SUBDIFFERENTIALS OF VALUE FUNCTIONS AND OPTIMALITY CONDITIONS FOR SOME CLASSES OF DC AND BILEVEL INFINITE AND SEMI-INFINITE PROGRAMS

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Abstract The paper concerns the study of new classes of parametric optimization problems of the so-called *infinite programming* that are generally defined on infinite-dimensional spaces of decision variables and contain, among other constraints, *infinitely many* of inequality constraints. These problems reduce to *semi-infinite programs* in the case of finite-dimensional spaces of decision variables. We focus on *DC* infinite programs with objectives given as the *difference of convex* functions subject to convex inequality constraints. The main results establish efficient upper estimates of certain subdifferentials of (intrinsically nonsmooth) *value functions* in DC infinite programs based on advanced tools of variational analysis and generalized differentiation. The value/marginal functions and their subdifferential estimates play a crucial role in many aspects of parametric optimization including *well-posedness* and *sensitivity*. In this paper we apply the obtained subdifferential estimates to establishing verifiable conditions for the local *Lipschitz continuity* of the value functions and deriving *necessary optimality conditions* in parametric DC infinite programs and their remarkable specifications. Finally, we employ the value function approach and the established subdifferential estimates to the study of *bilevel* finite and infinite programs with convex data on both lower and upper level of hierarchical optimization. The results obtained in the paper are new not only for the classes of infinite programs under consideration but also for their semi-infinite counterparts.

Keywords Variational analysis and parametric optimization · Well-posedness and sensitivity · Marginal and value functions · Generalized differentiation · Optimality conditions · Semi-infinite and infinite programming · Convex inequality constraints · Bilevel programming

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1 Introduction

This paper is devoted to the study of a broad class of *parametric constrained optimization* problems in Banach spaces with objectives given as the *difference of two convex functions* and constraints described by an *arbitrary* (possibly *infinite*) number of *convex* inequalities. We refer to such problems as to parametric *DC infinite programs*, where the abbreviation "DC" signifies the *difference of convex* functions, while the name "infinite" in this framework comes from the comparison with the class of *semi-infinite* programs that involve the same type of "infinite" inequality constraints but in finite-dimensional spaces; see, e.g., [10].

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Observe that the "infinite" terminology for constrained problems of this type has been recently introduced in [6] for the case of nonparametric problems with convex objectives.

Our approach to the study of infinite DC parametric problems is based on considering certain generalized differential properties of *marginal/value functions*, which have been recognized among the most significant objects of variational analysis and parametric optimization especially important for *well-posedness, sensitivity*, and *stability* issues in optimization-related problems, deriving *optimality conditions* in various problems of optimization and equilibria, control theory, viscosity solutions of partial differential equations, etc.; see, e.g., [13, 14, 20] and the references therein.

We mainly focus in this paper on a special class of marginal functions defined as *value functions* for *DC problems* of parametric optimization written in the form

$$\mu(x) := \inf\big\{\varphi(x,y) - \psi(x,y)\big|\; y \in F(x) \cap G(x)\big\} \tag{1.1}$$

with the moving/parameterized *geometric constraints* of the type

$$F(x) := \big\{y \in Y\big|\; (x,y) \in \Omega\big\} \tag{1.2}$$

and the moving *infinite inequality constraints* described by

$$G(x) := \big\{y \in Y\big|\; \varphi_t(x,y) \le 0,\; t \in T\big\}, \tag{1.3}$$

where T is an arbitrary (possibly infinite) index set. As usual, suppose by convention that $\inf \emptyset = \infty$ in (1.1) and in what follows.

Unless otherwise stated, we impose our *standing assumptions*: all the spaces under consideration are *Banach*; the functions φ, ψ, and φ_t in (1.1) and (1.3) defined on $X \times Y$ with their values in the *extended real line* $\overline{I\!R} := I\!R \cup \{\infty\}$ are *proper*, *lower semicontinuous* (l.s.c.), and *convex*; the set $\Omega \subset X \times Y$ in (1.2) is *closed* and *convex*. We use the convention that $\infty - \infty = \infty$ in (1.1), since we orient towards minimization.

It has been well recognized that marginal/value functions of type (1.1) are *intrinsically nonsmooth*, even in the case of simple and smooth initial data. Our primary goal in this paper is to investigate *generalized differential properties* of the value function $\mu(x)$ defined in (1.1)–(1.3) and utilize them in deriving verifiable *Lipschitzian stability* and *necessary optimality conditions* for parametric DC infinite programs and their remarkable specifications. Furthermore, we employ the obtained results for the value functions in the study of a new class of hierarchical optimization problems called *bilevel infinite programs*, which are significant for optimization theory and applications.

Since the value function $\mu(x)$ is generally *nonconvex*, despite the convexity of the initial data in (1.1)–(1.3), we need to use for its study appropriate generalized differential constructions for nonconvex functions. In this paper we focus on the so-called *Fréchet subdifferential* and the two subdifferential constructions by Mordukhovich: the *basic/limiting subdifferential* and the *singular subdifferential* introduced for arbitrary extended-real-valued functions; see [13] with the references and commentaries therein. These subdifferential constructions have been recently used in [13, 14, 15, 16, 17] for the study and applications of value functions in various classes of nonconvex optimization problems, mainly in the framework of

Asplund spaces. We are not familiar with any results in the literature for the classes of optimization problems considered in this paper, where the specific structures of the problems under consideration allow us to derive efficient results on generalized differential properties of the value function given in (1.1)–(1.3) and then apply them to establishing stability and necessary optimality conditions for such problems. The results obtained in this paper seem to be new not only for *infinite programs* treated in general *Banach space* settings, but also in finite-dimensional spaces, i.e., for *semi-infinite programming*.

The rest of the paper is organized as follows. In Section 2 we recall and briefly discuss *major constructions* and *preliminaries* broadly used in the sequel. Section 3 is devoted to *necessary optimality conditions* for *nonparametric DC infinite programs* in Banach spaces, which are certainly of their own interest while playing a significant role in deriving the main results of the next sections. Sections 4 and 5 contain the central results of the paper that provide *upper estimates* first for the *Fréchet subdifferential* and then for the *basic and singular subdifferentials* of the *value function* (1.1) in the general *parametric DC* framework with the *infinite convex constraints* under consideration. These results are specified for the class of *convex infinite programs*, which allows us to establish more precise subdifferential formulas in comparison with the general DC case. As consequences of the upper estimates obtained for the basic and singular subdifferentials of the value functions and certain fundamental results of variational analysis, we derive verifiable conditions of the local *Lipschitz continuity* of the value functions and new *necessary optimality conditions* for these classes of parametric infinite and semi-infinite programs.

The final Section 6 is devoted to applications of the results obtained in the preceding sections to a major class of *hierarchical optimization* problems known as *bilevel programming*, where the set of feasible solutions to the upper-level problem is built upon optimal solutions to the lower-level problem of parametric optimization. We assume the *convexity* of the initial data in both lower-level and upper-level problems, but—probably for the first time in the literature—consider bilevel programs with *infinitely many inequality constraints* on the lower-level of hierarchical optimization. Based on the *value function approach* to bilevel programming and on the results obtained in the preceding sections, we derive verifiable *necessary optimality conditions* for the bilevel programs under consideration, which are new not only for problems with infinite constraints but also for conventional bilevel programs with *finitely many* constraints in both finite and infinite dimensions.

Throughout the paper we use the standard *notation* of variational analysis; see, e.g., [13, 20]. Let us mention some of them often employed in what follows. For a Banach space X, we denote its *norm* by $\|\cdot\|$ and consider the topologically *dual* space X^* equipped with the *weak* topology* w^*, where $\langle\cdot,\cdot\rangle$ stands for the canonical pairing between X and X^*. The *weak* closure* of a set in the dual space (i.e., its closure in the weak* topology) is denoted by cl^*. The symbols $I\!B$ and $I\!B^*$ stand, respectively, for the *closed unit balls* in the space in question and its topological dual.

Given a set $\Omega\subset X$, the notation $\mathrm{bd}\,\Omega$ and $\mathrm{co}\,\Omega$ signify the *boundary* and *convex hull* of Ω, respectively, while $\mathrm{cone}\,\Omega$ stands for the *convex conic hull* of Ω, i.e., for the convex cone generated by $\Omega\cup\{0\}$. We use the symbol $F\colon X\rightrightarrows Y$ for a *set-valued* mapping defined on X with its values $F(x)\subset Y$ (in contrast to the standard notation $f\colon X\to Y$ for single-valued mappings) and denote the *domain* and *graph* of F by, respectively,

$$\mathrm{dom}\,F := \big\{x \in X \big|\; F(x) \neq \emptyset\big\} \;\text{ and }\; \mathrm{gph}\,F := \big\{(x,y) \in X \times Y \big|\; y \in F(x)\big\}.$$

Given a set-valued mapping $F\colon X \rightrightarrows X^*$ between X and X^*, recall that

$$\mathop{\mathrm{Lim\,sup}}_{x \to \bar{x}} F(x) := \Big\{x^* \in X^* \Big|\; \exists\, x_k \to \bar{x},\; \exists\, x_k^* \xrightarrow{w^*} x^* \;\text{ with }\; x_k^* \in F(x_k), \quad k \in I\!N\Big\} \tag{1.4}$$

signifies the *sequential Painlevé-Kuratowski outer/upper limit* of F as $x \to \bar{x}$ with respect to the norm topology of X and the weak* topology of X^*, where $I\!N := \{1, 2, \ldots\}$. Further, *sequential Painlevé-Kuratowski inner/lower limit* of F as $x \to \bar{x}$ is defined by

$$\mathop{\mathrm{Lim\,inf}}_{x \to \bar{x}} F(x) := \Big\{x^* \in X^* \Big|\; \forall\, x_k \to \bar{x}\; \exists\, x_k^* \xrightarrow{w^*} x^* \;\text{ with }\; x_k^* \in F(x_k), \quad k \in I\!N\Big\}. \tag{1.5}$$

Given an *extended-real-valued function* $\varphi\colon X \to \overline{I\!R}$, the notation

$$\mathrm{dom}\,\varphi := \big\{x \in X \big|\; \varphi(x) < \infty\big\} \;\text{ and }\; \mathrm{epi}\,\varphi := \big\{(x,\nu) \in X \times I\!R \big|\; \nu \geq \varphi(x)\big\}$$

is used, respectively, for the *domain* and the *epigraph* of φ. Depending on the context, the symbols $x \xrightarrow{\Omega} \bar{x}$ and $x \xrightarrow{\varphi} \bar{x}$ mean that $x \to \bar{x}$ with $x \in \Omega$ and $x \to \bar{x}$ with $\varphi(x) \to \varphi(\bar{x})$ for a set $\Omega \subset X$ and an extended-real-valued function $\varphi\colon X \to \overline{I\!R}$, respectively. Some other notation are introduced below when the corresponding notions are defined.

2 Basic Definition and Preliminaries

Let us start with recalling some basic definitions and presenting less standard preliminary facts for *convex* functions that play a fundamental role in this paper. Given $\varphi\colon X \to \overline{I\!R}$, we always assume that it is *proper*, i.e., $\varphi(x) \not\equiv \infty$ on X. The *conjugate function* $\varphi^*\colon X^* \to \overline{I\!R}$ to φ is defined by

$$\varphi^*(x^*) := \sup\big\{\langle x^*, x\rangle - \varphi(x) \big|\; x \in X\big\} = \sup\big\{\langle x^*, x\rangle - \varphi(x) \big|\; x \in \mathrm{dom}\,\varphi\big\}. \tag{2.1}$$

For any $\varepsilon \geq 0$, the *ε-subdifferential* (or *approximate subdifferential* if $\varepsilon > 0$) of a convex function $\varphi\colon X \to \overline{I\!R}$ at $\bar{x} \in \mathrm{dom}\,\varphi$ is

$$\partial_\varepsilon \varphi(\bar{x}) := \big\{x^* \in X^* \big|\; \langle x^*, x - \bar{x}\rangle \leq \varphi(x) - \varphi(\bar{x}) + \varepsilon \;\text{ for all }\; x \in X\big\}, \quad \varepsilon \geq 0. \tag{2.2}$$

If $\varepsilon = 0$ in (2.2), the set $\partial\varphi(\bar{x}) := \partial_0\varphi(\bar{x})$ is the classical *subdifferential of convex analysis*. As usual, the symbols $\partial_x\varphi(\bar{x},\bar{y})$ and $\partial_y\varphi(\bar{x},\bar{y})$ stand for the corresponding *partial subdifferentials* of $\varphi = \varphi(x,y)$ at $(\bar{x},\bar{y})$.

Observe the following useful representation [11] of the *epigraph of the conjugate function* (2.1) to a l.s.c. convex function $\varphi\colon X \to \overline{I\!R}$ via the ε-subdifferentials (2.2) of φ at any point $x \in \mathrm{dom}\,\varphi$ of the domain:

$$\mathrm{epi}\,\varphi^* = \bigcup_{\varepsilon \geq 0} \Big\{\big(x^*, \langle x^*, x\rangle + \varepsilon - \varphi(x)\big) \Big|\; x^* \in \partial_\varepsilon\varphi(x)\Big\}. \tag{2.3}$$

Further, it is well known in convex analysis that the *conjugate epigraphical rule*

$$\operatorname{epi}\big(\varphi_1+\varphi_2\big)^*=\operatorname{cl}^*\big(\operatorname{epi}\varphi_1^*+\operatorname{epi}\varphi_2^*\big) \tag{2.4}$$

is satisfied for any l.s.c. convex functions $\varphi_i\colon X\to\overline{I\!R}$, $i=1,2$, where the *weak** *closure operation* on the right-hand side of (2.4) can be *omitted* provided that one of the functions φ_i is *continuous* at some point $\bar{x}\in\operatorname{dom}\varphi_1\cap\operatorname{dom}\varphi_2$. More general results in this direction implying the fundamental subdifferential sum rule have been recently established in [1]. We summarize them in the following lemma broadly employed in this paper.

