On generalized Ekeland's variational principle for approximate Pareto minima of set-valued mappings*

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Abstract

We propose relaxed lower semicontinuity properties for set-valued mappings, using weak τ-functions, and employ them to weaken known lower semicontinuity assumptions to get enhanced Ekeland's variational principle for Pareto minimizers of set-valued mappings and underlying minimal-element principles. Our results improve or recover recent ones in the literature when applied to their particular cases.

Keywords: Ekeland's variational principle; Weak τ-functions; K-lower semicontinity from above; (k_0, K)-lower semicontinity from above

1. Introduction

The Ekeland variational principle (the EVP in short) is one of the most im-

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portant results in nonlinear analysis and optimization for the last three decades. It proves to be a powerful tool in variational analysis and optimization-related problems. A great number of generalizations of the EVP to more general problem settings and with less stringent assumptions have been obtained by many authors. In its original form, the EVP says that if a scalar function f on a complete metric space X is bounded from below and lower semicontinuous, then we can get a strict minimum of a slight perturbation of f.

Recently many efforts have been made to weaken the lower semicontinuity assumption by proposing generalized distances, additionally to the metric of X. First of these proposals are the Tataru distance in Tataru [18] and the w-distance in Kada et al. [9]. Then the τ-distance and τ-function were introduced in Suzuki [17] and Lin and Du [14], respectively. The latter two distances are more general than the former ones but are incomparable. In Khanh and Quy [10, 11] the weak τ-function was proposed to encompass these two definitions and to obtain improved formulations of the EVP for Pareto minimizers of a single-valued mapping and for Kuroiwa's minimizers of a set-valued mapping.

In the present paper, using the weak τ-function we first go into details to have a clearer insight of various relaxed lower semicontinuity properties of a set-valued mapping. Then imposing these semicontinuities we establish sufficient conditions for the existence of minimal elements and strict minimal elements of a set. From these results about underlying principles for the study of the EVP stated for minimization problems, we easily get enhanced versions of the EVP for approximate Pareto minimizers of a set-valued mapping. Our results either improve or agree with recent ones in [1-6, 8, 13-16] (and many previous results in the literature) when applied to the particular cases considered in those papers.

The layout of our paper is as follows. We recall needed notions and preliminaries in the rest of this section. Various relaxed lower semicontinuities for a set-valued mapping are discussed in Section 2, including several new definitions proposed here. Section 3 is devoted to the main results about minimal elements and enhanced versions of the EVP for approximate Pareto minimizers of multivalued mappings. In the last Section 4 we derive some consequences for special cases of single-valued mappings to see more advantages of our results.

We recall some notions. If Y is a topological vector space ordered by a convex cone K, then a subset $D \subseteq Y$ is said to be quasibounded from below if there is a bounded subset $M \subseteq Y$ such that $D \subseteq M + K$. D is called bounded from below if there is $\overline{y} \in Y$ such that $D \subseteq \overline{y} + K$. So boundedness (from below) implies quasiboundedness (from below) but not vice versa as one can easily find a counterexample. In the sequel we fix a point $k_0 \in K \setminus (-\text{cl}K)$, where cl(.) stands for the closure of the set (.). K is said to be closed in direction k_0 if $K \cap (z - R_+ k_0)$ is closed for each $z \in K$. We use K^+ to denote the positive polar of convex cone K, i.e.

$$K^+ := \{y^* \in Y^* : < y^*, k > \geq 0, \forall k \in K\}$$

and $K^\sharp$ to denote the quasi-interior of K^+, i.e.

$$K^\sharp := \{y^* \in Y^* : < y^*, k >> 0, \forall k \in K \setminus (-K)\}.$$

Now we pass to Pareto minimum notions. Let Y be a set partially ordered by a transitive ordering $\leq$ and $B \subseteq Y$. Then $y \in B$ is called a (Pareto) minimum of B if $z \leq y$ for some $z \in B$ implies $y \leq z$. Very often Y is a linear space and an order is defined by a convex cone $K \subseteq Y$ and is denoted by $\leq_K$. $y \in B$ is said to be a strict (Pareto) minimum of B if $z \not\leq y$ (or $z \not\leq_K y$), $\forall z \in B \setminus \{y\}$. We denote by $\text{Min}^{\leq} B$ (or $\text{Min}^K B$) the collection of all the (Pareto) minima of $B \subseteq Y$ with respect to (wrt) the ordering $\leq$ (or by the convex cone K, respectively). The notation $\text{SMin}^{\leq} B$ (or $\text{SMin}^K B$) is used for the set of all the strict minima of B. Note that $\text{SMin}^K B \subseteq \text{Min}^K B$ and if K is pointed, i.e. $K \cap (-K) = \{0\}$, then we have equality. A subset B of Y is said to have the domination property if, for any $y \in B$, there exists $y' \in \text{Min}^{\leq} B$ such that $y' \leq y$. B is said to have the strict domination property if, for any $y \in B$, there exists $y' \in \text{SMin}^{\leq} B$ such that $y' \leq y$. Note that B has the domination property if it is compact and $\leq$ is defined by a closed convex cone K. B has the strict domination property if, in addition, K is pointed.

Next we recall needed generalized distances. In this paper we use the following generalized distance proposed in Khanh and Quy [10, 11].

Definition 1.1. Let (X, d) be a metric space. A function $p : X \times X \to R_+$ is said to be a weak τ-function if the following conditions hold, for $x, y, z \in X$,

(τ1) (triangle inequality) $p(x, z) \leq p(x, y) + p(y, z)$;

($\tau 3$) for any sequences $\{x_n\}$ and $\{y_n\}$ in X with $\lim_{n\to\infty}\sup\{p(x_n, x_m) : m > n\} = 0$ and $\lim_{n\to\infty} p(x_n, y_n) = 0$, one has $\lim_{n\to\infty} d(x_n, y_n) = 0$;

($\tau 4$) $p(x, y) = 0$ and $p(x, z) = 0$ imply $y = z$.

Note that with the following additional condition

($\tau 2$) (lower semicontinuity) $\forall x \in X$, $p(x, .)$ is R_+-lsc;

a weak τ-function becomes a τ-function introduced in Lin and Du [14]. Observe further that (see Khanh and Quy [10, 11] all the recently introduced generalized distances: the w-distance, Tataru's distance, τ-distance and τ-function are particular cases of the weak τ-function. Example 2.1 in Khanh and Quy [11] shows that being a weak τ-function may be strictly weaker than being a kind of the mentioned distances. The main idea of omitting the lower semicontinuity of p (i.e. condition ($\tau 2$)) is that to prove the EVP we can impose a relaxed lower semicontinuity condition for the mapping F under consideration and p together instead of assuming the lower semicontinuity of F independently with the available continuity of the metric.

Concerning weak τ-functions we need the following fact which was proved in Lin and Du [14] for the case where p is a τ-function. But condition ($\tau 2$) was not used in the proof.

