Existence Conditions for Quasivariational Inclusion Problems in G-convex Spaces[1]

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Dedicated to Professor Van Hien Nguyen on his 65^{th} birthday

Abstract. We consider general quasivariational inclusion problems in G-convex spaces and prove sufficient conditions for the solution existence. The verifiability of the imposed assumptions is discussed in details. Corollaries and examples are supplied to show that the obtained results contain and improve recent existence conditions in the literature, even when applied to particular cases.

Key Words. Quasovariational inclusion problems, G-convex spaces, the solution existence, generalized G-quasiconvexity, generalized G-quasiconvexlikeness, fixed points.

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1. INTRODUCTION

The quasivariational inclusion problem is a general model which was introduced and studied recently, see [1-5, 11], to include many optimization-related problems such as quasiequilibrium problems, quasivariational inequalities, vector minimization problems, fixed-point and coincidence-point problems, etc.

On the other hand, spaces on which the considered problems are defined, have also been an object to be generalized in order to make the problems encompass a wide range of practical situations. A generalized convex space or G-convex space [13] is very general. Its particular forms are a convex subset of a topological vector space, a convex space [6], an S-contractive space, an H-space, a Komiya convex space, a metric space with the Michael convex structure, see [12] for more details.

The aim of this paper is to extend the quasivariational inclusion problem to being defined on G-convex spaces and to establish sufficient existence conditions under relaxed assumptions so that when applied even to particular cases these conditions improve recent existing results. The paper is structured as follows. The remaining part of this section consists of the problem setting and preliminaries. Section 2 is devoted to the main results. In Section 3 we deal with particular cases to compare in more details some recent results with our sufficient conditions when applied to the corresponding special cases.

Definition 1.1 (i) (see [6]) A convex set A in a vector space is called a convex space if it is equipped with a topology which includes the Euclidean topology on convex hulls of any nonempty finite subsets of A.

(ii) (see [13]) A generalized convex space or a G-convex space is a triple $(A,\ D, \Gamma)$ of a topological space A, a nonempty subset D of A and a generalized convex hull operator Γ from $\langle D\rangle$ (the set of all nonempty finite subsets of D) into 2^A (the space of all subsets of A) with nonempty values such that

(a) for each $M,\ N \in\ \langle D\rangle,\ M \subseteq N$ implies $\Gamma(M) \subseteq \Gamma(N)$;

(b) for each $N \in \langle D\rangle$ with $|N| = n+ 1$, there exists a continuous map Φ_N: $\Delta_n \longrightarrow \Gamma(N)$ such that, for each $J \in \langle N\rangle$, $\Phi_N(\Delta_J) \subset \Gamma(J)$, where Δ_n is the n - simplex with the vertices being the unit vectors $e_1, e_2, ..., e_{n+1}$ which form a basis of R^{n+1} and Δ_J is the face of Δ_n corresponding to $J \in \langle N\rangle$.

If $D = A$ we omit D writing simply $(A,\ D)$. (Note that for $N \in \langle D\rangle$, $\Gamma(N)$ does not need to contain N and a convex space A becomes a G-convex space $(A,\ \Gamma)$ by setting $\Gamma(N) = \mathrm{coN}$ for $N \in \langle A\rangle$.)

(iii) (see [13]) for a G-convex space (A, D, Γ), a subset B of A is said to be G-convex if, for each $N \in \langle D\rangle, N \subseteq B$ implies $\Gamma(N) \subseteq B$. (Note that any subset $C \subseteq A\backslash D$ is G-convex.)

(iv) for a G-convex space (A, D, Γ) and a nonempty subset B of A, the G-convex hull of B, denoted by G-co$B$, is $\bigcap\{C \subseteq A : C$ is a G-convex subset of A containing $B\}$.