Lemma 2.1 (refined epigraphical and subdifferential rules for convex function). *Let $\varphi_i\colon X\to\overline{I\!R}$, $i=1,2$, be l.s.c. and convex, and let $\operatorname{dom}\varphi_1\cap\operatorname{dom}\varphi_2\neq\emptyset$. Then the following conditions are equivalent:*

(i) *The set $\operatorname{epi}\varphi_1^*+\operatorname{epi}\varphi_2^*$ is weak* closed in $X^*\times I\!R$.*

(ii) *The refined conjugate epigraphical rule holds:*

$$\operatorname{epi}\big(\varphi_1+\varphi_2\big)^*=\big(\operatorname{epi}\varphi_1^*+\operatorname{epi}\varphi_2^*\big).$$

Furthermore, we have the subdifferential sum rule

$$\partial(\varphi_1+\varphi_2)(\bar{x})=\partial\varphi_1(\bar{x})+\partial\varphi_2(\bar{x}),\quad \bar{x}\in\operatorname{dom}\varphi_1\cap\operatorname{dom}\varphi_2, \tag{2.5}$$

provided that the afore-mentioned equivalent conditions are satisfied.

Since the above definitions and results are given for any extended-real-valued (l.s.c. and convex) functions, they encompass the case of *sets* by considering the *indicator function* $\delta(x;\Omega)$ of a set $\Omega\subset X$ equal to 0 when $x\in\Omega$ and ∞ otherwise. In this way, the *normal cone* to a convex set Ω at $\bar{x}\in\Omega$ is defined by

$$N(\bar{x};\Omega):=\partial\delta(\bar{x};\Omega)=\big\{x^*\in X^*\big|\ \langle x^*,x-\bar{x}\rangle\le 0\ \text{ for all }\ x\in\Omega\big\}. \tag{2.6}$$

In what follows we also use *projections* of the normal cone (2.6) to convex sets in *product* spaces. Given $\Omega\subset X\times Y$ and $(\bar{x},\bar{y})\in\Omega$, we define the corresponding projections by

$$\begin{aligned} N_X\big((\bar{x},\bar{y});\Omega\big)&:=\big\{x^*\in X^*\big|\ \exists y^*\in Y^*\ \text{ such that }\ (x^*,y^*)\in N\big((\bar{x},\bar{y});\Omega\big)\big\},\\ N_Y\big((\bar{x},\bar{y});\Omega\big)&:=\big\{y^*\in Y^*\big|\ \exists x^*\in Y^*\ \text{ such that }\ (x^*,y^*)\in N\big((\bar{x},\bar{y});\Omega\big)\big\}. \end{aligned} \tag{2.7}$$

Next we drop the convexity assumptions and consider, following [13], certain counterparts of the above subdifferential constructions for *arbitrary proper* extended-real-valued functions on Banach spaces. Given $\varphi\colon X\to\overline{I\!R}$ and $\varepsilon\ge 0$, define the *analytic ε-subdifferential* of φ at $\bar{x}\in\operatorname{dom}\varphi$ by

$$\widehat{\partial}_\varepsilon\varphi(\bar{x}):=\Big\{x^*\in X^*\Big|\ \liminf_{x\to\bar{x}}\frac{\varphi(x)-\varphi(\bar{x})-\langle x^*,x-\bar{x}\rangle}{\|x-\bar{x}\|}\ge-\varepsilon\Big\},\quad \varepsilon\ge 0, \tag{2.8}$$

and let for convenience $\widehat{\partial}_\varepsilon\varphi(\bar{x}):=\emptyset$ of $\bar{x}\notin\operatorname{dom}\varphi$. Note that if φ is *convex*, the analytic ε-subdifferential (2.8) admits the representation

$$\widehat{\partial}_\varepsilon\varphi(\bar{x})=\big\{x^*\in X^*\big|\ \langle x^*,x-\bar{x}\rangle\le\varphi(x)-\varphi(\bar{x})+\varepsilon\|x-\bar{x}\|\ \text{ for all }\ x\in\operatorname{dom}\varphi\big\}, \tag{2.9}$$

which is *different* from the ε-subdifferential of convex analysis (2.2) when $\varepsilon > 0$. If $\varepsilon = 0$, then $\widehat{\partial}\varphi(\bar{x}) := \widehat{\partial}_0\varphi(\bar{x})$ in (2.8) is known as the *Fréchet* (or *regular*, or *viscosity*) *subdifferential* of φ at $\bar{x}$ and reduces in the convex case to the classical subdifferential of convex analysis.

However, it turns out that in the *nonconvex* case neither the Fréchet subdifferential $\widehat{\partial}\varphi(\bar{x})$ nor its ε-enlargements (2.8) satisfy required calculus rules, e.g., the inclusion "$\subset$" in (2.5) needed for optimization theory and applications. Moreover, it often happens that $\widehat{\partial}\varphi(\bar{x}) = \emptyset$ even for nice and simple nonconvex functions as, e.g., for $\varphi(x) = -|x|$ at $\bar{x} = 0$. The picture dramatically changes when we employ the *sequential* regularization of (2.8) defined via the Painlevé-Kuratowski outer limit (1.4) by

$$\partial\varphi(\bar{x}) := \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{\varphi} \bar{x} \\ \varepsilon \downarrow 0}} \widehat{\partial}_\varepsilon \varphi(x) \tag{2.10}$$

and known as the *basic* (or *limiting*, or *Mordukhovich*) *subdifferential* of φ at $\bar{x} \in \operatorname{dom}\varphi$. It reduces to the subdifferential of convex analysis (2.2) as $\varepsilon = 0$ and, in contrast to $\widehat{\partial}\varphi(\bar{x})$ from (2.8), satisfies useful calculus rules in general nonconvex settings.

In particular, *full/comprehensive calculus* holds for (2.10) in the framework of *Asplund spaces*, which are Banach spaces whose separable subspaces have separable duals. This is a broad class of spaces including every Banach space admitting a Fréchet smooth renorm (hence every reflexive space), every space with a separable dual, etc.; see [13, 18] for more details on this remarkable class of spaces. Note that we can equivalently put $\varepsilon = 0$ in (2.10) for l.s.c. functions on Asplund spaces.

It is also worth observing that the basic subdifferential (2.10) is often a *nonconvex set* in X^* (e.g., $\partial\varphi(0) = \{-1, 1\}$ for $\varphi(x) = -|x|$), while vast calculus results and applications of (2.10) and related constructions for sets and set-valued mappings are based on *variational/extremal principles* of variational analysis that replace the classical convex separation in nonconvex settings. We refer the reader to [13, 14, 20, 21], with the extensive commentaries and bibliographies therein, for more details and discussions. Let us emphasize that most of the results obtained in this paper *do not require* the Asplund structure of the spaces in question and hold in *arbitrary Banach spaces*.

An additional subdifferential construction to (2.10) is needed to analyze *non-Lipschitzian* extended-real-valued functions $\varphi\colon X \to \overline{I\!R}$. It is defined by

$$\partial^\infty\varphi(\bar{x}) := \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{\varphi} \bar{x} \\ \lambda,\, \varepsilon \downarrow 0}} \lambda\widehat{\partial}_\varepsilon \varphi(x) \tag{2.11}$$

and is known as the *singular* (or *horizontal*) *subdifferential* of φ at $\bar{x} \in \operatorname{dom}\varphi$. We have $\partial^\infty\varphi(\bar{x}) = \{0\}$ if φ is locally Lipschitzian around $\bar{x}$, while the singular subdifferential (2.11) shares calculus and related properties of the basic subdifferential (2.10) in non-Lipschitzian settings. Given an arbitrary set $\Omega \subset X$ with $\bar{x} \in \Omega$ and applying (2.10) and (2.11) to the indicator function $\varphi(x) = \delta(x; \Omega)$ of Ω, we get

$$N(\bar{x}; \Omega) := \partial\delta(\bar{x}; \Omega) = \partial^\infty\delta(\bar{x}; \Omega),$$

where the latter general *normal cone* reduces to (2.6) if Ω is convex.

Finally in this section, recall an extended notion of inner semicontinuity for a general class of *marginal/value functions* defined by

$$\mu(x) := \inf\{\vartheta(x,y) \big| \; y \in S(x)\big\}, \tag{2.12}$$

where $\vartheta\colon X \times Y \to \overline{I\!R}$ and $S\colon X \rightrightarrows Y$. Denote

$$M(x) := \big\{y \in S(x) \big| \; \mu(x) = \vartheta(x,y)\big\} \tag{2.13}$$

the *argminimum mapping* generated by the marginal function (2.12). Given $\bar{y} \in M(\bar{x})$ and following [15], we say that $M(\cdot)$ in (2.13) is *μ-inner semicontinuous* at $(\bar{x},\bar{y})$ if for every sequence $x_k \stackrel{\mu}{\to} \bar{x}$ as $k \to \infty$ there is a sequence of $y_k \in M(x_k)$, $k \in I\!N$, which contains a subsequence converging to $\bar{y}$. This property is an extension of the more conventional notion of *inner/lower semicontinuity* for general multifunctions (see, e.g., [13, Definition 1.63] and the commentaries therein), where the convergence $x_k \stackrel{\mu}{\to} \bar{x}$ is replaced by $x_k \to \bar{x}$. In this paper we apply the defined μ-inner semicontinuity property to argminimum mappings generated by the marginal/value functions (1.1) for the infinite DC programs under consideration. Observe that the μ-inner semicontinuity assumption on the afore-mentioned argminimum mapping in the results obtained in Sections 5 can be replaced by a more relaxed *μ-inner semicompactness* requirement imposed on this mapping by the expense of weakening the resulting inclusions, which involve then *all* the points from the reference argminimum set; cf. [13, 15, 16] for similar devices in different settings. For brevity, we do not present the results of the latter type in this paper.

3 Optimality Conditions for DC Infinite Programs

In this section we consider a general class of *nonparametric DC infinite programs* with convex constraints of the type:

$$\left\{\begin{array}{l} \text{minimize } \; \vartheta(x) - \theta(x) \; \text{ subject to} \\ \vartheta_t(x) \le 0, \;\; t \in T, \;\; \text{and} \;\; x \in \Theta, \end{array}\right. \tag{3.1}$$

where T is a (possibly infinite) index set, where $\Theta \subset X$ is a closed convex subset of a Banach space X, and where $\vartheta\colon X \to \overline{I\!R}$, $\theta\colon X \to \overline{I\!R}$, and $\vartheta_t\colon X \to \overline{I\!R}$ are proper, l.s.c., convex functions. One can see that (3.1) is a nonparametric version of the infinite DC problem of parametric optimization defined in (1.1)–(1.3), which are of our primary concern in this paper. The results obtained in this section establish *necessary optimality conditions* for the nonparametric DC problem (3.1) and deduce from them some *calculus rules* for the initial data of (3.1) involving *infinite constraints*. These new results are certainly of independent interest in both finite and infinite dimensions, while the main intention of this paper is to apply them to the study of subdifferential properties of the value function in the *parametric* infinite DC problem (1.1)–(1.3); this becomes possible due to the intrinsic *variational structures* of the subdifferentials under consideration.

Denote the set of *feasible solutions* to (3.1) by

$$\Xi := \Theta \cap \big\{x \in X \big| \; \vartheta_t(x) \le 0 \;\text{ for all }\; t \in T\big\}. \tag{3.2}$$

Further, let $\mathbb{R}^T$ be the *product space* of $\lambda = (\lambda_t|\ t \in T)$ with $\lambda_t \in \mathbb{R}$ for all $t \in T$, let $\widetilde{\mathbb{R}}^T$ be collection of $\lambda \in \mathbb{R}^T$ such that $\lambda_t \neq 0$ for *finitely many* $t \in T$, and let $\widetilde{\mathbb{R}}^T_+$ be the *positive cone* in $\widetilde{\mathbb{R}}^T$ defined by

$$\widetilde{\mathbb{R}}^T_+ := \big\{\lambda \in \widetilde{\mathbb{R}}^T \big|\ \lambda_t \geq 0 \ \text{ for all } \ t \in T\big\}. \tag{3.3}$$

Observe that, given $u \in \mathbb{R}^T$ and $\lambda \in \widetilde{\mathbb{R}}^T$ and denoting $\operatorname{supp}\lambda := \{t \in T|\ \lambda_t \neq 0\}$, we have

$$\lambda u := \sum_{t\in T} \lambda_t u_t = \sum_{t\in \operatorname{supp}\lambda} \lambda_t u_t.$$

The following *qualification condition* plays a crucial role in deriving necessary optimality conditions for *DC infinite programs* considered in this section obtained in the so-called *qualified* (*Karush-Kuhn-Tucker*) *form* with a nonzero Lagrange multiplier corresponding to the cost function $\vartheta - \theta$. Furthermore, this qualification condition/requirement endures the validity of new *calculus rules* involving the *infinite* data of (3.1).

Definition 3.1 (closedness qualification condition). *We say that the triple* $(\vartheta, \vartheta_t, \Theta)$ *satisfies the* CLOSEDNESS QUALIFICATION CONDITION*, CQC in brief, if the set*

$$\operatorname{epi}\vartheta^* + \operatorname{cone}\Big\{\bigcup_{t\in T} \operatorname{epi}\vartheta_t^*\Big\} + \operatorname{epi}\delta^*(\cdot;\Theta)$$

is weak closed in the space* $X^* \times \mathbb{R}$.

If the *plus term* ϑ in cost function (3.1) is *continuous* at some point of the feasible set Ξ in (3.2), then the CQC requirement of Definition 3.1 holds provided that the set

$$\operatorname{cone}\Big\{\bigcup_{t\in T} \operatorname{epi}\vartheta_t^*\Big\} + \operatorname{epi}\delta^*(\cdot;\Theta) \ \text{ is weak}^* \text{ closed}$$

in $X^* \times \mathbb{R}$ (see [12]), or if the conical set $\operatorname{cone}(\operatorname{dom}\vartheta - \Xi)$ is a closed subspace of X; see [5, 6] for more details. Note also that the *dual* qualification conditions of the CQC type have been introduced and broadly used in [1, 5, 6, 7, 8, 12] and other publications of these authors for deriving duality results, stability and optimality conditions, and generalized Farkas-like relationships in various constrained problems of convex and DC programming. Furthermore, it has been proved in the afore-mentioned papers that the qualification conditions of the CQC type *strictly improved* more conventional *primal* constraint qualifications of the nonempty interior and relative interior types for problems considered therein.