Lemma 1.1. *Let p be a weak τ-function on a metric space X. If a sequence x_n satisfies the condition $\lim_{n\to\infty}\sup\{p(x_n, x_m) : m > n\} = 0$, then x_n is a Cauchy sequence.*

2. Relaxed Lower Semicontinuity Properties of Set-Valued Mappings

Recall that a set-valued mapping $F : X \to 2^Y$ between two topological spaces is said to be upper semicontinuous (usc) at $\bar{x} \in X$ if for any open superset U of $F(\bar{x})$, there exists a neighborhood V of $\bar{x}$ such that $F(V) \subseteq U$. If furthermore Y is ordered by a convex cone K, F is called upper K-continuous at $\bar{x} \in X$ if for any open superset U of $F(\bar{x})$, there exists a neighborhood V of $\bar{x}$ such that $F(V) \subseteq U + K$. Note that if F is usc at $\bar{x}$ then F is upper K-continuous at $\bar{x}$ but the converse is not true as shown by

Example 2.1. Let $X = Y = R$, $K = R_+$ and

$$F(x) = \begin{cases} [1, 2] & \text{if} \quad x \neq 0, \\ [0, 1] & \text{if} \quad x = 0. \end{cases}$$

Then F is upper R_+-continuous, but F is not usc at $\overline{x} = 0$.

From now on, if not otherwise stated, let X be a metric space, Y be a topological vector space ordered by a convex cone $K \subseteq Y$ and $k_0 \in K \setminus (-\mathrm{cl}K)$. $F : X \to 2^Y$ is said to be K-lower semicontinuous (K-lsc) at $\bar{x}$ (or K-level closed at $\bar{x}$) if, for $e \in Y$ and $x_n \to \bar{x}$, from $x_n \in \{x \in X : F(x) \cap (e - K) \neq \emptyset\}$ it follows that $\bar{x}$ belongs also to this e-level set. We say that F is K-lsc or K-level closed if F is K-lsc at all points in $\mathrm{dom}F := \{x \in X : F(x) \neq \emptyset\}$, or what is the same if, for all $e \in Y$, the e-level set is closed. It is clear also that F is K-epiclosed (i.e. $\mathrm{epi}F := \{(x, y) \in X \times Y : y \in F(x) + K\}$ is closed in $X \times Y$), then F is K-level closed, but not vice versa.

Observe that if K is a closed convex cone and $F : X \to 2^Y$ has $\mathrm{dom}F = X$, then F is upper K-continuous only if F is K-lsc. Indeed, let $x_n \in \{x \in X : F(x) \cap (e - K) \neq \emptyset\}$ and $x_n \to \bar{x}$. Let $y_n \in F(x_n) \cap (e - K)$. Suppose to the contrary that $F(\bar{x}) \subseteq Y \setminus (e - K)$. By the upper K-continuity of F there is a neighborhood U of $\bar{x}$ such that $F(U) \subseteq Y \setminus (e - K) + K$. Then, for large n, $y_n \in Y \setminus (e - K) + K$. Consequently, $y_n \notin (e - K)$, a contradiction.

The following example ensures that the converse is false.

Example 2.2. Let $X = R$, $Y = R^2$, $K = R^2_+$ and

$$F(x) = \begin{cases} \{(x, u) : u \geq 1\} & \text{if} \quad x \leq 0, \\ \{(\frac{1}{x}, u) : u \geq \frac{-x}{x+1}\} & \text{if} \quad x > 0. \end{cases}$$

Then it is not hard to see that F is $K-$lsc on R but it is not upper K-continuous at $x = 0$.

A more relaxed notion of K-lower semicontinuity from above at a point $\overline{x} \in X$ is proposed in Chen et al. [7] for scalar single-valued functions. In Khanh and Quy [10] we extended it to set-valued mappings, suitably for considering Kuroiwa's minima (Kuroiwa [12]) which are different from Pareto minima, as follows: F is called K-lower semicontinuous from above (K-lsca) at $\overline{x} \in X$ if for each convergent sequence $x_n \to \overline{x}$ with $F(x_n) \subseteq F(x_{n+1}) + K, \forall n \in \mathbb{N}$ (the set of the natural numbers), one has $F(x_n) \subseteq F(\overline{x}) + K, \forall n \in \mathbb{N}$. As any definition for a point is extended to a set, F is called K-lsca on $A \subseteq X$ if F is K-lsca at all $x \in A$. If $A = \mathrm{dom}F$ we omit "on A" in the statement.

Now we modify the mentioned definition of K-lower semicontinuity from above in a natural way, replacing the order of subsets with inclusion relations by another transitive relation (suitably for investigating Pareto minima). Since we investigate only Pareto minima, not Kuroiwa's minima, we retain the same term "K-lsca" without possible confusions.

Definition 2.1. $F : X \to 2^Y$ is said to be K-lsca at $\overline{x}$ if for any convergent sequence $x_n \to \overline{x}$ and any K-decreasing sequence $y_n \in F(x_n)$ there exists $\overline{y} \in F(\overline{x})$ such that, for each $n \in \mathbb{N}$, $\overline{y} \leq_K y_n$.

Note that being K-lsca is more relaxed than satisfying the limiting monotonicity condition defined in Bao and Mordukhovich [3], where $\overline{y} \in F(\overline{x})$ is replaced by $\overline{y} \in \mathrm{Min}^K F(\overline{\mathrm{x}})$. If $p = d$, F is K-lsca if and only if $\mathrm{gr}F := \{(x, y) \in X \times Y : y \in F(x)\}$ satisfies condition (H2) in Göpfert et al. [8].

Proposition 2.1. *Let K be closed. Then $F : X \to 2^Y$ is K-lsca if either of the following conditions holds*

(i) F is upper K-continuous and compact-valued;

(ii) F is K-lsc and closed-valued;

(iii) F is K-lsc; $F(x)$ has the domination property and $\mathrm{Min}^K F(x)$ is closed for every $x \in \mathrm{dom}F$;

(iv) F is K-lsc; $F(x)$ has the strict domination property and $\mathrm{SMin}^K F(x)$ is closed for every $x \in \mathrm{dom}F$.

Proof. Let $\{(x_n, y_n)\} \subseteq \mathrm{gr}F$ with $\{y_n\}$ being a K-decreasing sequence and $x_n \to x$.

(i) Let $\mathcal{U}$ be a basis of the neighborhoods of zero in Y and $U_n \in \mathcal{U}$ be such that, for each n, $U_{n+1} \subseteq U_n$ and, for each open superset U of $F(x)$, there is n such that $F(x) + U_n \subseteq U$. The upper K-continuity of F yields an open neighborhood V_n of x such that $F(V_n) \subseteq F(x) + U_n + K$. We can assume that $x_n \in V_n$ for all n. Hence $z_n \in F(x) + U_n$ exists with $z_n \leq_K y_n$. As $F(x)$ is compact, there is $t_n \in F(x)$ such that $t_n - z_n \to 0$ and $t_n \to y$ for some $y \in F(x)$ (using a subsequence if needed). Consequently, $z_n \to y$. One has $z_{n+q} \leq_K y_{n+q} \leq_K y_n$ for $q \in \mathbb{N}$. Since K is closed, passing $q \to \infty$ one gets $y \leq_K y_n$, i.e. F is K-lsca.

(ii) Let

$$T(y_n) = \{y \in F(x) : y \leq_K y_n\}.$$

Then $T(y_n)$ is nonempty and closed for all $n \in \mathbb{N}$. Indeed, since $y_{n+q} \leq_K y_n$, $x_{n+q} \in \{z : F(z) \cap (y_n - K) \neq \emptyset\}$ for all q. By the closedness of this level set, $x \in \{z : F(z) \cap (y_n - K) \neq \emptyset\}$, i.e. $T(y_n) \neq \emptyset$. The closedness of $T(y_n)$ follows directly from the closedness of $F(x)$ and K. Furthermore, $T(y_{n+1}) \subseteq T(y_n)$ for all $n \in \mathbb{N}$ as $y_{n+q} \leq_K y_n$. It follows that

$$\emptyset \neq \bigcap_{n=1}^{\infty} T(y_n) \subseteq F(x),$$

i.e. there exists $y \in F(x)$ with $y \leq_K y_n$ for all $n \in \mathbb{N}$.