Now, we pass to our problem setting. Let (A, Γ) be a G-convex space, A being Hausdorff, Y and Z be (real) Hausdorff topological vector spaces, $B \subseteq Z$ be nonempty. Let S_i: $A \longrightarrow 2^A$, $i = 1, 2$, T: $A \longrightarrow 2^B$, f: $B \times A \times A \longrightarrow 2^Y$ and $g : B \times A \longrightarrow 2^Y$ be multimaps (i.e. multivalued maps). We consider the following two quasivariational inclusion problems

(IP$_1$) Find $\bar{x} \in S_1(\bar{x})$ such that,$\forall y \in S_2(\bar{x}), \forall t \in T(\bar{x}, y)$,

$$f(t, \bar{x}, y) \subseteq g(t, \bar{x});$$

(IP$_2$) Find $\bar{x} \in S_1(\bar{x})$ such that,$\forall y \in S_2(\bar{x}), \forall t \in T(\bar{x}, y)$,

$$f(t, \bar{x}, y) \cap g(t, \bar{x}) \neq \emptyset.$$

Note that in [1-5], for quasivariational inclusion problems defined in A being a topological vector space, weaker models with "$\forall y \in S_2(\bar{x}), \forall t \in T(\bar{x}, y)$" replaced by "$\forall y \in S_2(\bar{x}), \exists t \in T(\bar{x}, y)$" or by "$\exists t \in T(\bar{x}, y), \forall y \in S_2(\bar{x})$" were also investigated. However, both the results and the proof techniques are similar as for the above model. Therefore, in this paper we are concerned only with the model of (IP$_1$) and (IP$_2$) for the sake of simple presentation. With this remark the reader will be convinced that our two problems include a wide range of optimization - related problems by refering to, e.g. [3, 4].

Definition 1.2 Let X, Y be topological spaces and F: $X \longrightarrow 2^Y$.

(i) F is said to be upper semicontinuous (usc in short, respectively lower semicontinuous, lsc in short) at $x_0 \in X$ if for every open subset $U \supseteq F(x_0)$ (respectively, $U \cap F(x_0) \neq \emptyset$), there is a neighborhood N of x_0 such that $\forall x \in N, U \supseteq F(x)$(respectively, $U \cap F(x) \neq \emptyset$). F is called usc in $A \subseteq X$ if F is usc at $\forall x \in A$. If $A = \text{dom}F = \{x \in X : F(x) \neq \emptyset\}$ we omit "in A" in the saying. We adopt this convention for each property of a multimap.

(ii) F is called transfer open (respectively transfer closed) if for every $x \in X$ and $y \in F(x)$ (respectively $y \notin F(x)$), there is x' $\in X$ such that $y \in \text{int}F(x')$ (respectively $y \notin \text{cl}F(x')$).

(iii) Let $C \subseteq Y$ be a cone. F is termed C-usc (respectively C-lsc) at $x_0 \in X$ if for every neighborhood V of 0 in Y, there is a neighborhood N of x_0 such that $F(x) \subseteq F(x_0) + U + C$ (respectively $F(x_0) \subseteq F(x) + U + C$) for every $x \in N$.

Note that F: $X \longrightarrow 2^Y$ is lsc at x_0 if and if $\forall x_\lambda \in \mathrm{domF}$: $x_\lambda \longrightarrow x_0, \forall y \in F(x_0), \exists y_\lambda \in F(x_\lambda), y_\lambda \longrightarrow y$.

Lemma 1.1 (see [7]) *Let* X, Y *be topological spaces and* F: $X \longrightarrow 2^Y$ *be a multimap. Then the following two assertions are equivalent*

(i) *the inverse* F^{-1} *is transfer open and* $\mathrm{dom}F = X$;

(ii) $X = \bigcup_{y \in Y} \mathrm{intF^{-1}(y)}$.

Lemma 1.2 *Let* (A, D, Γ) *be a* G*-convex space and* $B \subseteq A$ *be nonempty. Then* G-co$B = \{x \in A$: $\exists N \in \langle B \rangle, x \in G\mathrm{-coN}\}$.

Proof. For every $N \in \langle B \rangle$, it is clear that G-co$N \subseteq G\mathrm{-coB}$. Hence

$$B \subseteq \bigcup_{N \in \langle B \rangle} (G\mathrm{-coN}) \subseteq \mathrm{G-coB}.$$

Therefore, it suffices to show that the union in these inclusions, which is now denoted by C, is G-convex. Assume that $N_0 \in \langle D \rangle$ such that $N_0 \subseteq C$. By the definition of C, there is $\{N_1, N_2, ..., N_k\} \in \langle B \rangle$, such that $N_0 \subseteq \bigcup_{i=1}^{k}(G\mathrm{-coN_i}) \subseteq \mathrm{C}$. Since $\bigcup_{i=1}^{n} N_i \in \langle B \rangle$, one has $G\mathrm{-co}(\bigcup_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{N_i}) \subseteq \mathrm{C}$. As, $\forall j = 1, 2, ..., k$, $G\mathrm{-coN_j} \subseteq \mathrm{G-co}(\bigcup_{\mathrm{i=1}}^{\mathrm{k}} \mathrm{N_i})$, one has further $N_0 \subseteq G\mathrm{-co}(\bigcup_{\mathrm{i=1}}^{\mathrm{k}} \mathrm{N_i})$. Since each G-convex hull is G-convex, by Definition 1.1(iii), $\Gamma(N_0) \subseteq G$-co$(\bigcup_{\mathrm{i=1}}^{\mathrm{k}} \mathrm{N_i}) \subseteq C$. Hence, again by this definition, C is G-convex.