The next result establishes new necessary optimality conditions for the DC infinite program (3.1) under the CQC requirement introduced in Definition 3.1. In what follows we use the set of *active constraint multipliers* defined by

$$A(\bar{x}) := \big\{\lambda \in \widetilde{\mathbb{R}}^T_+ \big|\ \lambda_t\vartheta_t(\bar{x}) = 0 \ \text{ for all } \ t \in \operatorname{supp}\lambda\big\}. \tag{3.4}$$

Theorem 3.2 (qualified necessary optimality conditions for DC infinite programs). *Let* $\bar{x} \in \Xi \cap \operatorname{dom}\vartheta$ *be a local minimizer to problem* (3.1) *satisfying the CQC requirement. Then we have the inclusion*

$$\partial\theta(\bar{x}) \subset \partial\vartheta(\bar{x}) + \bigcup_{\lambda\in A(\bar{x})} \Big[\sum_{t\in\operatorname{supp}\lambda} \lambda_t \partial\vartheta_t(\bar{x})\Big] + N(\bar{x};\Theta). \tag{3.5}$$

Proof. Taking a local minimizer $\bar{x} \in \Xi \cap \operatorname{dom} \vartheta$ to (3.1), we suppose without loss of generality that $\bar{x} \in \operatorname{dom} \theta$ and that $\partial\theta(\bar{x}) \neq \emptyset$; otherwise (3.5) holds trivially. By (2.2) with $\varepsilon = 0$ there is $x^* \in X^*$ satisfying

$$\theta(x) - \theta(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle \text{ for all } x \in X.$$

This implies that the reference local minimizer $\bar{x}$ to (3.1) is also a local minimizer to the *convex* infinite program:

$$\begin{cases} \text{minimize } \widetilde{\vartheta}(x) := \vartheta(x) - \langle x^*, x - \bar{x} \rangle - \theta(\bar{x}) \\ \text{subject to } \vartheta_t(x) \leq 0, \quad t \in T, \quad \text{and } x \in \Theta. \end{cases} \tag{3.6}$$

Since (3.6) is convex, its local minimizer $\bar{x}$ is a *global solution* to this problem, i.e.,

$$\widetilde{\vartheta}(\bar{x}) \leq \widetilde{\vartheta}(x) \text{ for all } x \in \Xi.$$

By [6, Lemma 4] the latter is *equivalent* to the inclusion

$$\big(0, -\widetilde{\vartheta}(\bar{x})\big) \in \operatorname{cl}^*\Big(\operatorname{epi} \widetilde{\vartheta}^* + \operatorname{cone}\Big[\bigcup_{t \in T} \operatorname{epi} \vartheta_t^*\Big] + \operatorname{epi} \delta^*(\cdot; \Theta)\Big).$$

Observing from the structure of $\widetilde{\vartheta}$ in (3.6) that $\operatorname{epi} \widetilde{\vartheta}^* = (-x^*, \theta(\bar{x}) - \langle x^*, \bar{x} \rangle) + \operatorname{epi} \vartheta^*$, we get therefore the relationship

$$\big(0, -\widetilde{\vartheta}(\bar{x})\big) \in \big(-x^*, \theta(\bar{x}) - \langle x^*, \bar{x} \rangle\big) + \operatorname{cl}^*\Big(\operatorname{epi} \vartheta^* + \operatorname{cone}\Big[\bigcup_{t \in T} \operatorname{epi} \vartheta_t^*\Big] + \operatorname{epi} \delta^*(\cdot; \Theta)\Big). \tag{3.7}$$

Furthermore, the equivalence (i)$\Longleftrightarrow$(ii) in Lemma 2.1 ensures, under the assumed CQC condition, that (3.7) is *equivalent* to

$$\big(x^*, -\widetilde{\vartheta}(\bar{x}) - \theta(\bar{x}) + \langle x^*, \bar{x} \rangle\big) \in \Big(\operatorname{epi} \vartheta^* + \operatorname{cone}\Big[\bigcup_{t \in T} \operatorname{epi} \vartheta_t^*\Big] + \operatorname{epi} \delta^*(\cdot; \Theta)\Big). \tag{3.8}$$

Now applying the subdifferential representation (2.3) to each of the conjugate functions ϑ^*, ϑ_t^* as $t \in T$, and $\delta^*(\cdot; \Theta)$, taking then into account the construction of the convex cone "cone" in (3.8) as well as the structure of the positive cone $\widetilde{I\!R}_+^T$ in (3.3), and noting that $-\widetilde{\theta}(\bar{x}) - \theta(\bar{x}) + \langle x^*, \bar{x} \rangle = \langle x^*, \bar{x} \rangle - \vartheta(\bar{x})$, we find

$$\varepsilon \geq 0, \; u^* \in \partial_\varepsilon \vartheta(\bar{x}), \;\; \lambda \in \widetilde{I\!R}_+^T, \;\; \varepsilon_t \geq 0, \; u_t^* \in \partial_{\varepsilon_t} \vartheta_t(\bar{x}) \;\text{ as }\; t \in T, \;\; \gamma \geq 0, \;\text{ and }\; v^* \in \partial \delta_\gamma(\bar{x}; \Theta)$$

satisfying the following relationships:

$$\begin{cases} x^* = u^* + \displaystyle\sum_{t \in T} \lambda_t u_t^* + v^*, \\ \langle x^*, \bar{x} \rangle - \vartheta(\bar{x}) = \langle u^*, \bar{x} \rangle + \varepsilon - \vartheta(\bar{x}) + \displaystyle\sum_{t \in T} \lambda_t \Big[\langle u_t^*, \bar{x} \rangle + \varepsilon_t - \langle \vartheta_t^*, \bar{x} \rangle\Big] \\ + \langle v^*, \bar{x} \rangle + \gamma - \delta(\bar{x}; \Theta). \end{cases} \tag{3.9}$$

Since $\bar{x} \in \Theta$, the first equality in (3.9) allows us to reduce the second one therein to

$$\varepsilon + \sum_{t \in T} \lambda_t \varepsilon_t - \sum_{t \in T} \lambda_t \vartheta_t(\bar{x}) + \gamma = 0. \tag{3.10}$$

The feasibility of $\bar{x}$ to problem (3.1) and the above choice of $(\varepsilon, \lambda_t, \gamma)$ imply the relationships

$$\varepsilon \geq 0,\ \gamma \geq 0,\ \lambda_t \geq 0, \ \text{ and } \ \lambda_t \vartheta_t(\bar{x}) \leq 0 \ \text{ for all } \ t \in T,$$

and therefore we get from (3.10) that in fact $\varepsilon = 0$, $\gamma = 0$, $\lambda_t \vartheta_t(\bar{x}) = 0$, and $\lambda_t \varepsilon_t = 0$ for all $t \in T$. Furthermore, the latter allows us to conclude that $\varepsilon_t = 0$ for all $t \in \operatorname{supp} \lambda$. Thus

$$u^* \in \partial \vartheta(\bar{x}),\ \ u_t^* \in \partial \vartheta_t(\bar{x}), \ \text{ and } \ v^* \in \partial \delta(\bar{x}; \Theta) = N(\bar{x}; \Theta),$$

and so the first equality in (3.9) can be written as

$$x^* \in \partial \vartheta(\bar{x}) + \sum_{t \in \operatorname{supp} \lambda} \lambda_t \partial \vartheta_t(\bar{x}) + N(\bar{x}; \Theta) \ \text{ with } \ \lambda_t \vartheta_t(\bar{x}) = 0 \ \text{ for } \ t \in \operatorname{supp} \lambda. \tag{3.11}$$

This justifies (3.5) and completes the proof of the theorem. △

Let us next present two useful consequences of Theorem 3.2 that provide new *calculus rules* in the framework of (3.1) involving *infinite constraints* in both finite and infinite dimensions. As above, we use the set of active constraint multipliers $A(\bar{x})$ defined in (3.4).

Corollary 3.3 (subdifferential sum rule involving convex infinite constraints). *Let $\bar{x} \in \Xi$ be any feasible solution to problem* (3.1) *with $\theta(x) \equiv 0$ and $\vartheta(\bar{x}) < \infty$, and let $(\vartheta, \vartheta_t, \Theta)$ satisfy all the assumptions of Theorem* 3.2 *including the CQC condition. Then*

$$\partial\big(\vartheta + \delta(\cdot; \Xi)\big)(\bar{x}) \subset \partial \vartheta(\bar{x}) + \bigcup_{\lambda \in A(\bar{x})} \Big[\sum_{t \in \operatorname{supp} \lambda} \partial \vartheta_t(\bar{x}) \Big] + N(\bar{x}; \Theta). \tag{3.12}$$

Proof. For each $x^* \in \partial(\vartheta + \delta(\cdot; \Xi))(\bar{x})$ with $\bar{x} \in \Xi \cap \operatorname{dom} \vartheta$ we have

$$\vartheta(x) - \vartheta(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle \ \text{ whenever } \ x \in \Xi,$$

which means by the construction of Ξ in (3.2) that $\bar{x}$ is a (global) minimizer to the following *DC infinite program*:

$$\begin{cases} \text{minimize } \ \vartheta(x) - \widetilde{\theta}(x) \ \text{ with } \ \widetilde{\theta}(x) := \langle x^*, x - \bar{x} \rangle + \vartheta(\bar{x}) \\ \text{subject to } \ \vartheta_t(x) \leq 0 \ \text{ for all } \ t \in T, \ \text{ and } \ x \in \Theta. \end{cases} \tag{3.13}$$

Applying Theorem 3.2 to problem (3.13) and taking into account the structure of the linear function $\widetilde{\theta}$ therein, we get from (3.5) that

$$\partial \widetilde{\theta}(\bar{x}) = \big\{x^*\big\} \subset \partial \vartheta(\bar{x}) + \bigcup_{\lambda \in A(\bar{x})} \Big[\sum_{t \in \operatorname{supp} \lambda} \partial \vartheta_t(\bar{x}) \Big] + N(\bar{x}; \Theta),$$

which gives (3.12) and completes the proof of the corollary. △

The next corollary provides a constructive *upper estimate* of the normal cone to the feasible constraint set Ξ from (3.2) in terms of the initial data of (3.2) and the set of active constraint multipliers (3.4).

Corollary 3.4 (upper estimate of the normal cone to convex infinite constraints). *Assume that ϑ_t and Θ satisfy the assumptions of Theorem 3.2 with the condition CQC specified as follows:*

$$\text{the set } \Big\{\text{cone}\Big[\bigcup_{t\in T}\text{epi}\,\vartheta_t^*\Big]+\text{epi}\,\delta^*(\cdot;\Theta)\Big\} \text{ is weak}^* \text{ closed in } X^*\times I\!R.$$

Then for any $\bar{x}\in\Xi$ we have the inclusion

$$N(\bar{x};\Xi)\subset\bigcup_{\lambda\in A(\bar{x})}\Big[\sum_{t\in\operatorname{supp}\lambda}\partial\vartheta_t(\bar{x})\Big]+N(\bar{x};\Theta).$$

Proof. Follows from Corollary 3.3 by letting $\vartheta(x)\equiv 0$ therein. △

The final result of this section concerns establishing an improved version of Theorem 3.2 in the case the *convex infinite program* given by

$$\begin{cases}\text{minimize } \vartheta(x) \text{ subject to}\\ \vartheta_t(x)\le 0,\ \ t\in T, \ \text{ and } \ x\in\Theta,\end{cases}\tag{3.14}$$

which is of course a particular case of the DC infinite program (3.1). The next theorem shows that the specification of condition (3.5) in this case is not only *necessary* but also *sufficient* for optimality in (3.14) under the CQC requirement introduced in Definition 3.1 above. The result obtained is a refinement of the corresponding condition established recently in [6] under a more restrictive constraint qualification.

Theorem 3.5 (necessary and sufficient optimality conditions for convex infinite programs). *Let $\bar{x}\in\Xi$ be a feasible solution to problem* (3.14) *with $\vartheta(\bar{x})<\infty$, and let all the assumptions of Theorem* 3.2 *be satisfied. Then $\bar{x}$ is optimal to* (3.14) *if and only if there is $\lambda\in\widetilde{I\!R}^T_+$ such that the following generalized Karush-Kuhn-Tucker* (*KKT*) *condition holds:*

$$0\in\partial\vartheta(\bar{x})+\bigcup_{\lambda\in A(\bar{x})}\Big[\sum_{t\in\operatorname{supp}\lambda}\partial\vartheta_t(\bar{x})\Big]+N(\bar{x};\Theta),\tag{3.15}$$

where the set of active constraint multipliers is given in (3.4).