(iii) Define

$$R(y_n) := \{y \in \mathrm{Min}^K F(x) : y \leq_K y_n\}.$$

We claim that $R(y_n)$ is nonempty and closed for all $n \in \mathbb{N}$. Indeed, for a fixed n and each $i \in \mathbb{N}$, as for part (ii) there is $y^* \in F(x)$ such that

$$y^* \leq_K y_n.$$

As $F(x)$ has the domination property, there is $y \in \mathrm{Min}^K F(x)$ with $y \leq_K y^*$. Consequently, $y \leq_K y_n$, which implies that $y \in R(y_n)$. The closedness of $R(y_n)$ follows directly from the closedness of $\mathrm{Min}^K F(x)$ and K. We also have $R(y_{n+1}) \subseteq R(y_n)$ for all $n \in \mathbb{N}$. With the afore-mentioned properties we see that

$$\emptyset \neq \bigcap_{n=1}^{\infty} R(y_n) \subseteq \mathrm{Min}^K F(x).$$

Then, there is $y \in \mathrm{Min}^K F(x)$ with $y \leq_K y_n$ for all $n \in \mathbb{N}$.

(iv) We define

$$N(y_n) := \{y \in \mathrm{SMin}^K F(x) : y \leq_K y_n\}$$

and argue similarly as for part (iii). □

The following example says that the converse to any assertion of Proposition 2.1 is not valid.

Example 2.3. Let $X = Y = R$, $K = R_+$ and $F : X \to 2^Y$ be the multifunction

$$F(x) = \begin{cases} [1, 2] & \text{if } \quad x \leq 0, \\ \{(2+x)^{-1}\} & \text{if } \quad x > 0. \end{cases}$$

Then F is R_+-lsca, but F is not R_+-lsc at $\overline{x} = 0$.

Let X and Y be as above and p be a weak τ-function. A transitive relation $\leq_{k_0}$ in $X \times Y$ is defined by

$$(x_2, y_2) \leq_{k_0} (x_1, y_1) \quad \text{if and only if} \quad y_2 + k_0 p(x_1, x_2) \leq_K y_1.$$

If $p = d$, the metric of X, this relation is defined in Göpfert et al. [8]. For $A \subseteq X \times Y$ and $(x, y) \in A$ we denote by $S_A(x, y)$ the lower sector of A with respect to $\leq_{k_0}$, i.e.

$$S_A(x, y) := \{(x', y') \in A : (x', y') \leq_{k_0} (x, y)\}.$$

Definition 2.2. F is said to be (k_0, K)-lsc at $\overline{x}$ if for each $(x, y) \in \text{gr}F$, each sequence $\{(x_n, y_n)\}$ in $\text{gr}F$ with $x_n \to \overline{x}$ and $(x_n, y_n) \leq_{k_0} (x, y)$, there exists $\overline{y} \in F(\overline{x})$ such that, $(\overline{x}, \overline{y}) \leq_{k_0} (x, y)$.

Note that $F : X \to 2^Y$ is (k_0, K)-lsc if and only if for each $(x, y) \in \text{gr}F$, the set $\{x' \in X : \exists y' \in F(x'), (x', y') \leq_{k_0} (x, y)\}$ is closed.

Definition 2.3. F is said to be (k_0, K)-lsca at $\overline{x}$ if for any convergent sequence $x_n \to \overline{x}$ and any $\leq_{k_0}$-decreasing sequence $\{(x_n, y_n)\}$ in $\text{gr}F$, there exists $\overline{y} \in F(\overline{x})$ such that, for each $n \in \mathbb{N}$, $(\overline{x}, \overline{y}) \leq_{k_0} (x_n, y_n)$.

Observe for the case where $p = d$ that F is (k_0, K)-lsca at $\overline{x}$ if and only if $\text{gr}F$ satisfies condition (H1) stated in Göpfert et al. [8].

Proposition 2.2. *When K is closed, F is (k_0, K)-lsca if at least one of the following conditions holds*

(i) F is (k_0, K)-lsc and closed-valued;

(ii) F is (k_0, K)-lsc; $F(x)$ has the dominattion property and $\text{Min}^K F(x)$ *is closed for every $x \in \text{dom}F$;*

(iii) F is (k_0, K)-lsc; $F(x)$ has the strict domination property and $\text{SMin}^K F(x)$ *is closed for every $x \in \text{dom}F$.*

Proof. Let $\{(x_n, y_n)\}$ be a $\leq_{k_0}$-decreasing sequence in gr$F$ with $x_n \to x$.

(i) Let
$$T(x_n, y_n) := \{y \in F(x) : y + k_0 p(x_n, x) \leq_K y_n\}$$
and argue similarly as for Proposition 2.1(ii).

(ii) Put
$$R(x_n, y_n) := \{y \in \mathrm{Min}^K F(x) : y + k_0 p(x_n, x) \leq_K y_n\}.$$
and use a technique similar to that for Proposition 2.1(iii).

(iii) Set
$$N(x_n, y_n) := \{y \in \mathrm{SMin}^K F(x) : y + k_0 p(x_n, x) \leq_K y_n\}$$
and reason similarly as for Proposition 2.1(iv). □

The converse to Proposition 2.2 is false as explained by the following example.

Example 2.4. Let $X = [-2, 2], Y = R, K = R_+, k_0 = 1, p(x, y) = |x - y|$ and F be single-valued and defined by
$$F(x) = \begin{cases} 7 & \text{if} \quad x = 0, \\ x^2 & \text{otherwise.} \end{cases}$$

To check that F is (k_0, K)-lsca at any $x \in X$, i.e. if $x_n \to x$ and, for all $n \in \mathbb{N}$,
$$y_{n+1} + |x_{n+1} - x_n| \leq y_n, \tag{2.1}$$
then
$$\bar{y} + |\bar{x} - x_n| \leq y_n \tag{2.2}$$
for all n, where $\bar{y} = F(\bar{x})$. We split the consideration into three subcases. If $\bar{x} \neq 0$ and all $x_n \neq 0$, then by the continuity of F, (2.2) is evidently satisfied. If $(x_n, y_n) = (0, 7)$ for some n, then (2.1) means that $n = 1$ and $(x_m, y_m) = (0, 7)$ for all m and then (2.2) is fulfilled. Finally if $(\bar{x}, \bar{y}) = (0, 7)$ then (2.1) holds only if $(x_n, y_n) = (0, 7)$ for all n and hence (2.2) holds too.

To see that F is not (k_0, K)-lsc at $\bar{x} = 0$ take $x_n = \dfrac{1}{n} \to 0 = \bar{x}$ and $(x, y) = (2, 4)$. Then
$$\frac{1}{n^2} + |\frac{1}{n} - 2| \leq 4$$

for all n, but $7 + |0 - 2| \not\leq 4$.

Proposition 2.3. *Assume ($\tau 2$) and that K is closed in direction k_0.*

(i) If F is K-lsc at $\bar{x}$, then F is (k_0, K)-lsc at $\bar{x}$.