We propose the following relaxed generalized convexity for multimaps in G-convex spaces.

Definition 1.3 Let B, C be sets, (A,Γ) be a G-convex space and $M \subseteq A$. Let f: $B \times A \times A \longrightarrow 2^C$, g: $B \times A \longrightarrow 2^C$ and T: $A \times A \longrightarrow 2^B$ be multimaps. f is called g-G-quasiconvex relative to T on M if, for any finite subset $N = \{x_1, x_2, ..., x_n\} \in \langle M \rangle$ and for any $x \in G\text{-}\mathtt{co}N$, there is some $i \in \{1, 2, ..., n\}$ such that, $\forall t \in T(x, x_i)$,

$$f(t, x, x_i) \subseteq g(t, x). \qquad (1)$$

A definition of the corresponding g-G-quasiconvexlikeness is obtained by replacing (1) by $f(t, x, x_i) \cap g(t, x) \neq \emptyset$.

If A is a convex subset of a vector space and G-co$N$ is the usual convex hull of N, then the g-G-quasiconvexity and g-G-quasiconvexlikeness collapse respectively to the g-quasiconvexity and g-quasiconvexlikeness defined in [3]. If, more particular, $T(x, y) = \{x\}$ and $g(t, x) = K(x)$ with $K(x)$ being a convex cone, these notions become the strong type 1 and type 2 (respectively) K-diagonal quasiconvexities proposed in [8].

The following fixed-point theorem is the main tool for proving our results.

Theorem 1.3 (see [9, Theorem 1]) *Let (A,Γ) be a G-convex space and Q: $A \longrightarrow 2^A$ be a multimap satisfying the conditions*

(i) $A = \bigcup_{y \in A} \mathrm{int Q^{-1}(y)}$;

(ii) *there is a nonempty compact subset D of A such that, for each $N \in$*

$\langle A \rangle$, *there exists a compact G-convex subset* L_N *of* A, *containing* N *so that*

$$L_N \cap (A \setminus \bigcup_{y \in L_N} \mathrm{int Q^{-1}(y)}) \subseteq \mathrm{D}.$$

Then, G-co$Q(.)$ *has a fixed point in* A.

2. MAIN RESULTS

For problems (IP_1) and (IP_2) we set

$$E = \{x \in A : x \in S_1(x)\}.$$

Theorem 2.1 *Assume for problem* (IP_1) *that there are* h: $B \times A \times A \longrightarrow 2^Y$ *and* $k : B \times A \longrightarrow 2^Y$ *such that the following conditions hold*

(i) E *is nonempty and closed,* $S_2(x) \neq \emptyset$ *and* $G-\mathrm{coS_2(x)} \subseteq \mathrm{S_1(x)}$ *for each* $x \in A \setminus E$;

(ii) *for* $(x, y) \in E \times S_2(x)$, *if* $h(t, x, y) \subseteq k(t, x)$ $\forall t \in T(x, y)$ *then* $f(t, x, y) \subseteq g(t, x)$, $\forall t \in T(x, y)$;

(iii) *for each* $x \in E$, h *is* k*-G-quasiconvex relative to* T *on* $S_2(x)$;

(iv) S_2^{-1} *and* H^{-1} *are transfer open, where* $H : A \longrightarrow 2^A$ *is defined by*

$$H(x) = \{y \in S_2(x) : \exists t \in T(x, y), h(t, x, y) \not\subseteq k(t, x)\};$$

(v) *there is a nonempty compact subset* D *of* A *such that, for each* $N \in \langle A \rangle$, *there exists a compact G-convex subset* L_N *of* A *containing* N *so that,* $\forall x \in L_N \setminus D$, $\exists y \in L_N$: $x \in \mathrm{intS_2^{-1}(y)}$, *and that,* $\forall x \in S_1(x) \cap (L_N \setminus D)$, $\exists y \in L_N$: $x \in \mathrm{intH^{-1}(y)}$.