Proof. The *necessary* of the generalized KKT condition (3.15) for the optimality to (3.14) follows immediately from Theorem 3.2 with $\theta(x)\equiv 0$. To justify the *sufficiency* part of the theorem by conventional arguments in convex optimization (with *no qualification* conditions), assume that inclusion (3.15) holds with some $\lambda\in A(\bar{x})$; the latter implies, in particular, that $\partial\vartheta_t(\bar{x})\neq\emptyset$ whenever $t\in\operatorname{supp}\lambda$. Then we find $x^*\in X^*$ such that $-x^*\in N(\bar{x};\Theta)$ and

$$x^*\in\partial\vartheta(\bar{x})+\sum_{t\in\operatorname{supp}\lambda}\partial\vartheta_t(\bar{x})\subset\partial\Big(\vartheta+\sum_{t\in T}\lambda_t\vartheta_t\Big)(\bar{x}).\tag{3.16}$$

Construction (2.2) of the convex subdifferential yields by (3.16) that

$$\vartheta(x)+\sum_{t\in T}\lambda_t\vartheta_t(x)\ge\vartheta(\bar{x})+\sum_{t\in T}\lambda_t\vartheta_t(\bar{x})+\langle x^*,x-\bar{x}\rangle\ge 0 \ \text{ for all } \ x\in X.\tag{3.17}$$

Since $\lambda_t\vartheta_t(\bar{x})=0$ for all $t\in T$ while $\lambda\in A(\bar{x})$ due to (3.4) and since $-x^*\in N(\bar{x};\Theta)$, we get from (3.17) and the normal cone construction (2.6) that

$$\vartheta(x)+\sum_{t\in T}\lambda_t\vartheta_t(x)-\vartheta(\bar{x})\ge\langle x^*,x-\bar{x}\rangle\ge 0\ \text{ for all }\ x\in\Theta,$$

which implies in turn the inequality

$$\vartheta(x)\ge\vartheta(x)+\sum_{t\in T}\lambda_t\vartheta_t(x)\ge\vartheta(\bar{x})\ \text{ whenever }\ x\in\Xi$$

by (3.2) and (3.3). The latter justifies the (global) optimality of $\bar{x}$ to the convex infinite program (3.14) and thus completes the proof of theorem. △

4 Fréchet Subgradients of Value Functions in Parametric DC Infinite Programs

This and the next sections are devoted to the main topic of our study in the paper: *generalized differential properties* of the *value functions* for *parametric DC infinite programs* defined in (1.1)–(1.3). As discussed in Section 1, marginal/value functions of this type are *intrinsically nonsmooth*, and our primary goal is to obtain constructive *upper estimates* of their *subgradient sets*, i.e., subdifferentials. Despite the convexity of the initial data in (1.1)–(1.3), the value function (1.1) is generally *nonconvex* due to the DC nature of parametric optimization problems under consideration, and thus it requires the usage of the appropriate subdifferentials of nonconvex functions.

The main result of this section provides an efficient upper estimate for the *Fréchet subdifferential* $\widehat{\partial}\mu(\bar{x})$ of the value function (1.1) in terms of the initial data in (1.1)–(1.3) and the associated Lagrange/KKT multipliers. We derive this estimate using a *variational approach*: by reducing the calculus issue to a *nonparametric* infinite *optimization* problem and employing further *necessary optimality conditions* for such problems established in Section 3. This device is based on the *intrinsic variational nature* of Fréchet subgradients.

In the next theorem and subsequent results we strongly employ the *CQC condition* from Definition 3.1 applied to the triple $(\varphi,\varphi_t,\Omega)$ in the *parametric* problem (1.1)–(1.3): the set

$$\operatorname{epi}\varphi^*+\operatorname{cone}\Big(\bigcup_{t\in T}\operatorname{epi}\varphi_t^*\Big)+\operatorname{epi}\delta^*(\cdot;\Omega)\ \text{ is weak}^*\text{ closed in } X^*\times Y^*\times I\!R. \tag{4.1}$$

We also need the following three constructions associated with (1.1)–(1.3): the *argminimum mapping* $M\colon X\rightrightarrows Y$ defined by

$$M(x):=\big\{y\in F(x)\cap G(x)\big|\ \mu(x)=\varphi(x,y)-\psi(x,y)\big\}, \tag{4.2}$$

the *constraint set* in (1.2) and (1.3) given by

$$\Gamma:=\Omega\cap\big\{(x,y)\in X\times Y\big|\ \varphi_t(x,y)\le 0\ \text{ for all }\ t\in T\big\}, \tag{4.3}$$

and the set of *KKT multipliers* dependent on $(\bar{x},\bar{y})\in\operatorname{gph}M$ for M in (4.2) and on $y^*\in Y^*$:

$$\begin{aligned}\Lambda(\bar{x},\bar{y},y^*):=\Big\{\lambda\in\widetilde{I\!R}^T_+\Big|\quad & y^*\in\partial_y\varphi(\bar{x},\bar{y})+\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial_y\varphi_t(\bar{x},\bar{y})+N_Y\big((\bar{x},\bar{y});\Omega\big),\\ & \lambda_t\varphi_t(\bar{x},\bar{y})=0\ \text{ for all }\ t\in\operatorname{supp}\lambda\Big\}.\end{aligned} \tag{4.4}$$

Theorem 4.1 (upper estimate for the Fréchet subdifferential of value functions in DC programs). *In addition to the standing assumptions of Section* 1, *suppose that* $\operatorname{dom} M \neq \emptyset$ *in* (4.2) *and that the CQC qualification condition* (3.1) *is satisfied. Then, given any point* $(\bar{x}, \bar{y}) \in \operatorname{gph} M \cap \operatorname{dom} \partial\psi$ *and a number* $\gamma > 0$, *we have the inclusion*

$$\widehat{\partial}\mu(\bar{x}) \subset \bigcap_{(x^*,y^*)\in\partial\psi(\bar{x},\bar{y})} \Big\{\partial_x\varphi(\bar{x},\bar{y}) - x^* + \bigcup_{\lambda\in\Lambda(\bar{x},\bar{y},y^*)} \Big[\sum_{t\in\operatorname{supp}\lambda} \lambda_t\partial_x\varphi_t(\bar{x},\bar{y})\Big]\Big\} \\ +N_X\big((\bar{x},\bar{y});\Omega\big) + \gamma I\!B^*. \tag{4.5}$$

Proof. Fix $(\bar{x}, \bar{y}) \in \operatorname{gph} M \cap \operatorname{dom} \partial\psi$, $u^* \in \widehat{\partial}\mu(\bar{x})$, and $(x^*, y^*) \in \partial\psi(\bar{x}, \bar{y})$. Then pick an arbitrary number $\gamma > 0$. By definition (2.8) of the Fréchet subdifferential of μ at $\bar{x}$ as $\varepsilon = 0$ there is $\eta > 0$ such that

$$\mu(x) - \mu(\bar{x}) - \langle u^*, x - \bar{x}\rangle + \gamma\|x - \bar{x}\| \geq 0 \ \text{ for all } \ x \in \bar{x} + \eta I\!B. \tag{4.6}$$

Since $\mu(\bar{x}) = \varphi(\bar{x}, \bar{y}) - \psi(\bar{x}, \bar{y})$ by the choice of $\bar{y} \in M(\bar{x})$ and since $\mu(x) \leq \varphi(x, y) - \psi(x, y)$ for all $(x, y) \in \Gamma$ due to (1.1)–(1.3) and (4.3), we get from (4.6) by taking into account inequality (2.2) with $\varepsilon = 0$ defining the subgradient $(x^*, y^*) \in \partial\psi(\bar{x}, \bar{y})$ that

$$\begin{aligned} 0 &\leq \varphi(x,y) - \varphi(\bar{x},\bar{y}) - \psi(x,y) + \psi(\bar{x},\bar{y}) - \langle u^*, x-\bar{x}\rangle + \gamma\|x-\bar{x}\| \\ &\leq \varphi(x,y) - \varphi(\bar{x},\bar{y}) - \langle u^* + x^*, x-\bar{x}\rangle - \langle y^*, y-\bar{y}\rangle + \gamma\|x-\bar{x}\| \end{aligned}$$

for all $(x, y) \in \Omega \cap [(\bar{x} + \eta I\!B) \times Y]$ with $\varphi_t(x, y) \leq 0$ as $t \in T$. Consider the function

$$\vartheta(x,y) := \varphi(x,y) - \varphi(\bar{x},\bar{y}) - \langle u^* + x^*, x-\bar{x}\rangle - \langle y^*, y-\bar{y}\rangle + \gamma\|x-\bar{x}\|, \tag{4.7}$$

which is clearly proper, l.s.c., and convex on $X \times Y$. It follows from (4.6) and (4.7) that $(\bar{x}, \bar{y})$ is a solution to the (unconstrained) *convex infinite program*

$$\begin{cases} \text{minimize } \ \vartheta(x,y) \ \text{ subject to} \\ \varphi_t(x,y) \leq 0 \ \text{ as } \ t \in T, \quad (x,y) \in \Omega \cap \big[(\bar{x}+\eta I\!B) \times Y\big]. \end{cases} \tag{4.8}$$

It follows from Lemma 4.2, the rather technical proof of which is postponed and presented after the proof of the theorem, that the qualification condition (4.1) imposed in this theorem implies the fulfillment of the *CQC requirement* from Definition 3.1 for the corresponding data of (4.8), i.e., that the set

$$\operatorname{epi}\vartheta^* + \operatorname{cone}\Big[\bigcup_{t\in T}\operatorname{epi}\varphi_t^*\Big] + \operatorname{epi}\delta^*\big(\cdot;\Omega\cap[(\bar{x}+\eta I\!B)\times Y]\big) \tag{4.9}$$

is *weak** *closed* in the space $X^* \times Y^* \times I\!R$. Thus applying the optimality conditions from Theorem 3.5 to problem (4.8), we find $\lambda \in \widetilde{I\!R}^T_+$ such that

$$0 \in \partial\vartheta(\bar{x},\bar{y}) + \sum_{t\in\operatorname{supp}\lambda} \lambda_t\partial\varphi_t(\bar{x},\bar{y}) + N\big((\bar{x},\bar{y});\Omega\cap[(\bar{x}+\eta I\!B)\times Y]\big) \\ \text{with } \ \lambda_t\varphi_t(\bar{x},\bar{y}) = 0 \ \text{ for all } \ t \in \operatorname{supp}\lambda. \tag{4.10}$$

It easily follows from the subdifferential sum rule in (2.5) of Lemma 2.1 applied to the indicator functions $\delta((\bar{x}, \bar{y}); \Omega)$ and $\delta((\bar{x}, \bar{y}); (\bar{x} + \eta I\!B) \times Y)$ that

$$N\big((\bar{x},\bar{y});\Omega\cap[(\bar{x}+\eta I\!B)\times Y]\big) = N\big((\bar{x},\bar{y});\Omega\big).$$

Indeed, $(\bar{x},\bar{y})$ is an interior point of the set $(\bar{x}+\eta I\!\!B)\times Y$, and thus the indicator function of this set is continuous at $(\bar{x},\bar{y})$. Further, it follows from the construction of $\vartheta(x,y)$ in (4.7) and from the subdifferential sum rule of convex analysis (2.5) that

$$\partial\vartheta(\bar{x},\bar{y})=\partial\varphi(\bar{x},\bar{y})+(-u^*-x^*,-y^*)+(\eta I\!\!B^*)\times\{0\}.$$

Substituting the latter relationships into (4.10) and taking into account that

$$\partial\varphi(\bar{x},\bar{y})\subset\partial_x\varphi(\bar{x},\bar{y})\times\partial_y\varphi(\bar{x},\bar{y})\ \text{ and }\ \partial\varphi_t(\bar{x},\bar{y})\subset\partial_x\varphi_t(\bar{x},\bar{y})\times\partial_y\varphi_t(\bar{x},\bar{y}), \tag{4.11}$$

we arrive at the following two inclusions:

$$\begin{cases} u^*\in\partial_x\varphi(\bar{x},\bar{y})-x^*+\displaystyle\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial_x\varphi_t(\bar{x},\bar{y})+N_X\big((\bar{x},\bar{y});\Omega\big)+\gamma I\!\!B^*,\\ y^*\in\partial_y\varphi(\bar{x},\bar{y})+\displaystyle\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial_y\varphi_t(\bar{x},\bar{y})+N_Y\big((\bar{x},\bar{y});\Omega\big)\end{cases} \tag{4.12}$$

with $\lambda_t\varphi_t(\bar{x},\bar{y})=0$ for all $t\in\operatorname{supp}\lambda$. Using finally construction (4.4) of the KKT multipliers, we deduce from (4.12) the desired upper estimate (4.5) and thus complete the proof of the theorem provided that Lemma 4.2 is justified. △

Let us now justify the afore-mentioned technical lemma used in the proof of Theorem 4.1.

Lemma 4.2 (relationship between qualification conditions). *Let the qualification condition* (4.1) *imposed in Theorem* 4.1 *be satisfied. Then we have the CQC condition* (4.9) *for the nonparametric convex problem* (4.8) *with the cost function* ϑ *defined in* (4.7).

Proof. The arguments below are mainly based on the *refined epigraphical rule* for *conjugate functions* from Lemma 2.1(ii). First using the data defined in the proof of Theorem 4.1, construct the real-valued function

$$\xi(x,y):=-\varphi(\bar{x},\bar{y})-\langle u^*+x^*,x-\bar{x}\rangle-\langle y^*,y-\bar{y}\rangle+\gamma\|x-\bar{x}\|,$$

which is obviously convex and continuous on $X\times Y$ with $\vartheta=\varphi+\xi$. Substituting the latter into the qualification (4.9) and using several times the epigraphical rule from Lemma 2.1 with taking into account that the indicator function $\delta(\cdot;(\bar{x}+\eta I\!\!B^*)\times Y)$ is *continuous* at $(\bar{x},\bar{y})$, we conclude that the set in (4.9) reduces to

$$\operatorname{epi}\varphi^*+\operatorname{cone}\Big[\bigcup_{t\in T}\operatorname{epi}\varphi_t^*\Big]+\operatorname{epi}\delta^*(\cdot;\Omega)+\operatorname{epi}\big[\xi+\delta\big(\cdot;(\bar{x}+\eta I\!\!B)\times Y\big)\big]^*. \tag{4.13}$$

On the other hand, the qualification condition (4.1) implies by Lemma 2.1 that

$$\operatorname{epi}\big(\varphi+\delta(\cdot;\Gamma)\big)^*=\operatorname{epi}\varphi^*+\operatorname{cone}\Big[\bigcup_{t\in T}\operatorname{epi}\varphi_t^*\Big]+\operatorname{epi}\delta^*(\cdot;\Omega) \tag{4.14}$$

for the constraint set Γ defined in (4.3). Substituting (4.14) into (4.13) and using Lemma 2.1 again, we observe that the set in (4.9) equals to

$$\operatorname{epi}\big(\varphi+\delta(\cdot;\Gamma)+\xi+\delta(\cdot;(\bar{x}+\eta I\!\!B)\times Y)\big)^*,$$

which is *weak** *closed* in the space $X^*\times Y^*\times I\!R$ as the epigraph of the conjugate function to the proper, l.s.c., convex function $\varphi+\delta(\cdot;\Gamma)+\xi+\delta(\cdot;(\bar{x}+\eta I\!B)\times Y)$. This justifies the qualification condition (4.9) and completes the proof of the lemma. $\triangle$

Next we derive an easy consequence of Theorem 4.1 that establishes new *necessary optimality conditions* for *parametric DC infinite programs*. In the terminology of [14, Chapter 5], these conditions are of the *upper subdifferential* type for *minimization* problems, since they employ *all upper subgradients* of the cost function $-\psi$, which reduce to (lower) subgradients of ψ, in the DC setting under consideration; see more discussions in [14] for general (not particularly DC) minimization problems.