(ii) If F is K-lsca at $\bar{x}$, then F is (k_0, K)-lsca at $\bar{x}$.

Proof. (i) Let (x, y) and (x_n, y_n) in gr$F$ be such that $x_n \to \bar{x}$ and $(x_n, y_n) \leq_{k_0} (x, y)$ for all $n \in \mathbb{N}$. By ($\tau 2$), for each $i \in \mathbb{N}$, there exists $Q(i) \in \mathbb{N}$ such that, $\forall q > Q(i)$,

$$p(x, x_q) \geq p(x, \bar{x}) - 1/i.$$

Therefore,

$$y_q + k_0(p(x, \bar{x}) - 1/i) \leq_K y_q + k_0 p(x, x_q) \leq_K y.$$

Hence $x_q \in \{x' \in X : F(x') \cap (y - k_0(p(x, \bar{x}) - 1/i) - K) \neq \emptyset\}$. As F is K-lsc at $\bar{x}$ and $x_q \to \bar{x}$, $\bar{x}$ belongs to this level set as well, i.e. there exists $\bar{y} \in F(\bar{x})$ such that

$$\bar{y} + k_0(p(x, \bar{x}) - 1/i) \leq_K y.$$

The closedness of K in direction k_0 implies that

$$\bar{y} + k_0 p(x, \bar{x}) \leq_K y,$$

i.e. F is (k_0, K)-lsc at $\bar{x}$.

(ii) Let $\{(x_n, y_n)\}$ in gr$F$ be a $\leq_{k_0}$-decreasing sequence with $x_n \to \bar{x}$. Since F is K-lsca at $\bar{x}$ and $\{y_n\}$ is clearly $\leq_K$-decreasing, there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y} \leq_K y_n$ for every $n \in \mathbb{N}$. It follows that, $\forall n, q \in \mathbb{N}$,

$$\bar{y} + k_0 p(x_n, x_{n+q}) \leq_K y_{n+q} + k_0 p(x_n, x_{n+q}) \leq_K y_n.$$

Hence

$$k_0 p(x_n, x_{n+q}) \leq_K y_n - \bar{y}.$$

Then ($\tau 2$) and the closedness of K in direction k_0 give

$$k_0 p(x_n, \bar{x}) \leq_K y_n - \bar{y},$$

i.e. F is (k_0, K)-lsca at $\bar{x}$. □

Note that Proposition 2.3(ii) was proved in Göpfert et al. [8] for the case, where p is the metric of X. Reexamining Example 2.4 it is not hard to see that F is not K-lsca at 0 although it is (k_0, K)-lsca. The following example ensures a similar situation for being K-lsc.

Example 2.5. Let $X = Y = R, K = R_+, k_0 = 1, p(x, y) = |x - y|$ and f be a single-valued function defined by

$$f(x) = \begin{cases} 2 & \text{if} \quad x = 0, \\ \dfrac{1}{|x| + 1} & \text{otherwise.} \end{cases}$$

Then direct checking shows that f is (k_0, K)-lsc on X but not K-lsc at 0.

In the sequel we also say that gr$F$ satisfies the property of (k_0, K)-lower semicontinuity from above instead of saying that F is (k_0, K)-lsca (on dom$F$). Sometimes we concern a subset $A \subseteq X \times Y$, instead of dealing with the graph of a set-valued mapping, and then we may impose that A satisfies the property of (k_0, K)-lower semicontinuity from above.

3. Enhanced Versions of Ekeland's Variational Principle

From now on we fix a point $z^* \in K^+$ such that $z^*(k_0) = 1$ (the existence of such a z^* is guaranteed by the separation theorem). We use P_X to denote the projection from $X \times Y$ on X. A subset $A \subseteq X \times Y$ is said to be $\leq_{k_0}$-complete if, for every $\leq_{k_0}$-decreasing sequence $\{(x_n, y_n)\}$ in A with $\{x_n\}$ being a Cauchy sequence, $x_n \to \bar{x}$ for some $\bar{x}$ in X. Note that if X is complete then any $A \subseteq X \times Y$ is $\leq_{k_0}$-complete.

Theorem 3.1 (Minimal elements). *Let X, Y, K, k_0, p, z^* and $\leq_{k_0}$ be as above. Assume that $A \subseteq X \times Y$ satisfies the property of (k_0, K)-lower semicontinuity from above and, for some $(x_0, y_0) \in A$, $S_A(x_0, y_0)$ is nonempty, $\leq_{k_0}$-complete and $z^*(P_Y(S_A(x_0, y_0)))$ is bounded from below. Then there exists $(\bar{x}, \bar{y}) \in S_A(x_0, y_0)$ such that, if $(x, y) \in A$ with $(x, y) \leq_{k_0} (\bar{x}, \bar{y})$, then $x = \bar{x}$ and $z^*(y) = z^*(\bar{y})$.*

Moreover, if $z^* \in K^\sharp$ then, for $(x, y) \in A$ with $(x, y) \leq_{k_0} (\bar{x}, \bar{y})$, one has $(x, y) = (\bar{x}, \bar{y})$.

Proof. We note first that if $S_A(x^*, y^*) = \emptyset$ for some $(x^*, y^*) \in S_A(x_0, y_0)$, then $(\bar{x}, \bar{y}) = (x^*, y^*)$ is a required point. Considering now an arbitrarily fixed $(x, y) \in$

$S_A(x_0, y_0)$ we can assume that $S_A(x, y) \neq \emptyset$. Let $x' \in P_X(S_A(x, y))$. Then there exists $y' \in P_Y(S_A(x_0, y_0))$ such that $(x', y') \leq_{k_0} (x, y)$. Hence

$$k_0 p(x, x') \leq_K y - y'.$$

Since $z^*(P_Y(S_A(x_0, y_0)))$ is bounded from below, we have

$$p(x, x^{'}) \leq z^*(y) - z^*(y^{'}) \leq z^*(y) - \inf\{z^*(v) : v \in P_Y(S_A(x_0, y_0))\}.$$

Then

$$\sup\{p(x, x') : x' \in P_X(S_A(x, y))\} < +\infty.$$

Starting with (x_0, y_0), let us construct a sequence $\{(x_n, y_n)\} \subseteq S_A(x_0, y_0)$ in the following way: having $(x_n, y_n) \in S_A(x_{n-1}, y_{n-1})$ we take a point $(x_{n+1}, y_{n+1}) \in S_A(x_n, y_n)$ such that

$$p(x_n, x_{n+1}) \geq \tfrac{1}{2} \sup\{p(x_n, x') : x' \in P_X(S_A(x_n, y_n))\}. \tag{3.1}$$

We obtain in this way a sequence $\{(x_n, y_n)\} \subseteq S_A(x_0, y_0)$ such that, for all $n \in \mathbb{N}$, $(x_{n+1}, y_{n+1}) \leq_{k_0} (x_n, y_n)$. Hence $P_X(S_A(x_{n+1}, y_{n+1})) \subseteq P_X(S_A(x_n, y_n))$. Now, let us show that there are two cases for the sequence $\{(x_n, y_n)\}$: either $\sup\{p(x_{n_0}, x') : x' \in P_X(S_A(x_{n_0}, y_{n_0}))\} = 0$ for some n_0 or there exists a subsequence $\{(x_{n_k}, y_{n_k})\}$ satisfying $\lim_{k\to\infty} \sup\{p(x_{n_k}, x') : x' \in P_X(S_A(x_{n_k}, y_{n_k}))\} = 0$.