Then (IP_1) *is solvable.*

Proof. Since $E \neq \emptyset$, reasoning ab absurdo, suppose $\forall z \in E$, $\exists y \in S_2(x), \exists t \in$

$T(x,y)$,

$$f(t,x,y) \not\subseteq g(t,x). \tag{2}$$

Let Φ, Ψ, $P : A \longrightarrow 2^A$ be defined by

$$\Phi(x) = \begin{cases} H(x) & \text{if } x \in E, \\ S_2(x) & \text{if } x \in A\backslash E, \end{cases}$$

$$\Psi(x) = \{y \in S_2(x) : \exists t \in T(x,y), f(t,x,y) \not\subseteq g(t,x)\},$$

$$P(x) = \{y \in A : \exists t \in T(x,y), h(t,x,y) \not\subseteq k(t,x)\}. \tag{3}$$

By (2) and assumption (ii), $\emptyset \neq \Psi(x) \subseteq H(x), \forall x \in E$. Hence, by (i), $\Phi(x) \neq \emptyset, \forall x \in A$. For any $y \in A$, one has

$$\Phi^{-1}(y) = [E \cap H^{-1}(y)] \cup [(A \setminus E) \cap S_2^{-1}(y)]$$

$$= H^{-1}(y) \cup [(A \setminus E) \cap S_2^{-1}(y)].$$

So there are two possibilities for $x \in \Phi^{-1}(y)$. If $x \in H^{-1}(y)$, then by (iv), there is $y' \in A$ such that

$$x \in \mathrm{int H^{-1}(y')} \subseteq \mathrm{int}\Phi^{-1}(\mathrm{y'}).$$

If $x \in (A\backslash E) \cap S_2^{-1}(y)$, (iv) implies also the existence of $y' \in A$ such that

$$x \in (A\backslash E) \cap \mathrm{int S_2^{-1}(y')} \subseteq \mathrm{int[((A\backslash E) \cap S_2^{-1}(y')) \cup H^{-1}(y')]} = \mathrm{int}\Phi^{-1}(\mathrm{y'}).$$

Consequently, Φ^{-1} is transfer open and hence $A = \bigcup_{y \in A} \mathrm{int}\Phi^{-1}(\mathrm{y})$ by Lemma 2.1.

We claim that assumption (ii) of Theorem 1.3 is satisfied for $Q = \Phi$ with D and L_N obtained from assumption (v) of this theorem. let $x \in L_N$ and, $\forall y \in L_N$,

$$x \notin \mathrm{int}\Phi^{-1}(\mathrm{y}). \tag{4}$$

Suppose to the contrary that $x \notin D$. If $x \in A\backslash E$, assumption (v) yields $y \in L_N$ with $x \in \mathrm{intS}_2^{-1}(\mathrm{y}) \cap (\mathrm{A}\backslash\mathrm{E}) \subseteq \mathrm{int}\Phi^{-1}(\mathrm{y})$ as $A\backslash E$ is open. This contradicts (4). If $x \in E$, then $x \in S_1(x) \cap (L_N \backslash D)$ and (v) gives $y \in L_N$ with $x \in \mathrm{intH}^{-1}(\mathrm{y}) \subseteq \mathrm{int}\Phi^{-1}(\mathrm{y})$, a contradiction again. Thus, both assumptions of Theorem 1.3 are fulfilled.

Finally, suppose there exists $x_0 \in A$ such that $x_0 \in G\text{−co}\Phi(\mathrm{x}_0)$. If $x_0 \in A\backslash E, x_0 \in G\text{−coS}_2(\mathrm{x}_0) \subseteq \mathrm{S}_1(\mathrm{x}_0)$, i.e, $x_0 \in E$, which is impossible. If $x_0 \in E$, then $x_0 \in G\text{−coH}(\mathrm{x}_0)$. By Lemma 2.2, $N \in \langle H(x_0)\rangle \subseteq \langle S_2(x_0)\rangle$ exists such that $x_0 \in G\text{−coN}$. According to assumption (iii), one has $x_i \in N$ such that, $\forall t \in T(x_0, x_i), h(t, x_0, x_i) \subseteq k(t, x_0)$. This contradiction with the fact that $x_i \in H(x_0)$ completes the proof.