Corollary 4.3 (necessary conditions for parametric DC infinite programs from Fréchet subgradients of value functions). *Given a parameter value* $\bar{x}\in\operatorname{dom}M$ *in* (4.2)*, let* $\bar{y}$ *be an optimal solution to the parametric DC problem*

$$\textit{minimize}\ \ \varphi(\bar{x},y)-\psi(\bar{x},y)\ \ \textit{subject to}\ \ y\in F(x)\cap G(x), \tag{4.15}$$

where F *and* G *are defined in* (1.2) *and* (1.3)*, respectively, under the standing assumptions made. Suppose in addition that* $\widehat{\partial}\mu(\bar{x})\neq\emptyset$ *for the value function* (1.1) *and that the qualification condition* (4.1) *is satisfied. Then for each* $(x^*,y^*)\in\partial\psi(\bar{x},\bar{y})$ *and* $\gamma>0$ *there are* $u^*\in X^*$ *and* $\lambda\in\widetilde{I\!R}^T_+$ *from* (3.3) *such that we have the relationships*

$$\begin{cases} u^*+x^*\in\partial_x\varphi(\bar{x},\bar{y})+\displaystyle\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial_x\varphi_t(\bar{x},\bar{y})+N_X\big((\bar{x},\bar{y});\Omega\big)+\gamma I\!B^*,\\ y^*\in\partial_y\varphi(\bar{x},\bar{y})+\displaystyle\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial_y\varphi_t(\bar{x},\bar{y})+N_Y\big((\bar{x},\bar{y});\Omega\big),\\ \lambda_t\varphi_t(\bar{x},\bar{y})=0\ \ \textit{for all}\ \ t\in\operatorname{supp}\lambda. \end{cases} \tag{4.16}$$

Proof. This follows direction from inclusion (4.5) in Theorem 4.1 with $\widehat{\partial}\mu(\bar{x})\neq\emptyset$ due to the construction of the KKT multiplier set (4.4). $\triangle$

The most restrictive and not easily verifiable assumption in Corollary 4.3 is that of $\widehat{\partial}\mu(\bar{x})\neq\emptyset$. In fact it holds on the *dense* set of parameters if the space X is *Asplund*; see, e.g., [13, Corollary 2.29]. However, the Fréchet subdifferential may often be *empty* (even in simple finite-dimensional settings) at *individual* points of the domains for nonconvex functions; see discussions in Section 2. It is worth mentioning here that the restrictive assumption $\widehat{\partial}\mu(\bar{x})\neq\emptyset$ can be *dropped* with keeping necessary optimality conditions for DC infinite programs similar to those in Corollary 4.3, which are valid for *every* parameter $\bar{x}\in\operatorname{dom}M$; see Theorem 5.10. This is derived from the upper estimates for the *limiting* (*basic and singular*) *subdifferentials* of the value function obtained in the next section.

5 Basic and Singular Subgradients of Value Functions in Parametric DC Infinite Programs

This section is devoted to establishing verifiable upper estimates for the basic subdifferential (2.10) and the singular subdifferential (2.11) of the value function (1.1) and deriving from them necessary optimality conditions for the DC infinite programs under consideration. We start with upper estimates for the *basic subdifferential* of the value function in (1.1)–(1.3) and obtain two independent results in this direction.

The first result provides a tight upper estimate for the basic subdifferential of (1.1) under the following rather restrictive assumption on the *minus term* ψ in the cost function of (1.1) introduced and needed in this paper for proper convex functions.

Definition 5.1 (inner subdifferential stability). *We say that a proper convex function* $\psi\colon X \to \overline{I\!R}$ *is* INNER SUBDIFFERENTIALLY STABLE *at* $\bar{x} \in \operatorname{dom}\psi$ *if*

$$\operatorname*{Lim\,inf}_{x \xrightarrow{\operatorname{dom}\psi} \bar{x}} \partial\psi(x) \neq \emptyset, \tag{5.1}$$

where Lim inf *stands for the Painlevé-Kuratowski inner limit* (1.5).

If ψ is w^*-*continuously Gâteaux differentiable* around $\bar{x} \in \operatorname{int}(\operatorname{dom}\psi)$—i.e., it is Gâteaux differentiable on a neighborhood of $\bar{x}$ including this point, and its Gâteaux derivative operator $d\psi\colon X \to X^*$ is continuous with respect to the weak* topology of X^*—then the "Lim inf" in (5.1) reduces to the singleton $\{d\psi(\bar{x})\}$ in any Banach space. The next proposition relaxes the smoothness assumption in the neighborhood of $\bar{x}$ provided that the closed unit ball $I\!B^*$ in X^* is *weak** *sequentially compact*. This latter property holds for general classes of Banach spaces X; in particular, for those admitting an equivalent norm Gâteaux differentiable at nonzero points, for weak Asplund spaces (including every Asplund space and every weakly compactly generated space, and hence every reflexive and every separable space), etc. We refer the reader to [9] for more information on this property and the afore-mentioned classes of Banach spaces.

Proposition 5.2 (sufficient conditions for inner subdifferential stability). *Let* X *be a Banach space such that the closed unit ball* $I\!B^*$ *is weak** *sequentially compact in* X^**, and let* ψ *be convex, continuous, and Gâteaux differentiable at* $\bar{x} \in \operatorname{int}(\operatorname{dom}\psi)$*. Then* ψ *is inner subdifferentially stable at* $\bar{x}$*.*

Proof. Take any sequence $x_k \to \bar{x}$ as $k \to \infty$ and suppose that it entirely belongs to U. Employing the well-known *boundedness* of the subdifferential mapping $\partial\psi(\cdot)$ around $\bar{x}$ (see, e.g., [18, Proposition 1.11]) and using the assumed weak* sequential compactness of the dual ball $I\!B^*$, we conclude that every subset of the set

$$V^* := \big\{x^* \in X^* \big|\, \exists\, x \in U \;\text{ with }\; x^* \in \partial\psi(x)\big\}$$

contains a *subsequence* converging in the weak* topology of X^*. Then picking any sequence of subgradients $x_k^* \in \partial\psi(x_k)$, we assume with no loss of generality that there is $x^* \in X^*$ such that $x_k^* \xrightarrow{w^*} x^*$ as $k \to \infty$. It follows directly from (2.2) that $x^* \in \partial\psi(\bar{x})$. Since ψ is continuous and Gâteaux differentiable at $\bar{x}$, we have from convex analysis [18] that $\partial\psi(\bar{x}) = \{d\psi(\bar{x})\}$, and therefore $x_k^* \xrightarrow{w^*} d\psi(\bar{x})$ as $k \to \infty$. By definition of the inner limit

(1.5) the latter ensures (5.1) and thus justifies the inner subdifferential stability of ψ at $\bar{x}$ under the assumptions made. $\triangle$

It is not hard to give various examples of functions, which are *not differentiable* at the reference point while *inner subdifferentially stable* at it. Such functions can be constructed in the following general way. Take a proper closed convex subset Ω of a Gâteaux smooth space X, a point $\bar{x} \in \mathrm{bd}\,\Omega$, and a function $\theta(x)$ that is convex, continuous, and Gâteaux differentiable on an open set containing $\bar{x}$. Then define $\psi\colon X \to \overline{I\!R}$ by

$$\psi(x) = \begin{cases} \theta(x) & \text{if } \ x \in \Omega, \\ \infty & \text{otherwise.} \end{cases} \tag{5.2}$$

It follows from Proposition 5.2 that $\operatorname{Lim\,inf} \partial\psi(x)$ in (5.1) reduces to $\{d\theta(\bar{x})\}$. Note that

$$\partial\psi(\bar{x}) = d\theta(\bar{x}) + N(\bar{x}; \Omega)$$

by the subdifferential sum rule (2.5) held due to the continuity of θ. Observe also that, by our convention that $\infty - \infty = \infty$, a boundary domain point $\bar{x} \in \mathrm{bd}(\mathrm{dom}\,\psi)$ can give a local minimizer to the DC function $\varphi - \psi$ provided that $\mathrm{dom}\,\varphi \subset \mathrm{dom}\,\psi$.

Remark 5.3 (inner subdifferential stability in finite dimension). Note that any function $\psi(x)$ constructed in the way of (5.2) is *extended-real-valued* around the reference point $\bar{x} \in \mathrm{dom}\,\psi$. This choice is motivated by the following observation: if ψ is a convex function defined on $I\!R^n$ with $\mathrm{int}(\mathrm{dom}\,\psi) \neq \emptyset$, then "Lim inf" in (5.1) is *empty* at any point $\bar{x} \in \mathrm{int}(\mathrm{dom}\,\psi)$ where $\partial\psi(\bar{x})$ is *not a singleton*, i.e., where ψ is *not differentiable*; in this case Gâteaux and Fréchet derivatives agree at $\bar{x}$. Indeed, this follows from the well-known fact in finite-dimensional convex analysis (see, e.g., [19, Theorem 25.5]) that such a function ψ is *differentiable* in the classical sense on a *dense* subset of $\mathrm{int}(\mathrm{dom}\,\psi)$ and, moreover, its subdifferential at $\bar{x} \in \mathrm{int}(\mathrm{dom}\,\psi)$ admits the representation

$$\partial\psi(\bar{x}) = \mathrm{co}\,\big\{ \lim_{k\to\infty} \nabla\psi(x_k) \big|\ \psi \ \text{ is differentiable at } \ x_k \to \bar{x} \big\}$$

via the classical gradients $\nabla\psi(x)$ on the afore-mentioned dense subset; see, e.g., [19, Theorem 25.6]. Taking into account Proposition 5.2 and the automatic continuity of convex functions on the interior of their domains in finite dimensions by [19, Theorem 10.1], we thus conclude that the inner subdifferential stability of ψ at $\bar{x} \in \mathrm{int}(\mathrm{dom}\,\psi) \subset I\!R^n$ is *equivalent* to its differentiability at this point. It is *not* the case for $\bar{x} \in \mathrm{bd}(\mathrm{dom}\,\psi)$ as shown in (5.2).

Now we are ready to formulate and prove a *tight upper estimate* for the basic subdifferential of the value function in (1.1)–(1.3) under the *inner subdifferential stability* of the minus function ψ in (1.1).

Theorem 5.4 (basic subgradients of value functions in DC programs under inner subdifferential stability). *In addition to the standing assumptions, suppose that the argminimum mapping $M(\cdot)$ in (4.2) is μ-inner semicontinuous at $(\bar{x}, \bar{y}) \in \mathrm{gph}\,M$, that ψ in*

(1.1) *is inner subdifferentially stable at* $(\bar{x},\bar{y})$*, and that the qualification condition* (4.1) *is satisfied. Then given any* $(x^*,y^*)\in \displaystyle\mathop{\rm Lim\,inf}_{(x,y)\stackrel{\rm dom\,\psi}{\to}(\bar{x},\bar{y})}\partial\psi(x,y)$*, we have the inclusion*

$$\partial\mu(\bar{x})\subset\partial_x\varphi(\bar{x},\bar{y})-x^*+\bigcup_{\lambda\in\Lambda(\bar{x},\bar{y},y^*)}\Big[\sum_{t\in{\rm supp}\,\lambda}\lambda_t\partial_x\varphi_t(\bar{x},\bar{y})\Big]+N_X\big((\bar{x},\bar{y});\Omega\big) \tag{5.3}$$

with the set of KKT multipliers $\Lambda(\bar{x},\bar{y},y^*)$ *defined in* (4.4).

Proof. To justify inclusion (5.3) for any fixed $(x^*,y^*)\in \displaystyle\mathop{\rm Lim\,inf}_{(x,y)\stackrel{\rm dom\,\psi}{\to}(\bar{x},\bar{y})}\partial\psi(x,y)$, pick an arbitrary basic subgradient $u^*\in\partial\mu(\bar{x})$ and by definition (2.10) find sequences $\varepsilon_k\downarrow 0$, $x_k\stackrel{\mu}{\to}\bar{x}$, and $u_k^*\in\widehat{\partial}_{\varepsilon_k}\mu(x_k)$ satisfying $u_k^*\stackrel{w^*}{\to}u^*$ as $k\to\infty$. Then applying definition (2.8) to the ε_k-subgradient $u_k^*\in\widehat{\partial}_{\varepsilon_k}\mu(x_k)$ for any fixed $k\in I\!\!N$, we get $\eta_k>0$ such that

$$\langle u_k^*,x-x_k\rangle\le\mu(x)-\mu(x_k)+2\varepsilon_k\|x-x_k\|\ \text{ whenever }\ x\in x_k+\eta_k I\!\!B. \tag{5.4}$$

Since the argminimum mapping $M(\cdot)$ is μ-inner semicontinuous at $(\bar{x},\bar{y})$ and since $x_k\stackrel{\mu}{\to}\bar{x}$, there is a sequence of $y_k\in M(x_k)$ that contains a subsequence converging to $\bar{y}$; we can assume that $y_k\to\bar{y}$ for all $k\to\infty$. Taking (x^*,y^*) fixed in the theorem and using definition (1.5) of the inner limit, for the chosen sequence (x_k,y_k) we find a sequence of subgradients $(x_k^*,y_k^*)\in\partial\psi(x_k,y_k)$ such that $(x_k^*,y_k^*)\stackrel{w^*}{\to}(x^*,y^*)$ as $k\to\infty$. It follows from (5.4), from definitions (4.2) of the argminimum mapping $M(\cdot)$ and (4.3) of the feasible solution set Γ to (1.1)–(1.3), and from the subdifferential construction (2.2) that

$$\begin{array}{rl}\langle u_k^*,x-x_k) & \le\varphi(x,y)-\psi(x,y)-\varphi(x_k,y_k)+\psi(x_k,y_k)+2\varepsilon_k\big(\|x-x_k\|+\|y-y_k\|\big)\\ & \le\varphi(x,y)-\varphi(x_k,y_k)-\langle x_k^*,x-x_k\rangle-\langle y_k^*,y-y_k\rangle\\ & +2\varepsilon_k\big(\|x-x_k\|+\|y-y_k\|\big)\ \text{ for all }\ (x,y)\in\Gamma\cap\big((x_k,y_k)+\eta_k I\!\!B\big).\end{array}$$

