Irish. Acad. Sci.,* **A98**(1998), 131-138.

32. L. C. Zeng, J. C. Yao, Existence theorems for variational inequalities in Banach spaces, *J. Optim*, *Theory Appl.*, **132**(2007), 321-337.