Indeed, suppose $\sup\{p(x_n, x') : x' \in P_X(S_A(x_n, y_n))\} \geq \delta$ for some $\delta > 0$ and every $n \in \mathbb{N}$. Then, by (3.1),

$$\frac{1}{2}\delta k_0 \leq_K k_0 p(x_n, x_{n+1}) \leq_K y_n - y_{n+1},$$

which implies that

$$\frac{1}{2}\delta \leq z^*(y_n) - z^*(y_{n+1}).$$

Adding these relations from 0 to $n-1$, we obtain

$$\frac{1}{2}n\delta \leq z^*(y_0) - z^*(y_n) \leq z^*(y_0) - \inf\{z^*(v) : v \in P_Y(S_A(x_0, y_0))\},$$

which yields a contradiction as $n \to +\infty$.

Case 1 ($\sup\{p(x_{n_0}, x') : x' \in P_X(S_A(x_{n_0}, y_{n_0}))\} = 0$ for some n_0). By (τ4) there exists $(\bar{x}, y^*) \in S_A(x_{n_0}, y_{n_0})$ such that $P_X(S_A(x_{n_0}, y_{n_0})) = \{\bar{x}\}$. Hence

$P_X(S_A(\bar{x}, y^*)) = \{\bar{x}\}$.

Case 2 (there exists a subsequence $\{(x_{n_k}, y_{n_k})\}$ satisfying $\lim_{k\to\infty} \sup\{p(x_{n_k}, x') : x' \in P_X(S_A(x_{n_k}, y_{n_k}))\} = 0$). Since $\lim_{k\to\infty} \sup\{p(x_{n_k}, x_{n_m}) : n_m > n_k\} = 0$, Lemma 1.1 implies that $\{x_{n_k}\}$ is a Cauchy sequence and then $x_{n_k} \to \bar{x}$ for some $\bar{x} \in X$ as $S_A(x_0, y_0)$ is $\leq_{k_0}$-complete. By the property of (k_0, K)-lower semicontinuity from above there exists $y^* \in Y$ such that $(\bar{x}, y^*) \in A$ and, for each $k \in \mathbb{N}$, $(\bar{x}, y^*) \leq_{k_0} (x_{n_k}, y_{n_k})$.

Now we show that $\bigcap_{k\in\mathbb{N}} P_X(S_A(x_{n_k}, y_{n_k})) = \{\bar{x}\}$. It is easy to see that $\bar{x} \in \bigcap_{k\in\mathbb{N}} P_X(S_A(x_{n_k}, y_{n_k}))$. If $w \in \bigcap_{k\in\mathbb{N}} P_X(S_A(x_{n_k}, y_{n_k}))$, then $\lim_{k\to\infty} p(x_{n_k}, w) = 0$. We have $\lim_{k\to\infty} \sup\{p(x_{n_k}, x_{n_m}) : m > k\} = 0$ as well. From $(\tau 3)$ it follows that $\lim_{k\to\infty} d(x_{n_k}, w) = 0$, i.e. $w = \bar{x}$. Hence, we also have $P_X(S_A(\bar{x}, y^*)) = \{\bar{x}\}$ as for Case 1.

Starting with $t_0 = y^*$, we construct a sequence $t_n \in P_Y(S_A(\bar{x}, y^*))$ as follows: having $t_n \in P_Y(S_A(\bar{x}, t_{n-1}))$, we take $t_{n+1} \in P_Y(S_A(\bar{x}, t_n))$ such that

$$z^*(t_{n+1}) \leq \inf\{z^*(t) : t \in P_Y(S_A(\bar{x}, t_n))\} + \frac{1}{n+1}.$$

The sequence $\{(\bar{x}, t_n)\}$ is clearly $\leq_{k_0}$-decreasing. Again by the property of (k_0, K)-lower semicontinuity from above, there exists $\bar{y} \in Y$ such that $(\bar{x}, \bar{y}) \in A$ and $(\bar{x}, \bar{y}) \leq_{k_0} (\bar{x}, t_n)$ for every $n \in \mathbb{N}$.

Of course $(\bar{x}, \bar{y}) \in S_A(x_0, y_0)$. If $(x, y) \in A$ and $(x, y) \leq_{k_0} (\bar{x}, \bar{y})$, then $x = \bar{x}$ as $P_X(S_A(\bar{x}, \bar{y})) = \{\bar{x}\}$. Since $(x, y) \leq_{k_0} (\bar{x}, t_n)$, we have $y \in P_Y(S_A(\bar{x}, t_n))$ for every $n \in \mathbb{N}$. Therefore, $\forall n \geq 1$,

$$0 \leq p(\bar{x}, x) \leq z^*(\bar{y}) - z^*(y) \leq z^*(t_n) - \inf\{z^*(t) : t \in P_Y(S_A(\bar{x}, t_{n-1}))\} \leq \frac{1}{n},$$

whence $z^*(y) = z^*(\bar{y})$.

Moreover, if $z^* \in K^\sharp$, we have $y = \bar{y}$, as $\bar{y} - y \in K$ and $z^*(\bar{y} - y) = 0$. $\square$

Theorem 3.2 (Strict minimal element). *Additionally to the assumptions of Theorem 3.1, for each $x \in X$, impose that $\{y' : (x, y') \in A\}$ has the strict domination property. Then there exists $(\bar{x}, \bar{y}) \in S_A(x_0, y_0)$ with $\bar{y} \in \mathrm{SMin}^K\{y' : (\bar{x}, y') \in A\}$ and $(\bar{x}, \bar{y}) \in \mathrm{SMin}^{k_0} A$, i.e. if $(x, y) \in A$ and $(x, y) \leq_{k_0} (\bar{x}, \bar{y})$, then $(x, y) = (\bar{x}, \bar{y})$.*

Proof. Theorem 3.1 implies the existence of $(\bar{x}, y^*) \in S_A(x_0, y_0)$ with the property that if $(x, y) \in A$ and $(x, y) \leq_{k_0} (\bar{x}, y^*)$ then $x = \bar{x}$. For $(\bar{x}, y^*)$, as $\{y' : (\bar{x}, y') \in A\}$ has the strict domination property, there is $\bar{y} \in \mathrm{SMin}^K\{y' :$

$(\bar{x}, y') \in A\}$ such that $\bar{y} \leq_K y^*$. Let us show that $(\bar{x}, \bar{y})$ is a desired element. Of course $(\bar{x}, \bar{y}) \in S_A(x_0, y_0)$. Let $(x, y) \in A$ and $(x, y) \leq_{k_0} (\bar{x}, \bar{y})$. Since $\bar{y} \leq_K y^*$, $(x, y) \leq_{k_0} (\bar{x}, y^*)$, which implies that $x = \bar{x}$, $(\bar{x}, y) \in A$ and $(\bar{x}, y) \leq_{k_0} (\bar{x}, \bar{y})$. Hence $y \leq_K \bar{y}$. By the definition of $\mathrm{SMin}^K\{.\}$, one gets $\bar{y} = y$. $\square$

Remark 3.1. We note that for every $(x_0, y_0) \in A$, the condition $S_{\mathrm{A}}(x_0, y_0) \neq \emptyset$ in Theorems 3.1 and 3.2 is satisfied if $0 \in K$ and $p(x_0, x_0) = 0$, since $(x_0, y_0) \in S(x_0, y_0)$. A special case is: K is pointed and p is a metric. Theorem 3.1 brings some improvements to Theorem 1 and Corollary 5 of Göpfert et al. [8].