The latter implies in turn that the relationship

$$\langle u_k^*+x_k^*,x-x_k\rangle+\langle y_k^*,y-y_k\rangle\le\varphi(x,y)-\varphi(x_k,y_k)+2\varepsilon_k\big(\|x-x_k\|+\|y-y_k\|\big)$$

valid for all such (x,y), which can be written via the analytic ε-subdifferentials (2.8) as

$$(u_k^*+x_k^*,y_k^*)\in\widehat{\partial}_{2\varepsilon_k}\big(\varphi+\delta(\cdot;\Gamma)\big)(x_k,y_k)\ \text{ for all }\ k\in I\!\!N. \tag{5.5}$$

Passing to the limit in (5.5) as $k\to\infty$ and taking into account the weak* convergence $(u_k^*+x_k^*,y_k^*)\stackrel{w^*}{\to}(u^*+x^*,y^*)$, we get from definition (2.10) of the basic subdifferential that

$$(u^*+x^*,y^*)\in\partial\big(\varphi+\delta(\cdot;\Gamma)\big)(\bar{x},\bar{y}). \tag{5.6}$$

Since the function $\varphi+\delta(\cdot;\Gamma)$ is obviously convex on $X\times Y$, the basic subdifferential in (5.6) reduces to the subdifferential (2.2) as $\varepsilon=0$ of convex analysis on the Banach space in question; see [13, Theorem 1.93]. Further, the *subdifferential sum rule* from Corollary 3.3 held under the assumed qualification condition (4.1) gives

$$\partial\big(\varphi+\delta(\cdot;\Gamma)\big)(\bar{x},\bar{y})\subset\partial\varphi(\bar{x},\bar{y})+\bigcup_{\lambda\in A(\bar{x},\bar{y})}\Big[\sum_{t\in{\rm supp}\,\lambda}\lambda_t\partial\varphi_t(\bar{x},\bar{y})\Big]+N\big((\bar{x},\bar{y});\Omega\big) \tag{5.7}$$

with $A(\bar{x},\bar{y})=\{\lambda\in\widetilde{I\!R}^T_+|\ \lambda_t\varphi_t(\bar{x},\bar{y})=0\ \text{ for all }\ t\in\operatorname{supp}\lambda\}$. Substituting now (5.7) into (5.6) and taking into account relationships (4.11) between the full and partial subdifferentials of convex functions, we arrive at the inclusions

$$\begin{cases} u^*\in\partial_x\varphi(\bar{x},\bar{y})-x^*+\displaystyle\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial_x\varphi_t(\bar{x},\bar{y})+N_X\big((\bar{x},\bar{y});\Omega\big),\\ y^*\in\partial_y\varphi(\bar{x},\bar{y})+\displaystyle\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial_y\varphi_t(\bar{x},\bar{y})+N_Y\big((\bar{x},\bar{y});\Omega\big)\end{cases}$$

for some $\lambda\in A(\bar{x},\bar{y})$, which imply (5.3) due to construction (4.4) of the KKT multiplier set $\Lambda(\bar{x},\bar{y},y^*)$. This completes the proof of the theorem. △

As discussed above, the inner subdifferential stability of the minus function ψ required in Theorem 5.4 is a rather restrictive assumption. In the next theorem we replace it by much more flexible assumption on ψ that holds, in particular, for any *continuous* convex functions. The upper estimate for the basic subdifferential of the value function (1.1) obtained under this assumption is less precise than in Theorem 5.4 while is still sufficient for the majority of applications including those in this paper. The new condition is formulated as follows.

Definition 5.5 (subdifferential boundedness). *We say that a proper convex function $\psi\colon X\to\overline{I\!R}$ is* SUBDIFFERENTIALLY BOUNDED *around $\bar{x}\in\operatorname{dom}\psi$ if for any sequences $\varepsilon_k\downarrow 0$ and $x_k\overset{\operatorname{dom}\psi}{\to}\bar{x}$ as $k\to\infty$ there is a sequence of $x^*_k\in\partial_{\varepsilon_k}\psi(x_k)$, $k\in I\!N$, such that the set $\{x^*_k|\ k\in I\!N\}$ is bounded in X^*.*

Of course, this definition can be applied to nonconvex functions as well (which is not needed in this paper) if we appropriately modify the constructions of the ε-subdifferentials (2.2). The following sufficient condition for the subdifferential boundedness is entirely based on the local Lipschitzian property of ψ *around* $\bar{x}$ that is a consequence of just the usual *continuity at* the reference point in the *convex* setting.

Proposition 5.6 (sufficient condition for subdifferential boundedness of convex functions). *Let $\psi\colon X\to\overline{I\!R}$ be a convex function, which is continuous at $\bar{x}\in\operatorname{int}(\operatorname{dom}\psi)$. Then ψ is subdifferentially bounded around this point.*

Proof. It is well known in convex analysis that the *continuity* of a convex function ψ *at* the reference point $\bar{x}\in\operatorname{int}(\operatorname{dom}\psi)$ yields that ψ is *locally Lipschitzian around* $\bar{x}$; see, e.g., [18, Proposition 1.6]. On the other hand, the local Lipschitz continuity of ψ around $\bar{x}$ easily implies by (2.2) with $\varepsilon=0$ that the subdifferential sets $\partial\psi(x)$ are *uniformly bounded.* Furthermore, $\partial\psi(x)\subset\partial_\varepsilon\psi(x)$ for any $\varepsilon>0$. Now taking arbitrary sequences $\varepsilon_k\downarrow 0$ and $x_k\overset{\operatorname{dom}\psi}{\to}\bar{x}$ as $k\to\infty$, we have $x^*_k\in\partial_{\varepsilon_k}\psi(x_k)$ for any sequence of subgradients $x^*_k\in\partial\psi(x_k)$, $k\in I\!N$. This justifies the subdifferential boundedness of ψ. △

The following theorem provides a result *largely independent* of Theorem 5.4. The upper estimate (5.8) obtained below reduces to (5.3) in Theorem 5.4 if the minus function ψ in (1.1) is Gâteaux differentiable at $(\bar{x},\bar{y})$ and the closed unit balls in X^* and Y^* are sequentially weak* compact in X^*. Observe that Theorem 5.7 is free of the restrictive (in the nonsmooth case) requirement on the inner subdifferential stability of ψ providing however a less precise estimate of $\partial\mu(\bar{x})$ when ψ is not Gâteaux differentiable at the reference point $(\bar{x}, \bar{y})$. The proof of Theorem 5.7 is significantly different and more involved in comparison with that of Theorem 5.4. In particular, we use below the fundamental *Brøndsted-Rockafellar theorem* on *subdifferential density* in convex analysis, which is a predecessor and convex counterpart of the seminal *Ekeland variational principle* in variational analysis.

Theorem 5.7 (basic subgradients of value functions in DC programs under subdifferential boundedness). *In addition to the standing assumptions, suppose that for both spaces X and Y the dual unit balls are sequentially weak* compact in X^* and Y^*, respectively, that the argminimum mapping $M(\cdot)$ in* (4.2) *is μ-inner semicontinuous at some point $(\bar{x}, \bar{y}) \in \operatorname{gph} M$, that ψ in* (1.1) *is subdifferentially bounded around $(\bar{x}, \bar{y})$, and that the qualification condition* (4.1) *is satisfied. Then we have the upper estimate*

$$\begin{aligned}\partial\mu(\bar{x}) \subset \;& \partial_x\varphi(\bar{x}, \bar{y}) + \bigcup_{(x^*, y^*) \in \partial\psi(\bar{x}, \bar{y})} \Big\{ -x^* + \bigcup_{\lambda \in \Lambda(\bar{x}, \bar{y}, y^*)} \Big[\sum_{t \in \operatorname{supp}\lambda} \lambda_t \partial_x \varphi_t(\bar{x}, \bar{y}) \Big] \Big\} \\ & + N_X\big((\bar{x}, \bar{y}); \Omega\big)\end{aligned} \tag{5.8}$$

with the set of KKT multipliers $\Lambda(\bar{x}, \bar{y}, y^*)$ defined in (4.4).

Proof. Pick any $u^* \in \partial\mu(\bar{x})$ and similarly to the proof of Theorem 5.4 find sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\mu} \bar{x}$, and $u_k^* \in \widehat{\partial}_{\varepsilon_k}\mu(x_k)$ satisfying $u_k^* \xrightarrow{w^*} u^*$ as $k \to \infty$. Then we get $\eta_k \downarrow 0$ such that inequality (5.4) holds and, by the assumed μ-inner semicontinuity of $M(\cdot)$, obtain a sequence of $y_k \in M(x_k)$ converging to $\bar{y}$ as $k \to \infty$.

Select further $\nu_k > 0$ satisfying $2\sqrt{\nu_k} < \eta_k$. Taking into account that $\nu_k \downarrow 0$ and $(x_k, y_k) \to (\bar{x}, \bar{y})$ as $k \to \infty$ and employing the *subdifferential boundedness* condition imposed on ψ, we find a sequence of $(x_k^*, y_k^*) \in \partial_{\nu_k}\psi(x_k, y_k)$, $k \in I\!N$, such that the set $\{(x_k^*, y_k^*) \in X^* \times Y^* |\; k \in I\!N\}$ is bounded. The assumed *sequential* weak* compactness of the dual balls in X^* and Y^* allows us to select a *subsequence* of $\{(x_k^*, y_k^*)\}$ that weak* converges (with no relabeling) to some $(x^*, y^*) \in X^* \times Y^*$ as $k \to \infty$. The well-known *closed-graph* property of subdifferential and ε-subdifferential mappings in convex analysis (see, e.g., [23, Theorem 2.4.2]) implies that $(x^*, y^*) \in \partial\psi(\bar{x}, \bar{y})$. Similarly to the proof of Theorem 5.4 we derive from (5.4) the inequality

$$\langle u_k^* + x_k^*, x - x_k \rangle + \langle y_k^*, y - y_k \rangle - \nu_k \le \varphi(x, y) - \varphi(x_k, y_k) + 2\varepsilon_k\big(\|x - x_k\| + \|y - y_k\|\big)$$

held for all $(x, y) \in \Gamma \cap \big((x_k, y_k) + \eta_k I\!B\big)$ with $\Gamma \subset X \times Y$ given in (4.3). This implies that

$$(u_k^* + x_k^*, y_k^*) \in \partial_{\nu_k}\vartheta_k(x_k, y_k), \quad k \in I\!N, \tag{5.9}$$

via the ε-subdifferentials (2.2) of the proper, l.s.c., and convex function $\vartheta_k \colon X \times Y \to \overline{I\!R}$ constructed for each $k \in I\!N$ in the form

$$\begin{aligned}\vartheta_k(x, y) :\; &= \varphi(x, y) + \delta\big((x, y); \Gamma \cap [(x_k, y_k) + \eta_k I\!B]\big) \\ & -\varphi(x_k, y_k) + 2\varepsilon_k\big(\|x - x_k\| + \|y - y_k\|\big).\end{aligned} \tag{5.10}$$

Applying now to the elements in (5.9), for each $k \in I\!N$, the afore-mentioned *Brøndsted-Rockafellar density theorem* (see, e.g., [18, Theorem 3.17]), we find pairs $(\widetilde{x}_k, \widetilde{y}_k) \in \operatorname{dom} \vartheta_k$ and $(\widetilde{x}^*_k, \widetilde{y}^*_k) \in \partial\vartheta_k(\widetilde{x}_k, \widetilde{y}_k)$ satisfying the estimates

$$\|\widetilde{x}_k - x_k\| + \|\widetilde{y}_k - y_k\| \le \sqrt{\nu_k} \;\text{ and }\; \|\widetilde{x}^*_k - (u^*_k + x^*_k)\| + \|\widetilde{y}^*_k - y^*_k\| \le \sqrt{\nu_k}. \tag{5.11}$$

It follows from the latter relationships, constructions (2.2) and (5.10), and the choice of ν_k with $0 < 2\sqrt{\nu_k} < \eta_k$ that

$$\begin{array}{l}\langle \widetilde{x}^*_k, x - \widetilde{x}_k\rangle + \langle \widetilde{y}^*_k, y - \widetilde{y}_k\rangle \le \vartheta_k(x,y) - \vartheta_k(\widetilde{x}_k, \widetilde{y}_k) \le \varphi(x,y) - \varphi(\widetilde{x}_k, \widetilde{y}_k)\\ +2\varepsilon_k\big(\|x - x_k\| + \|y - y_k\|\big) - 2\varepsilon_k\big(\|\widetilde{x}_k - x_k\| + \|\widetilde{y}_k - y_k\|\big)\\ \le \varphi(x,y) - \varphi(\widetilde{x}_k, \widetilde{y}_k) + 2\varepsilon_k\big(\|x - \widetilde{x}_k\| + \|y - \widetilde{y}_k\|\big)\end{array}$$

for all $(x,y) \in \Gamma \cap \big((x_k, y_k) + \eta_k I\!B\big)$, which yields the inclusions

$$(\widetilde{x}^*_k, \widetilde{y}^*_k) \in \widehat{\partial}_{2\varepsilon_k}\big(\varphi + \delta(\cdot;\Gamma)\big)(\widetilde{x}_k, \widetilde{y}_k), \quad k \in I\!N, \tag{5.12}$$

via the analytic ε-subdifferentials (2.8) of the convex l.s.c. function $\varphi + \delta(\cdot;\Gamma)$.