Now we go further into details of the assumptions of Theorem 2.1 to see that although they look seemingly complicated, they are in fact relaxed and satisfied in many situations which are often met. We begin with assumption (iii).

Proposition 2.2 *Let* A, B, Y *and* T *be as in problem* (IP_1). *Let* h: $B \times A \times A \longrightarrow 2^Y$ *and* $k : B \times A \longrightarrow 2^Y$. *Assume that for each* $x \in A$, $h(t, x, x) \subseteq k(t, x)$ *for all* $t \in T(x, x)$ *and the set*

$$\mathcal{U}_x = \{y \in A : \exists t \in T(x, y), h(t, x, y) \not\subseteq k(t, x)\}$$

is G-convex. Then h is k-G-quasiconvex relative to T on A.

Proof. Suppose to the contrary the existence of $N \in \langle A\rangle$ and $x \in G\text{−coN}$ such that, $\forall x_i \in N$, $\exists t \in T(x, x_i), h(t, x, x_i) \not\subseteq k(t, x)$. Then $N \subseteq \mathcal{U}_x$ and, as $\mathcal{U}_x$ is G-convex, $x \in \mathcal{U}_x$, i.e. $h(t, x, x_i) \nsubseteq k(t, x)$ for some $t \in T(x, x)$, a contradiction.

Passing to the transfer openness assumption we have

Proposition 2.3 *Let A, T be as in problem (IP_1) and h, k, H be defined as in Theorem* 2.1. *For $y \in A$, if $S_2^{-1}(y)$ is open and the set*

$$\mathcal{V}_y = \{x \in A \colon \forall t \in T(x, y), h(t, x, t) \subseteq k(t, x)\}$$

is closed, then $H^{-1}(y)$ is open. Hence, if this is satisfied for all $y \in A$, H^{-1} is transfer open.

Proof. For $y \in A$ we have $H^{-1}(y) = S_2^{-1}(y) \cap P^{-1}(y)$, where P is defined by (3). We also have

$$\begin{aligned} P^{-1}(y) &= \{x \in A \colon \exists t \in T(x, y), h(t, x, y) \nsubseteq k(t, x)\} \\ &= A \backslash \mathcal{V}_y, \end{aligned}$$

which is open. By the assumption, H^{-1} is open.

To ensure the closedness of $\mathcal{V}_y$ we have

Proposition 2.4 *Let A, T, h, k, and $\mathcal{V}_y$ be defined as in Proposition* 2.3. *Then $\mathcal{V}_y$ is closed if one of the following conditions holds*

(i) *$T(\cdot, y)$ and $h(\cdot, \cdot, y)$ are lsc and k has a closed graph*;

(ii) *Y is a locally convex space*; *k has the form $k(t, x) = k_1(t, x) + K$ with $K \subseteq Y$ being a closed convex cone* ; *$T(\cdot, y)$ is lsc, $h(\cdot, \cdot, y)$ is K-lsc and k_1 is K-usc and compact-valued.*

Proof. (i) Let $x_\gamma \in \mathcal{V}_y, x_\gamma \longrightarrow x_0$. Then by the assumption, $\forall t_0 \in T(x_0, y)$,

$\forall v_0 \in h(t_0, x_0, y), \exists t_\gamma \in T(x_\gamma, y)$: $t_\gamma \longrightarrow t_0, \exists v_\gamma \in h(t_\gamma, x_\gamma, y) \subseteq k(t_\gamma, x_\gamma)$: $v_\gamma \longrightarrow v_0$. Since the graph of k is closed, $v_0 \in k(t_0, x_0)$. Hence, $\forall t_0 \in T(x_0, y)$, $h(t_0, x_0, y) \subseteq k(t_0, x_0)$, i.e. $x_0 \in \mathcal{V}_y$.