Theorem 3.3 (An enhanced Ekeland's variational principle). *Let X, Y, K, k_0, p and $\leq_{k_0}$ be as in Theorem 3.1. Consider a set-valued mapping $F : X \to 2^Y$ which is quasibounded from below and (k_0, K)-lsca. Assume that $(x_0, y_0) \in \mathrm{gr}F$ and $S_{\mathrm{grF}}(x_0, y_0)$ is nonempty and $\leq_{k_0}$-complete. Then, for any $\varepsilon > 0$ and $\lambda > 0$, there exists $(\bar{x}, \bar{y}) \in \mathrm{gr}F$ such that*

(i) $\bar{y} - y_0 + \dfrac{\varepsilon}{\lambda} p(x_0, \bar{x}) k_0 \leq_K 0$;

(ii) $y - \bar{y} + \dfrac{\varepsilon}{\lambda} p(\bar{x}, x) k_0 \not\leq_K 0$ *for all* $(x, y) \in \mathrm{gr}F$ *with* $x \neq \bar{x}$.

If (x_0, y_0) is an εk_0-minimizer of F (i.e. $y + \varepsilon k_0 \not\leq_K y_0$, for all $y \in \mathrm{Im}F := \{y \in F(x)$ for some $x \in X\}$), then $\bar{x}$ can be chosen to satisfy $p(x_0, \bar{x}) \leq \lambda$.

Proof. Applying Theorem 3.1 with $\mathrm{gr}F$, εk_0 and $\frac{1}{\lambda}p$ in the places of A, k_0 and p, respectively, one sees the existence of $(\bar{x}, \bar{y}) \in \mathrm{gr}F$ satisfying (i) and (ii). If $p(x_0, \bar{x}) > \lambda$, we would have

$$\bar{y} + \varepsilon k_0 \leq_K \bar{y} + \frac{\varepsilon}{\lambda} p(x_0, \bar{x}) k_0 \leq_K y_0,$$

which contradicts the fact that (x_0, y_0) is an εk_0-minimizer. $\square$

Theorem 3.4 (An enhanced Ekeland's variational principle). *Let X, Y, K, k_0, p and $\leq_{k_0}$ be as in Theorem 3.1. Assume that K is closed, p satisfies $(\tau 2)$ and F is quasibounded from below and K-lsca. Assume further that $(x_0, y_0) \in \mathrm{gr}F$ and $S_{\mathrm{grF}}(x_0, y_0)$ is nonempty and $\leq_{k_0}$-complete. Then, the conclusions of Theorem 3.3 also hold.*

Proof. Proposition 2.3(ii) implies that F is (k_0, K)-lsca. Apply now Theorem

3.3 to complete the proof. $\square$

Theorem 3.5 (An extended version of the EVP). *Impose additionally to the assumptions of Theorem 3.3, that for every $x \in \text{dom}F$, $F(x)$ has the strict domination property. Then there exists $(\bar{x}, \bar{y}) \in \text{gr}F$ satisfying*

(i') $\bar{y} - y_0 + \dfrac{\varepsilon}{\lambda} p(x_0, \bar{x}) k_0 \leq_K 0$, *with* $\bar{y} \in \text{SMin}^K F(\bar{x})$;

(ii') $y - \bar{y} + \dfrac{\varepsilon}{\lambda} p(\bar{x}, x) k_0 \not\leq_K 0$ *for all* $(x, y) \in \text{gr}F$ *with* $(x, y) \neq (\bar{x}, \bar{y})$.

If (x_0, y_0) is an εk_0-minimizer of F then $\bar{x}$ can be chosen such that $p(x_0, \bar{x}) \leq \lambda$.

Proof. The usage of Theorem 3.2 with $A = \text{gr}F$, εk_0 in the place of k_0 and $\frac{1}{\lambda}p$ in the place of p, yields $(\bar{x}, \bar{y}) \in \text{gr}F$ satisfying (i') and (ii'). The second assertion was checked in Theorem 3.3. $\square$

Note that (ii') of Theorem 3.5 is strictly stronger than (ii) of Theorem 3.3. Assertions like (ii') were discussed first in Bao and Mordukhovich [2, 3] with additional assumptions. The following example shows the essentialness of the strict domination property of $F(x)$.

Example 3.1. Let $X = R, Y = R^2$, $K = \{(a, b) : a \in R, b \geq 0\}$, $k_0 = (0, 1)$, $z^*(a, b) = b, p(x_1, x_2) = |x_2 - x_1|$ and

$$F(x) = \begin{cases} \{(a, b) : a \geq 0, b \geq -1\} & \text{if} \quad x = 0, \\ \{(0, 0)\} & \text{otherwise.} \end{cases}$$

We see that the assumptions of Theorem 3.3 are fulfilled and $\bar{x} = 0, \bar{y} = (0, -1)$ satisfy (ii) but not (ii') of Theorem 3.5. The reason is that $F(x)$ does not have the strict domination property. If now $K = \{(a, b) : a \geq 0, b \geq 0)\}$ then $F(x)$ does, and in fact this point $(\bar{x}, \bar{y})$ meets condition (ii'). If, we retain the same K as at the beginning but modify $F(0)$ as $\{(a, b) : a \geq 0, b \geq a^2 - 1\}$, then $F(x)$ also has the mentioned property and $(\bar{x}, \bar{y})$ does not violate (ii').

Theorem 3.6 (An extended version of the EVP). *Let X, Y, K, k_0, p and $\leq_{k_0}$ be as in Theorem 3.1. Assume ($\tau 2$), that K is closed, pointed and that F is quasi-bounded from below, upper K-continuous and nonempty-compact-valued. Assume further that $(x_0, y_0) \in \text{gr}F$ and $S_{\text{grF}}(x_0, y_0)$ is nonempty and $\leq_{k_0}$-complete. Then* *we have the same conclusions of Theorem 3.5.*

Proof. By Proposition 2.2(i) F is K-lsca. Then Proposition 2.4(ii) yields that F is (k_0, K)-lsca. On the other hand, for $x \in \mathrm{dom}F$, $F(x)$ has the strict domination property. Applying Theorem 3.5 achieves the proof. $\square$

Remark 3.2. For the special case where X is complete and $p = d$, Theorem 3.6 improves Corollary 3.2 of Li and Zhang [13], where F is assumed to be bounded from below. It brings some improvements also to Theorem 3.1 of Chen and Huang [6] as well as Corollary 3.1 of Li and Zhang [13], where F is required additionally to be usc.

The following example indicates that Theorem 3.6 is more advantegeous than Theorem 3.1 of Chen and Huang [6].

Example 3.2. Let $X = Y = R$, $p(x, y) = |y - x|$, $K = R_+$, $k_0 = 1$, $(x_0, y_0) = (\frac{1}{2}, 0)$ and

$$F(x) = \begin{cases} [-1, 1] & \text{if} \quad x = 0, \\ [0, 2] & \text{if} \quad x \neq 0. \end{cases}$$

Then F is not usc at 0 and Theorem 3.1 of Chen and Huang [6] is not applicable. But all the assumptions of Theorem 3.6 are fulfilled. Direct calculations show that $(\bar{x}, \bar{y}) = (0, -1) \in \mathrm{gr}F$ satisfies (i') and (ii').