It easily follows from the convergences $(x_k, y_k) \to (\bar{x}, \bar{y})$, $(u^*_k + x^*_k, y^*_k) \xrightarrow{w^*} (u^* + x^*, y^*)$ and from the norm estimates in (5.11) that

$$(\widetilde{x}_k, \widetilde{y}_k) \to (\bar{x}, \bar{y}) \;\text{ and }\; (\widetilde{x}^*_k, \widetilde{y}^*_k) \xrightarrow{w^*} (u^* + x^*, y^*) \;\text{ as }\; k \to \infty.$$

Thus passing to the limit in (5.12) as $k \to \infty$ and using construction (2.10) of the basic subdifferential, we arrive at inclusion (5.6) as in the proof of Theorem 5.4, where the basic subdifferential agrees with the subdifferential of convex analysis (2.2) with $\varepsilon = 0$ due to the convexity of the function $\varphi + \delta(\cdot;\Gamma)$. Proceeding finally as in the proof of Theorem 5.4 by employing the subdifferential sum rule from Corollary 3.3 held under the assumed qualification condition (4.1), we justify (5.8) and complete the proof of the theorem. $\triangle$

Our next results gives an upper estimate for the *singular subdifferential* (2.11) of the value function in the general parametric *DC infinite program* (1.1)–(1.3) under consideration. This is a singular counterpart of Theorem 5.7 that particularly plays a crucial role in establishing the local *Lipschitz continuity* of the value function and deriving *necessary optimality conditions* for (1.1)–(1.3); see below.

Theorem 5.8 (singular subgradients of value functions in DC programs). *Suppose that the assumptions of Theorem* 5.7 *are satisfied with replacing the qualification condition* (4.1) *by the following one: the set*

$$\operatorname{cone}\Big[\bigcup_{t\in T} \operatorname{epi}\varphi^*_t\Big] + \operatorname{epi}\delta^*(\cdot;\Omega) \;\text{ is weak}^* \text{ closed in } X^* \times Y^* \times I\!R. \tag{5.13}$$

Assume in addition that $\Gamma \subset \operatorname{dom}\varphi$ *for the set of feasible solutions* Γ *defined in* (4.3). *Then*

$$\partial^\infty\mu(\bar{x}) \subset \bigcup_{\lambda\in\Lambda^\infty(\bar{x},\bar{y})} \Big[\sum_{t\in\operatorname{supp}\lambda} \lambda_t \partial_x\varphi_t(\bar{x},\bar{y})\Big] + N_X\big((\bar{x},\bar{y});\Omega\big), \tag{5.14}$$

where the set of singular multipliers in (5.14) *is defined by*

$$\Lambda^\infty(\bar{x},\bar{y}) := \Big\{\lambda \in \widetilde{I\!R}^T_+ \Big|\ \ 0 \in \sum_{t\in \operatorname{supp}\lambda} \lambda_t \partial_y \varphi_t(\bar{x},\bar{y}) + N_Y\big((\bar{x},\bar{y});\Omega\big),\quad \lambda_t\varphi_t(\bar{x},\bar{y}) = 0 \ \ \textit{for all}\ \ t \in \operatorname{supp}\lambda\Big\}. \tag{5.15}$$

Proof. Take any singular subgradient $u^* \in \partial^\infty \mu(\bar{x})$ and by definition (2.11) find sequences

$$\lambda_k \downarrow 0,\ \ \varepsilon_k \downarrow 0,\ \ x_k \xrightarrow{\mu} \bar{x},\ \ u_k^* \in \widehat{\partial}_{\varepsilon_k}\mu(x_k)\ \ \text{with}\ \ \ \lambda_k u_k^* \xrightarrow{w^*} u^* \ \text{as}\ \ k\to\infty.$$

Following the corresponding arguments of Theorem 5.7, we select sequences

$$\nu_k \downarrow 0\ \text{ as }\ k\to\infty,\quad y_k \in M(x_k),\ \ \text{and}\ \ (x_k^*,y_k^*) \in \partial_{\nu_k}\psi(x_k,y_k),\quad k \in I\!N,$$

such that the one of $\{(x_k^*,y_k^*)\}$ weak* converges in $X^*\times Y^*$ to some $(x^*,y^*) \in \partial\psi(\bar{x},\bar{y})$. Further, the application of the Brøndsted-Rockafellar theorem to the function $\vartheta_k(x,y)$ from (5.10) gives us sequences of $(\widetilde{x}_k,\widetilde{y}_k) \in \operatorname{dom}\vartheta_k$ and $(\widetilde{x}_k^*,\widetilde{y}_k^*) \in \partial\vartheta_k(\widetilde{x}_k,\widetilde{y}_k)$ satisfying the estimates in (5.11) and the subdifferential inclusions (5.12) for all $k\in I\!N$. Since the function $\varphi+\delta(\cdot;\Gamma)$ is convex, its analytic ε-subdifferential in (5.12) can be written in form (2.9). By the assumption on $\Gamma\subset\operatorname{dom}\varphi$ we therefore have from (5.12) that

$$\langle \widetilde{x}_k^*, x-\widetilde{x}_k\rangle + \langle \widetilde{y}_k^*, y-\widetilde{y}_k\rangle \le \varphi(x,y) - \varphi(\widetilde{x}_k,\widetilde{y}_k) + 2\varepsilon_k\big(\|x-\widetilde{x}_k\| + \|y-\widetilde{y}_k\|\big)$$

for all $(x,y)\in\Gamma$ and $k\in I\!N$. The latter implies, by picking any $\gamma>0$ and using the l.s.c. of φ around $(\bar{x},\bar{y})$, that

$$\begin{aligned}\lambda_k\big[\langle \widetilde{x}_k^*, x-\widetilde{x}_k\rangle + \langle \widetilde{y}_k^*, y-\widetilde{y}_k\rangle\big] &\le \lambda_k\big[\varphi(x,y) - \varphi(\widetilde{x}_k,\widetilde{y}_k) + 2\varepsilon_k\big(\|x-\widetilde{x}_k\| + \|y-\widetilde{y}_k\|\big)\big]\\ &\le \lambda_k\big[\varphi(x,y) - \varphi(\bar{x},\bar{y}) + \gamma + 2\varepsilon_k\big(\|x-\widetilde{x}_k\| + \|y-\widetilde{y}_k\|\big)\big]\end{aligned}$$

for all $(x,y)\in\Gamma$ and all $k\in I\!N$ sufficiently large. Passing there to the limit as $k\to\infty$ and taking into account that the sequence $\{\widetilde{y}_k^*\}$ is bounded in Y^*, that $\lambda_k\downarrow 0$, and that $\lambda_k\widetilde{x}_k^* \xrightarrow{w^*} u^*$ by (5.11), we get the relationship

$$\langle u^*, x-\bar{x}\rangle \le 0\ \ \text{for all}\ \ (x,y)\in\Gamma,$$

which is equivalent to $(u^*,0)\in N((\bar{x},\bar{y});\Gamma)$ by (2.6). Applying now the *normal cone calculus* from Corollary 3.4 valid under the assumed qualification condition (5.13), we arrive at

$$(u^*,0) \in \bigcup_{\lambda\in A(\bar{x},\bar{y})} \Big[\sum_{t\in\operatorname{supp}\lambda} \lambda_t\partial\varphi_t(\bar{x},\bar{y})\Big] + N\big((\bar{x},\bar{y});\Omega\big)$$

with $A(\bar{x},\bar{y}) = \{\lambda\in\widetilde{I\!R}^T_+ |\ \lambda_t\varphi_t(\bar{x},\bar{y}) = 0,\ t\in\operatorname{supp}\lambda\}$. The latter yields (5.14) with $\Lambda^\infty(\bar{x},\bar{y})$ defined in (5.15) by using the arguments similar to the proof of the last part of Theorem 5.4. This completes the proof of the theorem. △

Next we obtain efficient applications of the upper estimates for the basic and singular subdifferentials of the value function $\mu(\cdot)$ given in Theorem 5.4 and Theorem 5.7 to the local *Lipschitz continuity* of $\mu(\cdot)$ and *necessary optimality conditions* for the parametric

DC infinite (and hence also semi-infinite) programs (1.1)–(1.3). These two types of results (Lipschitz stability and optimality conditions) are very much interrelated and are both based on the two fundamental issues in variational analysis and generalized differentiation in the framework of *Asplund spaces* [13]:

(a) *nonemptiness of the basic subdifferential for locally Lipschitzian functions*;
(b) *full subdifferential characterization of Lipschitz continuity.*

We summarize these results in the following lemma, with more specific references and comments in the lines of its proof. Note that for any Asplund space X the dual ball $I\!B^*$ is sequentially weak* compact in X^*, i.e., we meet the requirements of Theorem 5.4 and Theorem 5.7 assuming that the spaces X and Y in (1.1)–(1.3) are Asplund.

Lemma 5.9 (nonemptiness of the basic subdifferential and subdifferential characterization of Lipschitz continuity in Asplund spaces). *Let X be Asplund, and let $\varphi\colon X \to \overline{I\!R}$ be finite at $\bar{x}$. Then the following hold:*

(i) $\partial\varphi(\bar{x}) \neq \emptyset$ *provided that φ is locally Lipschitzian around $\bar{x} \in \mathrm{int}(\mathrm{dom}\,\varphi)$.*

(ii) *φ is locally Lipschitzian around $\bar{x} \in \mathrm{int}(\mathrm{dom}\,\varphi)$ if and only if it is l.s.c. around this point, the singular subdifferential of φ is trivial at $\bar{x}$, i.e.,*

$$\partial^\infty\varphi(\bar{x}) = \{0\}, \tag{5.16}$$

and for any sequences $\lambda_k \downarrow 0$, $x_k \xrightarrow{\varphi} \bar{x}$, and $x_k^* \in \lambda_k\widehat{\partial}\varphi(x_k)$ as $k \in I\!N$ we have the implication

$$\big[x_k^* \xrightarrow{w^*} 0\big] \Longrightarrow \big[\|x_k^*\| \to 0\big] \quad \text{as} \quad k \to \infty. \tag{5.17}$$

Proof. Assertion (i) is established in [13, Corollary 2.25] as a direct consequence of the *extremal principle*. The Lipschitzian characterization in (ii) is a combination of the two results from [13]: Theorem 3.52 where the local Lipschitz continuity is characterized via the simultaneous fulfillment of (5.16) and the so-called "sequential normal epi-compactness" (SNEC) property of l.s.c. functions, and Corollary 2.39 where the SNEC property is characterized in terms (5.17). In general, assertion (ii) of the theorem is a consequence of the *coderivative characterization* of the Lipschitz-like/Aubin property of set-valued mapping given in [13, Theorem 4.10]. Observe that the SNEC part (5.17) of this lemma holds automatically in finite dimensions, where the local Lipschitz continuity of l.s.c. functions is thus fully characterized by (5.16); cf. [20, Therem 9.13 and Theorem 9.40]. $\triangle$

Now based on Lemma 5.9 and the subdifferential estimates of Theorems 5.4 and 5.7, we obtain verifiable conditions for the local Lipschitz continuity of the value function $\mu(\cdot)$ in (1.1)–(1.3) and necessary optimality conditions for the class of parametric DC infinite programs under consideration. Recall that a set-valued mapping $S\colon X \rightrightarrows Y$ is *Lipschitz-like* around $(\bar{x}, \bar{y}) \in \mathrm{gph}\,S$ if there are $\ell \geq 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$S(x) \cap\ V \subset S(u) + \ell\|x - u\|I\!B \ \text{ for all } \ x, u \in U.$$

This property has been well recognized in nonlinear analysis and optimization as the most natural extension of the classical Lipschitz continuity to set-valued mappings, which is equivalent to the *metric regularity* and *linear openness* properties of the inverse S^{-1}.

Theorem 5.10 (Lipschitz continuity of value functions and necessary optimality conditions for parametric DC infinite programs.) *Let in the assumptions of Theorem* 5.8 *the parameter space* X *be Asplund* (*which implies the weak** *sequential compactness of the unit ball in* X^*) *and suppose in addition that*

$$\Big\{\bigcup_{\lambda\in\Lambda^\infty(\bar{x},\bar{y})}\Big[\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial_x\varphi_t(\bar{x},\bar{y})\Big]+N_X\big((\bar{x},\bar{y});\Omega\big)\Big\}=\{0\} \tag{5.18}$$

with the set of singular multipliers defined in (5.15)*. Then the value function* $\mu(\cdot)$ *is locally Lipschitzian around* $\bar{x}$ *provided that it is l.s.c. around this point* (*which is ensured by the inner semicontinuity of of* $M(\cdot)$ *around* $(\bar{x},\bar{y})$) *in each of the following cases:*

(a) *either* X *is finite dimensional,*

(b) *or both* φ *and* ψ *are continuous at* $(\bar{x},\bar{y})$ *and the mapping* $F(x)\cap G(x)$ *given in* (1.2) *and* (1.3) *is Lipschitz-like around* $(\bar{x},\bar{y})$*.*

If furthermore the qualification condition (4.1) *holds, then we have the following necessary optimality conditions for the minimizer* $\bar{y}$ *to the parametric DC infinite program* (4.15)*: there are* $(x^*,y^*)\in\partial\psi(\bar{x},\bar{y})$*,* $u^*\in X^*$*, and* $\lambda\in\widetilde{I\!R}^T_+$ *from* (3.3) *satisfying the relationships*

$$\begin{cases} u^*+x^*\in\partial_x\varphi(\bar{x},\bar{y})+\displaystyle\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial_x\varphi_t(\bar{x},\bar{y})+N_X\big((\bar{x},\bar{y});\Omega\big),\\ y^*\in\partial_y\varphi(\bar{x},\bar{y})+\displaystyle\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial_y\varphi_t(\bar{x},\bar{y})+N_Y\big((\bar{x},\bar{y});\Omega\big),\\ \lambda_t\varphi_t(\bar{x},\bar{y})=0\ \ \textit{for all}\ \ t\in\operatorname{supp}\lambda.\end{cases} \tag{5.19}$$

Proof. If (5.18) holds, then $\partial^\infty\mu(\bar{x})=\{0\}$ by Theorem 5.8. We can easily check by definitions that the lower semicontinuity of $\mu(\cdot)$ around $\bar{x}$ follows from the inner semicontinuity of $M(\cdot)$ around $(\bar{x},\bar{y})$. Thus the local Lipschitz continuity of $\mu(\cdot)$ around $\bar{x}$ in case (a) of the theorem follows directly from condition (5.16) of Lemma 5.9(ii), since the SNEC property (5.17) is automatic in finite-dimensional spaces.