(ii) Suppose the existence of $x_\gamma \in \mathcal{V}_y, x_\gamma \longrightarrow x_0$ but $x_0 \notin \mathcal{V}_y$, i.e., $\exists t_0 \in T(x_0, y), h(t_0, x_0, y) \not\subseteq k_1(t_0, x_0) + K$. Then there is $h_0 \in h(t_0, x_0, y)$ such that $0 \notin k_1(t_0, x_0) + K - h_0 =: M$. Since $k_1(t_0, x_0)$ is compact, $Y \backslash M$ is open. Consequently there exists a neighborhood U of 0 such that $U \subseteq Y \backslash M$. As Y is a locally convex space we can assume that U is convex and U = -U. Then

$$(-\tfrac{1}{2}U - \tfrac{1}{2}U) \cap (k_1(t_0, x_0) + K - h_0) = \emptyset. \tag{5}$$

By the lower semicontinuity of $T(\cdot, y)$ and K-lower semicontinuity of $h(\cdot, \cdot, y)$, there is $t_\gamma \in T(x_\gamma, y) : t_\gamma \longrightarrow t_0$ such that, for all γ,

$$h(t_0, x_0, y) \subseteq h(t_\gamma, x_\gamma, y) + \tfrac{1}{2}U + K. \tag{6}$$

As $x_\gamma \in \mathcal{V}_y, h(t_\gamma, x_\gamma, y) \subseteq k_1(t_\gamma, x_\gamma) + K$. The K-upper semicontinuity of k_1 in turn implies that, for all γ,

$$k_1(t_\gamma, x_\gamma) \subseteq k_1(t_0, x_0) + \tfrac{1}{2}U + K. \tag{7}$$

(6) and (7) imply that, $\forall h \in h(t_0, x_0, y)$,

$$(-\tfrac{1}{2}U - \tfrac{1}{2}U) \cap (k_1(t_0, x_0) + K - h) \neq \emptyset,$$

contradicting (5).

Theorem 2.1 together with Theorem 2.5 below are easily modified to become corresponding results for the other variants of problems (IP_1), (IP_2) mentioned in Section 1.

Observe that E is closed if S_1 has closed graph (but not vice versa) and that assumption (v) of Theorem 3.1 becomes an usual coercivity assumption if A is a convex subset of a real topological vector space. Therefore, taking into account Propositions 2.2 and 2.3 one sees that Theorem 2.1 includes Theorems 3.1 and 3.3 of [3] and Theorem 2.1 of [4].

Passing to problem (IP_2) we have the following result.

Theorem 2.5 *Impose the assumptions of Theorem* 2.1 *with the following replacements*: $h(t,x,y) \subseteq k(t,x)$ *replaced by* $h(t,x,y) \cap k(t,x) \neq \emptyset$; *the generalized quasiconvexity is replaced by the generalized quasiconvexlikeness and* H *is replaced by* H_1 *defined as*

$$H_1(x) = \{y \in S_2(x) : \exists t \in T(x,y), h(t,x,y) \cap k(t,x) = \emptyset\}.$$

Then problem (IP_2) *has solutions.*

By the similarity we omit the proofs of this theorem and of the conditions below for its assumptions to hold, corresponding to Propositions 2.2 - 2.4. Notice that $\mathcal{V}_y$ is now replaced by $\mathcal{V}_y^1 = \{x \in A : \forall t \in T(x,y), h(t,x,y) \cap k(t,x) \neq \emptyset\}$. Corresponding to Proposition 2.4, conditions for the closedness of $\mathcal{V}_y^1$ are given in the following.

Proposition 2.6 *Let* A, T, h, k *be as in Proposition* 2.4 *and* $\mathcal{V}_y^1$ *be as above. Then each of the following conditions is sufficient for* $\mathcal{V}_y^1$ *to be closed.*

(i) $T(\cdot,y)$ *and* $h(\cdot,\cdot,y)$ *are usc and compact-valued and* k *has a closed graph*;

(ii) Y *is a locally convex space*; k *has the form* $k(t,x) = k_1(t,x) + K$ *with*

$K \subseteq Y$ being a closed convex cone; $T(\cdot, y)$ is usc and compact-valued, $h(\cdot, \cdot, y)$ and k_1 are K - usc and k_1 has compact-values.

With these propositions, it is easy to see that Theorem 2.5 improves Theorem 3.2 and 3.4 of [3] and Theorem 2.1 of [4] when it is applied to the particular cases considered in [3, 4]. The following example shows that Theorems 2.1 and 2.5 contain properly the mentioned recent results.