The next examples give cases where Theorem 3.5 is more suitable for application than Corollary 3.2 of Li and Zhang [13], Theorem 3.1 of Chen and Huang [6] and also the above Theorem 3.6.

Example 3.3. Let X, Y, p, K, k_0 and (x_0, y_0) be as in Example 3.2 and

$$F(x) = \begin{cases} [-1, 1) & \text{if} \quad x = 0, \\ [0, +\infty) & \text{if} \quad x \neq 0. \end{cases}$$

Then F is not usc and $F(x)$ is noncompact for $x \neq 0$. Hence all Corollary 3.2 of Li and Zhang [13], Theorem 3.1 of Chen and Huang [6] and our Theorem 3.6 do not work. However, the conditions imposed in Theorem 3.5 are met. Direct computations ensure that $(\bar{x}, \bar{y}) = (0, -1) \in \mathrm{gr}F$ fulfils (i') and (ii').

Example 3.4. Let $X = R, Y = R^2, k_0 = (0, 1), p(x_1, x_2) = |x_2 - x_1|, x_0 = 0$ and

$y_0 = (1,1)$. Let

$$F(x) = \begin{cases} \{(a,b) : 0 \le a \le 2, a^2 - 1 \le b \le 3\} & \text{if} \quad x = 0, \\ (0,0) & \text{otherwise,} \end{cases}$$

$$K = \{(a,b) : a \in R, b \ge 0)\}.$$

Then F is not usc and K is not pointed, but Theorem 3.5 is applicable since its assumptions are satisfied, while the other results mentioned in Example 3.3 are not in use. In fact $\bar{x} = 0, \bar{y} = (0,-1)$ satisfies (i') and (ii').

Similarly as for Theorem 3.6, but using Proposition 2.1(iv) instead of Proposition 2.1(i), we obtain

Theorem 3.7 (An extended version of the EVP). *Let X, Y, K, k_0, p and $\le_{k_0}$ be as in Theorem 3.1. Assume that K is closed, p satisfies ($\tau 2$), F is quasibounded from below, K-lsc, $F(x)$ has the strict domination property and* $\mathrm{SMin}^K F(x)$ *is closed for every $x \in \mathrm{dom}F$. Assume further that $(x_0, y_0) \in \mathrm{gr}F$ and $S_{\mathrm{grF}}(x_0, y_0)$ is nonempty and $\le_{k_0}$-complete. Then the conclusions of Theorem 3.5 are still valid.*

Note that Theorem 3.7 includes properly Theorem 3.2 of Bao and Mordukhovich [2], where additional conditions that $p = d$ and K is pointed are imposed. A case where Theorem 3.7 is more suitable for application than many recent known results is given as follows.

Example 3.5. Let X, Y, p, K, k_0 and $(x_0, y_0$ be as in Example 3.4 and

$$F(x) = \begin{cases} \{(a,b) : a \ge 0, b \ge a^2 - 1\} & \text{if} \quad x = 0, \\ (0,0) & \text{otherwise.} \end{cases}$$

Then K is not pointed and $F(0)$ is noncompact. Then both the results encountered in Example 3.4 and Theorem 3.2 of Bao and Mordukhovich [2] cannot be applied, but Theorem 3.7 can be. It is easy to check directly that $(\bar{x}, \bar{y}) = (0, (0,0))$ satisfies (i') and (ii').

Theorem 3.8 (An extended version of the EVP). *Let X, Y, K, k_0, p and $\le_{k_0}$ be as in Theorem 3.1. Assume that K is closed, p satisfies ($\tau 2$), F is quasibounded from below, K-lsca and $F(x)$ has the strict domination property for* *every* $x \in \mathrm{dom}F$. *Assume further that* $(x_0, y_0) \in \mathrm{gr}F$ *is such that* $S_{\mathrm{grF}}(x_0, y_0)$ *is nonempty and* $\leq_{k_0}$*-complete. Then we have the same conclusions of Theorem 3.5.*

Proof. Argue similarly as in the proof of Theorem 3.6, using Proposition 2.4(ii) and Theorem 3.5. □

Remark 3.3. (i) The condition that F is K-lsca and $F(x)$ has the strict domination property imposed in Theorem 3.8 is easily seen to be equivalent to the limiting monotonicity condition assumed in Theorem 3.5 of Bao and Mordukhovich [3] (and recalled after Definition 2.1). Hence, when applied to the case where $p = d$, Theorem 3.8 improves that Theorem 3.5, since we omit the assumption that F is K-level closed.

(ii) Applying assertions of Propositions 2.1 - 2.3, different from those applied in Theorems 3.6 - 3.8, we can get other new versions of the EVP, similar to these theorems.

4. Some Consequences for Single-Valued Cases

In this section we apply our main results to single-valued cases. To have a generalization of a linear space Y, which is similar to the extended real line $R \cup \{+\infty\}$, we add to Y an additional element, denoted also by $+\infty$, with the usual rules for addition and multiplication with reals. We avoid indeterminate expressions like $0.(+\infty)$ and adopt that $y \leq_K +\infty, \forall y \in Y$. Now we consider a mapping $f : X \to Y \cup \{+\infty\}$ and denote $\mathrm{dom}f := \{x \in X : f(x) \neq +\infty\}$. We say that f is proper if $\mathrm{dom}f \neq$ *emptyset*. We use now the relation $\leq_{k_0}$ on $\mathrm{dom}f$ by the definition

$$x_2 \leq_{k_0} x_1 \quad \Leftrightarrow \quad f(x_2) + k_0 p(x_1, x_2) \leq_K f(x_1).$$

The following corollary is a direct consequence of Theorem 3.3.

Corollary 4.1. *Let* X, Y, p, K *and* k_0 *be as specified in Section 3, with the additional completeness of* X. *Assume that* $f : X \to Y \cup \{+\infty\}$ *be proper and quasibounded from below. Let* $S(x)$:=$\{x' \in X : f(x') + k_0 p(x, x') \leq_K f(x)\}$ *be closed for every* $x \in X$. *Then for every* $x_0 \in domf$ *with* $S(x_0) \neq \emptyset$ *there exists* $v \in X$ *such that,* $\forall x \neq v$,

$$f(v) + k_0 p(x_0, v) \leq_K f(x_0), \tag{4.1}$$

$$f(x) + k_0 p(v,x) \not\leq_K f(v). \tag{4.2}$$

Proof. Since $S(x)$ is closed for all $x \in X$, f is (k_0, K)-lsca. Applying Theorem 3.3 we have (4.1) and (4.2). □

This corollary properly includes Corollary 2 of Göpfert et al. [8], since $p \neq d$ and f may not be bounded from below.

The following corollary is a direct consequence of Corollary 4.1.

Corollary 4.2. *Let X, Y, K, p, k_0 and f be as in Corollary 4.1, with K being closed in the direction k_0 and ($\tau 2$) being satisfied. Assume that if $x_n \in domf, x_n \to x$ and $f(x_n)$ is $\leq_K$ decreasing, then $f(x) \leq_K f(x_n), \forall n \in \mathbb{N}$. Assume further that $x_0 \in domf$ and $S(x_0) \neq \emptyset$. Then there exists $v \in X$ such that, $\forall x \neq v$, (4.1) and (4.2) hold.*

If f is not only quasibounded from below but also bounded from below and $p = d$, this corollary coincides with Corollary 3 of Göpfert et al. [8].