Example 2.1 Let A = [-1, 1], $Y = Z = R_+$, $T(x, y) \equiv$ R, $g(t, x) \equiv (-\infty, 0)$,

$$S_1(x) = \begin{cases} [-1, x] & \text{if } \ -1 \leq \mathrm{x} \leq -0.5, \\ (0, 1) & \text{if } -0.5 < \mathrm{x} < 0, \\ (0, x) & \text{if } \ 0 \leq \mathrm{x} \leq 1, \end{cases}$$

$$S_2(x) = \begin{cases} [-1, x) & \text{if } \ -1 \leq \mathrm{x} \leq -0.5, \\ (0, 1) & \text{if } -0.5 < \mathrm{x} < 0, \\ (0, x) & \text{if } \ 0 \leq \mathrm{x} \leq 1, \end{cases}$$

$$f(t, x, y) = \begin{cases} [0, 1] & \text{if } \ 0 \leq \mathrm{y} \leq 1, \\ \{-xy\} & \text{if } -1 \leq \mathrm{y} < 0. \end{cases}$$

Then for y = -1, $\mathcal{V}_y = \mathcal{V}_y^1$ = [-1, 0) are not closed. Taking arbitrarily $N = \{x_1, x_2, ..., x_n\} \subseteq (0, 1)$, for every $x \in$ coN and $i \in \{1, 2, ..., n\}$ one sees that $\forall t \in T(x, x_i), f(t, x, x_i) \subseteq Y \backslash g(t, x)$, i.e. f is neither g-quasiconvex relative to T on A nor g-quasiconvexlike relative to T on A. Hence, Theorems 3.1 - 3.4 of [3] cannot be applied. Moreover, for any $x \in [0, 1]$, any $t \in T(x, x), f(t, x, x) \subseteq Y \backslash g(t, x)$. So Theorem 2.1 of [4] cannot be employed either.

Now we check the assumptions of Theorems 2.1 and 2.5 with $h = f$ and $k = g$. (v) and (i) are satisfied since A is compact and E = [-1, -0.5]. For (iii), taking any $x \in E, N \in \langle S_2(x) \rangle = \langle [-1, x) \rangle$, $x_0 \in \mathrm{coN}, \mathrm{x_i} \in \mathrm{N}$ and $t \in T(x_0, x_i)$ one has $f(t, x_0, x_i) \subseteq g(t, x_0)$. To verify assumption (iv) one easily computes the following preimages:

$$S_2^{-1}(y) = \begin{cases} (y, -0.5) & \text{if} \quad -1 \leq \mathrm{y} < -0.5, \\ \emptyset & \text{if} \quad -0.5 \leq \mathrm{y} \leq 0, \\ (-0.5, 0) \cup (y, 1] & \text{if} \quad \quad 0 < \mathrm{y} \leq 1, \end{cases}$$

$$H^{-1}(y) = S_2^{-1}(y) \cap \{x \in A : \exists t \in T(x, y), f(t, x, y) \not\subseteq g(t, x)\}$$

$$= \begin{cases} S_2^{-1}(y) \cap [-1, 1] & \text{if} \quad 0 \leq \mathrm{y} \leq 1, \\ S_2^{-1}(y) \cap [0, 1] & \text{if} -1 \leq \mathrm{y} < 0, \end{cases}$$

$$= \begin{cases} \emptyset & \text{if} \ -1 \leq \mathrm{y} \leq 0, \\ (y, 1] & \text{if} \quad 0 < \mathrm{y} \leq 1, \end{cases}$$

to see that these sets are open in A for all $y \in A$. Furthermore it is easily seen that $H_1^{-1}(y) = H^{-1}(y)$ for all $y \in A$. Thus Theorems 2.1 and 2.5 say that both the problems (IP_1) and (IP_2) have solutions.

3. PARTICULAR CASES

As mentioned in Section 1, our problems (IP_1) and (IP_2) (and their modified weaker models) include many quasivariational inclusion problems, quasiequilibrium problems, etc, considered recently in the literature, since both the formulation and the involved spaces are general. Therefore, we can derive from Theorems 2.1 and 2.5 consequences for special cases to include many recent results. In Section 2 we mentioned several theorems of [3, 4] as examples (these theorems were shown in [3, 4] to contain many other results, see also [5, 11]).

In this section we derive several theorems for cases of G-convex spaces and convex spaces of [8, 10] also as examples. Let (A, Γ) be a G-convex space, Y be a Hausdorff topological vector space, F: $A \times A \longrightarrow 2^Y$ and C: $A \longrightarrow 2^Y$. Consider the following problems

(I) Find $\bar{x} \in A$ such that, $\forall y \in A$,

$$F(\bar{x}, y) \subseteq C(\bar{x});$$

(II) Find $\bar{x} \in A$ such that, $\forall y \in A$,

$$F(\bar{x}, y) \cap C(\bar{x}) \neq \emptyset;$$

(III) Find $\bar{x} \in A$ such that, $\forall y \in A$,

$$F(\bar{x}, y) \nsubseteq -\mathrm{intC(\bar{x})};$$

(IV) Find $\bar{x} \in A$ such that, $\forall y \in A$,

$$F(\bar{x}, y) \cap (-\mathrm{intC(\bar{x})}) = \emptyset.$$

In [8], problems (I) - (IV) were studied for the case where A is convex space. The following problem

(II') Find $\bar{x} \in A$ such that, $\forall y \in A$,

$$F(\bar{x}, y) \nsubseteq \mathcal{C}(\bar{x}),$$

where $\mathcal{C}$: $A \longrightarrow 2^Y$, was investigated in [10] for the case where (A, Γ) is a G-convex space. By setting $C(x) = Y \backslash \mathcal{C}(x)$, problem (II') becomes problem (II).

Corollary 3.1 *Assume the existence of* H: $A \times A \longrightarrow 2^Y$ *and* K: $A \longrightarrow 2^Y$ *such that*

(i) *for* $x, y \in A, H(x,y) \subseteq K(x)$ *implies* $F(x,y) \subseteq C(x)$;

(ii) H *is* K*-*G*-quasiconvex relative to* T, *where* $T(x) = \{x\}$; (iii) $\mathcal{H}^{-1}$ *is transfer open, where* $\mathcal{H}$: $A \longrightarrow 2^A$ *is defined by*

$$\mathcal{H}(x) = \{y \in A\colon\ H(x,y) \not\subseteq K(x)\};$$

(iv) *there is a nonempty compact subset* D *of* A *such that for each* $N \in \langle A \rangle$, *there exists a compact* G*-convex subset* L_N *of* A, *containing* N *so that,* $\forall x \in L_N \backslash D$, $\exists y \in L_N$, $x \in \mathrm{int}\mathcal{H}^{-1}(\mathrm{y})$.

Then Problem (I) *has solutions.*

Corollary 3.1 is the special case of Theorem 2.1 with $A \equiv B$, $f(t,x,y) = F(x,y)$, $g(t,x) = C(x)$, $h(t,x,y) = H(x,y)$ and $k(t,x) = K(x)$. This corollary contains properly Theorem 4.2 and Corollary 4.2 of [8], where A is a convex space and the assumptions are stronger.

Corollary 3.2 *Assume for Problem* (II) *the conditions of Corollary* 3.1 *with the following modifications*: *"quasiconvex" in* (ii) *is replaced by "quasiconvex-like"*; *"*$H(x,y) \not\subseteq K(x)$*" in* (iii) *is replaced by "*$H(x,y) \cap K(x) = \emptyset$*" and* (i) *is replaced by "for* x, $y \in A$, $H(x,y) \cap K(x) \neq \emptyset$ *implies* $F(x,y) \cap C(x) \neq \emptyset$*".* *Then Problem* (II) *is solvable.*

This corollary is derived from Theorem 2.5 with the same setting as for Corollary 3.1. It includes Theorem 4.6 of [8] and Theorems 3.3 - 3.4 of [10] (the convexity assumption in (iii) here is more relaxed than in [10]).

Corollary 3.3 *Assume for Problem* (III) *the conditions*(ii), (iii) *and* (iv) *and replace* (i) *by "for* x, $y \in A$, $H(x,y) \cap K(x) \neq \emptyset$ *implies* $F(x,y) \not\subseteq -\text{intC(x)}$". *Then Problem* (III) *has a solution.*

To prove this corollary simply set B, f, h, k as for Corollary 3.1 and $g(x,y) = Y \backslash \texttt{-int} C(x)$ in Theorem 2.5. This corollary contains Theorem 4.8 of [8].

Corollary 3.4 *Impose for Problem* (IV) *assumptions*(ii), (iii) *and* (iv) *as in Corollary* 3.1 *and replace* (i) *by "for* x, $y \in A$, $H(x,y) \subseteq K(x)$ *implies* $F(x,y) \cap (-\text{intC(x)}) = \emptyset$ ". *Then Problem* (IV) *is solvable.*

To prove this corollary set B, f, h, k and g as for Corollary 3.3 into Theorem 2.1. This corollary includes Theorem 4.10 of [8].

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