Corollary 4.3. *Let X, Y, K and k_0 be as specified in Section 3, with the additional completeness of X and closedness of K. Let p be a τ-function and Φ: $X \times X \to Y \cup \{+\infty\}$ satisfy the assumptions*

(i) there is $x_0 \in X$ such that $\Phi(x_0, x_0) = 0$ and $\Phi(x_0, .)$ is K-lsca and quasibounded from below;

(ii) if $\Phi(x,z) \in -K$ and $\Phi(z,y) \in -K$, then

$$\Phi(x,y) \leq \Phi(x,z) + \Phi(z,y).$$

Then there exists $v \in X$ such that, $\forall x \neq v$,

(a) $\Phi(x_0, v) \in -K$, if $p(x_0, x_0) = 0$.

(b) $\Phi(v,x) + k_0 p(v,x) \notin -K$.

Proof. Set $f(.) = \Phi(x_0, .)$. Then the assumptions of Theorem 3.4 are clearly satisfied. Hence, this theorem gives $v \in X$ such that, $\forall x \neq v$, (4.1), (4.2) hold.

As $f(x_0) = \Phi(x_0, x_0) = 0$, (4.1) implies that $\Phi(x_0, v) \in -K$. For any $x \in X \setminus \{v\}$, if $\Phi(v,x) \notin -K$ then (b) is fulfilled. If $\Phi(v,x) \in -K$, (4.2) implies that

$$\Phi(x_0, x) - \Phi(x_0, v) + k_0 p(v,x) \notin -K,$$

and (ii) yields

$$\Phi(v,x)+k_0p(v,x)\notin -K.$$

$\square$

Corollary 4.4 (Lin and Du [14]). *Let X be a complete metric space and p be a τ-function. Let $f\colon X\to R\cup\{+\infty\}$ be proper, R_+-lsca and bounded from below. Let $\varphi\colon R\to(0,\infty)$ be nondecreasing. Then there exists $v\in X$ such that, $\forall x\neq v$,*

$$p(v,x)>\varphi(f(v))(f(v)-f(x)).$$

Proof. Setting $\Phi(x,y)=\varphi(f(x))(f(y)-f(x))$ we see that, $\forall x\in X$, $\Phi(x,.)$ is proper, R_+-lsca, bounded from below and $\Phi(x,x)=0$. We claim that Φ satisfies (ii) of Corollary 4.3. Indeed, if $\Phi(x,z)\leq 0$ and $\Phi(z,y)\leq 0$ then

$$f(y)\leq f(z)\leq f(x).$$

Hence, as $\varphi(f(z))\leq\varphi(f(x))$,

$$\begin{aligned}\Phi(x,z)+\Phi(z,y)&\geq\varphi(f(x))(f(z)-f(x))+\varphi(f(x))(f(y)-f(z))\\&=\Phi(x,y).\end{aligned}$$

Now applying Corollary 4.3 with $k_0=1$ one obtains $v\in X$ such that, $\forall x\neq v$,

$$\Phi(v,x)+p(v,x)>0.$$

Therefore

$$p(v,x)>\varphi(f(v))(f(v)-f(x)).$$ $\square$

Corollary 4.5. *Let X,p,f and φ be as in Corollary 4.4. Let $\varepsilon>0$ and $x_0\in X$ satisfy $f(x_0)\leq\inf_{x\in X}f(x)+\varepsilon$. Then $v\in X$ exists such that, $\forall x\neq v$,*

(i) $0\leq f(x_0)-f(v)\leq\varepsilon$, *if* $p(x_0,x_0)=0$;

(ii) $\varepsilon p(v,x)>\varphi(f(v))(f(v)-f(x))$.

Proof. Set $\Phi(x,y)=\varphi(f(x))(f(y)-f(x))$. Taking εp for p in Corollary 4.4 gives (ii) and $\Phi(x_0,v)\leq 0$. Hence, $f(x_0)-f(v)\geq 0$. Since $f(x_0)\leq\inf_{x\in X}f(x)+\varepsilon\leq f(v)+\varepsilon$, we obtain (i). $\square$

For the special case, where p is a w-distance, Corollary 4.5 coincides with Theorem 2.4 of Lin and Du [15].

Corollary 4.6. *Let X, Y, K, p and k_0 be as in Corollary 4.3. Let $x_0\in X$, $\varepsilon>0$ and $\Phi\colon X\times X\to Y$ satisfy the conditions* *(i)* $\Phi(x_0, x_0) = 0$ *and* $z^*(\Phi(x_0, .))$ *is bounded from below;*

(ii) $\Phi(x_0, .)$ *is* K*-lsca;*

(iii) $\Phi(x, y) \leq \Phi(x, z) + \Phi(z, y)$ *for any* $x, y, z \in X$.

Then there exists $v \in X$ *such that,* $\forall x \neq v$,

(a) $\Phi(x_0, v) + \varepsilon k_0 p(x_0, v) \in -K$, *if* $p(x_0, x_0) = 0$*;*

(b) $\Phi(v, x) + \varepsilon k_0 p(v, x) \notin -K$.

Proof. Setting $f(.) = \Phi(x_0, .)$ we see that $f(x_0) = 0$ and $f(.)$ is K-lsca. Applying now Theorem 3.1 with $A = \mathrm{gr} f$ yields $v \in X$ such that, $\forall x \neq v$,

$$f(x) + k_0 p(v, x) \notin f(v) - K.$$

Consequently, by (iii), we arrive at (b). Conclusion (a) is obvious. □

Note that Corollary 4.6 contains properly Theorem 3.1 of Ansari [1] (since in this theorem p is a w-distance, and (i) is required to be fulfilled for all $x \in X$), Theorem 1 of Bianchi et al. [5] (since in that theorem $p = d$, (i) is required to be fulfilled for all $x \in X$ and (ii) is replaced by the condition that $\Psi(x, .)$ is K-lsc for all $x \in X$) and Theorem 2.1 of Bianchi et al. [4], which is the special case with $Y = R$ of the mentioned Theorem 1.

Corollary 4.7. *Let* X, Y, K, p *and* k_0 *be as specified in Section 3, with the additional closedness of* K *and* $(\tau 2)$. *Let* $\Psi \colon X \times X \to Y$ *satisfy conditions (ii) and (iii) of Corollary 4.6.*

Define a binary relation $\leq'_{k_0}$ *on* X *by*

$$y \leq'_{k_0} x \quad \Leftrightarrow \quad y = x \quad \text{or} \quad y \leq_{k_0} x.$$

Assume that there exists a nonempty subset M *of* X *such that*

(i) M *is* $\leq'_{k_0}$ *complete;*

(ii) *there exists* $x_0 \in M$ *such that* $\Psi(x_0, x_0) = 0$ *and* $z^*(\Psi(x_0, .))$ *is bounded from below.*

Then $\leq'_{k_0}$ *is a quasi-order and there exists* $v \in X$ *such that, for all* $x \neq v$,

(1) $\{y \in M : y \leq'_{k_0} v\} = \{v\}$;

(2) $\Psi(x_0, v) + \varepsilon p(x_0, v)k_0 \in -K$, *if* $p(x_0, x_0) = 0$;

(3) $\Psi(v, x) + \varepsilon p(v, x)k_0 \notin -K$.

This corollary is derived directly from Corollary 4.7 and properly includes Theorems 2.1 - 2.2 of Lin and Du [16]. Furthermore applying Theorem 3.3 we obtain also Theorem 3.1 of that paper.

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