In the Asplund case (b) of the theorem, observe that the continuity assumptions on the convex functions φ and ψ at $(\bar{x},\bar{y})$ imply their Lipschitz continuity around this point. Then we employ [15, Theorem 5.2(i)], which ensures the SNEC property (5.17) of the value function $\mu(\cdot)$ in (1.1) at the point $\bar{x}$ provided that the cost function $\varphi-\psi$ is locally Lipschitzian around $(\bar{x},\bar{y})$ and the constraint mapping $F(\cdot)\cap G(\cdot)$ is Lipschitz-like around this point. Thus we conclude from Lemma 5.9(ii) that the value function $\mu(\cdot)$ is locally Lipschitzian around $\bar{x}$ under the assumptions imposed in case (b) of the theorem.

If furthermore the qualification condition (4.1) is satisfied, then we can use the upper estimate (5.8) for the basic subdifferential of the value function $\mu(\cdot)$ obtained in Theorem 5.7. Since $\partial\mu(\bar{x})\neq\emptyset$ by Lemma 5.9(i), the right-hand side of (5.8) is nonempty as well. Taking into account construction (4.4) of the KKT multiplier set $\Lambda(\bar{x},\bar{y},y^*)$, we arrive at the necessary optimality conditions (5.19) and complete the proof of the theorem. △

Note that verifiable pointwise conditions ensuring the Lipschitz-like property of the constraint mapping $F(x)\cap G(x)$ imposed in case (b) of Theorem 5.10 easily follow from [13, Theorem 4.37] in the case of *finitely many* inequalities in (1.3). In particular, in the case of smooth functions φ_t this property holds for such constraint systems under the classical

Mangasarian-Fromovitz constraint qualification; see [13, Corollary 4.39]. The case of *infinite constraints* in (1.3) is more challenging and requires further investigation.

All the results obtained above can be specified for the two remarkable subclasses of the general DC programs (1.1)–(1.3): *convex infinite programs* with $\psi = 0$ in (1.1) and *concave infinite programs* with $\varphi = 0$ in (1.1). In this way we do not observe any special phenomena for the case of *concave* programming in comparison with the general DC case, while the specifications of all the results derived in Sections 4 and 5 by putting $\varphi = 0$ therein seem to be new for this important and nonconventional class of infinite and semi-infinite programs.

The *convex* case is different from this viewpoint: it does provide specific results, which are *improvements* of those for the general case of DC infinite programs. First of all, for convex programs we do *not need* imposing any subdifferential inner semicontinuity and/or subdifferential boundedness conditions and the corresponding requirements on the *sequential weak* compactness* of the dual balls in the results of Section 5. Furthermore, the value function in (1.1)–(1.3) happens to be *convex* when $\psi = 0$, and thus both the Fréchet subdifferential $\widehat{\partial}\mu(\bar{x})$ in Section 4 and the basic subdifferential $\partial\mu(\bar{x})$ in Section 5 reduce to the subdifferential of convex analysis for which the condition $\partial\mu(\bar{x}) \neq \emptyset$ imposed, in particular, in Corollary 4.3 is *not restrictive*. We refer the reader to [5], where a comprehensive study of the latter condition is given for some important special classes of convex infinite programs.

Finally, the case of convex infinite programs allows us to establish the following *precise formula* for computing the subdifferential $\partial\mu(\bar{x})$ of the value function in (1.1)–(1.3) with $\psi = 0$, which does *not* have analogs in the general framework of DC infinite programs.

Theorem 5.11 (precise formula for computing subgradients of value functions in convex infinite programming). *Let $\psi = 0$ in problem (1.1)–(1.3) formulated in arbitrary Banach spaces, where the other data of this problem satisfy the standing assumptions of Section 1 that imply the convexity of the value function $\mu(\cdot)$. Suppose also that the qualification condition (4.1) holds and that $\operatorname{dom} M \neq \emptyset$ for the argminimum mapping $M(\cdot)$ defined in (4.2) with $\psi = 0$. Then given any $(\bar{x}, \bar{y}) \in \operatorname{gph} M$, the subdifferential of $\mu(\cdot)$ at $\bar{x}$ in the sense of convex analysis is computed by*

$$\partial\mu(\bar{x}) = \Big\{x^* \in X^* \Big| \; (x^*, 0) \in \partial\varphi(\bar{x}, \bar{y}) + \bigcup_{\lambda \in A(\bar{x}, \bar{y})} \Big[\sum_{t \in \operatorname{supp}\lambda} \lambda_t \partial\varphi_t(\bar{x}, \bar{y})\Big] + N\big((\bar{x}, \bar{y}); \Omega\big)\Big\}, \tag{5.20}$$

where the set $A(\bar{x}, \bar{y})$ of active constraint multipliers at $(\bar{x}, \bar{y})$ is defined by

$$A(\bar{x}, \bar{y}) := \big\{\lambda \in \widetilde{I\!R}^T_+ \big| \; \lambda_t \varphi_t(\bar{x}, \bar{y}) = 0 \;\; \text{for all} \;\; t \in \operatorname{supp}\lambda\big\}. \tag{5.21}$$

Proof. It is not hard to derive from the definition of convexity that the value function $\mu(\cdot)$ in (1.1)–(1.3) as $\psi = 0$ is *convex* under the standing convexity assumptions on the initial data of this problem; see, e.g., [2, Lemma 4.2.2], where it is done in the case when $F(x) \cap G(x)$ is a constant set. Let us first justify the inclusion "$\subset$" in (5.20).

Take any $x^* \in \partial\mu(\bar{x})$ and get by the subdifferential definition of convex analysis that

$$\mu(x) - \mu(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle \;\; \text{for any} \;\; x \in X,$$

which corresponds to (4.6) in the proof of Theorem 4.1 with $\gamma = 0$ and $\eta = \infty$ therein. Taking this into account and repeating the proof of Theorem 4.1 till using the partial subdifferential representations in (4.11) not needed now, we get

$$(x^*, 0) \in \partial\varphi(\bar{x}, \bar{y}) + \bigcup_{\lambda \in A(\bar{x},\bar{y})} \Big[\sum_{t \in \operatorname{supp}\lambda} \lambda_t \partial\varphi_t(\bar{x}, \bar{y}) \Big] + N\big((\bar{x}, \bar{y}); \Omega\big),$$

which justifies the inclusion "$\subset$" in (5.20).

To prove the opposite inclusion, take any $x^* \in X^*$ such that $(x^*, 0)$ belongs to the right-hand side of (5.20) and thus find $\lambda \in A(\bar{x}, \bar{y})$, $(u^*, v^*) \in \partial\varphi(\bar{x}, \bar{y})$, $(u_t^*, v_t^*) \in \partial\varphi_t(\bar{x}, \bar{y})$, and $(\widetilde{u}^*, \widetilde{v}^*) \in N((\bar{x}, \bar{y}); \Omega)$ such that

$$(x^*, 0) = (u^*, v^*) + \sum_{t \in \operatorname{supp}\lambda} \lambda_t (u_t^*, v_t^*) + (\widetilde{u}^*, \widetilde{v}^*). \tag{5.22}$$

Then using (5.21), definition (4.3) of the feasible solution set Γ, and the underlying definitions of convex analysis for the subgradients and normals in (5.22), we have

$$\begin{cases} \varphi(x, y) - \mu(\bar{x}) = \varphi(x, y) - \varphi(\bar{x}, \bar{y}) \ge \langle u^*, x - \bar{x} \rangle + \langle v^*, y - \bar{y} \rangle, \\ 0 \ge \lambda_t \varphi_t(x, y) - \lambda_t \varphi_t(\bar{x}, \bar{y}) \ge \lambda_t \langle u_t^*, x - \bar{x} \rangle + \lambda_t \langle v_t^*, y - \bar{y} \rangle, \quad t \in \operatorname{supp}\lambda, \\ 0 \ge \langle \widetilde{u}^*, x - \bar{x} \rangle + \langle \widetilde{v}^*, y - \bar{y} \rangle \ \text{ for all } \ (x, y) \in \Gamma. \end{cases}$$

The latter inequalities together with representation (5.22) immediately imply that

$$\varphi(x, y) + \delta\big((x, y); \Gamma\big) - \mu(\bar{x}) \ge \langle x^*, x - \bar{x} \rangle \ \text{ for all } \ (x, y) \in X \times Y,$$

which gives $\mu(x) - \mu(\bar{x}) \ge \langle x^*, x - \bar{x} \rangle$ for all $x \in X$ due to the construction of the value function $\mu(\cdot)$ in (1.1)–(1.3) with $\psi = 0$. Therefore $x^* \in \partial\mu(\bar{x})$, and we thus justify the inclusion "$\supset$" in (5.20) and complete the proof of the theorem. $\triangle$

In the next section we give efficient applications of the latter theorem and other results of this paper to a new class of hierarchical optimization problems labeled as *bilevel infinite programs*. The necessary optimality conditions obtained in this way *essentially improve* known results *even for standard bilevel programs* with *finitely many* constraints in both finite-dimensional and infinite-dimensional spaces.

6 Applications to Bilevel Programming

Bilevel programming concerns a broad class of *two-level* hierarchical optimization problems, where the set of *feasible solutions* to the *upper-level* problem consists of *optimal solutions* to the *lower-level* problem of *parametric optimization*; see the book [3] and the extended introduction to the recent paper [4] for comprehensive discussions, various examples, results, and references. In this paper we study the so-called *optimistic version* of bilevel programming dealing with optimization problems of the following type:

$$\begin{cases} \text{minimize } \ f(x, y) \ \text{ subject to} \\ y \in M(x) := \big\{ y \in G(x) \big| \ \varphi(x, y) = \mu(x) \big\}, \end{cases} \tag{6.1}$$

where $M(x)$ is a parameter-dependent set of *optimal solutions* to the *lower-level problem*

$$\text{minimize } \varphi(x,y) \text{ subject to } y \in G(x) := \big\{ y \in Y \big|\, \varphi_t(x,y) \le 0,\ t \in T \big\}, \tag{6.2}$$

and where $\mu(\cdot)$ is the *value function* to the parametric lower-level problem:

$$\mu(x) := \inf \big\{ \varphi(x,y) \big|\, y \in G(x) \big\}. \tag{6.3}$$

As above, the index set T in the inequality constraints of the lower-level problem (6.2) is *arbitrary*, and thus we generally refer to (6.1) as to a *bilevel infinite program*. Of course, this includes the standard case in bilevel programming when T is finite; in the latter case we specify (6.1) as a bilevel program with *finitely many* constraints.

Our *standing assumptions* on the initial data $\varphi\colon X \times Y \to \overline{I\!R}$ and $\varphi_t\colon X \times Y \to \overline{I\!R}$ of the lower-level problem (6.2) are the same as those imposed in Section 1 for the whole paper: *properness, lower semicontinuity, and convexity.* We impose the same assumptions on the cost/objective function $f\colon X \times Y \to \overline{I\!R}$ of the upper-level problem in (6.1). Bilevel programs of this type are called *fully convex.* The spaces X and Y under consideration in this paper are *arbitrary Banach.*

The reader immediately recognizes that the lower-level problem (6.2) is a parametric *convex infinite program*, which is a particular case of the parametric DC infinite program formulated in (1.1) and (1.3) with $\psi = 0$ and the absence of the geometric constraints (1.2). Note that we can easily include the latter constraints into the lower-level problems as well as include additional *convex* geometric and/or functional constraints into the upper-level problem in (6.1); they are dropped for *simplicity.*

It turns out that, involving a certain "partial calmness" qualification assumption, the fully convex bilevel problem under consideration can be *equivalently* reduced to a *DC infinite program*, which contains (as the "minus" function in the DC objective) the *convex value function* (6.3) to the lower-level problem (6.2). Applying further the necessary optimality conditions for DC programs and the subdifferential formula for the value function obtained above, we derive in this way verifiable necessary optimality conditions in bilevel programming, which seem to be the *first results* in the literature for *infinite* bilevel programs while also *significantly improve* previously known optimality conditions for bilevel programs with *finitely many* constraints of this type; see the results and comments below.

To proceed, we rewrite the bilevel problem (6.1) in the (globally) *equivalent* form

$$\begin{cases} \text{minimize } f(x,y) \text{ subject to} \\ \varphi(x,y) - \mu(x) \le 0, \quad y \in G(x) \end{cases}$$

and consider its *perturbed* version linearly parameterized by $p \in I\!R$:

$$\begin{cases} \text{minimize } f(x,y) \text{ subject to} \\ \varphi(x,y) - \mu(x) + p = 0, \quad y \in G(x). \end{cases} \tag{6.4}$$

Following [22], we say that the unperturbed problem problem (6.1) is *partially calm* at its feasible solution $(\bar{x},\bar{y})$ if there are a constant $\nu > 0$ and a neighborhood U of the triple $(\bar{x},\bar{y},0) \in X \times Y \times I\!R$ such that

$$f(x,y) - f(\bar{x},\bar{y}) + \nu|p| \ge 0 \text{ for all } (x,y,p) \in U \text{ feasible to (6.4).} \tag{6.5}$$

In this case we also say that $(\bar{x},\bar{y})$ is a *partially calm feasible solution* to (6.1). In the original paper [22] and in the recent one [4], the reader can find various discussions on partial calmness, its relationships with other constraint qualifications, and efficient conditions for its validity for important classes of optimization problems. In particular, this condition always holds at *optimal solutions* to the lower-level problem when the latter is *either linear or* admits a *uniform weak sharp minimizer*, for classes of nonlinear problems allowing the so-called *exact penalization*, etc.

The following lemma justifies the possibility to reduce, under partial calmness, the initial bilevel program (6.1) to a *one-level DC* optimization problem with infinitely many constraints. In fact, this result needs only the *continuity* assumption on the (nonconvex) upper level objective in (6.1) with no other requirements on the initial data; cf. [22, Proposition 3.3], where a similar penalization statement is formulated without proof for a standard bilevel program with Lipschitzian data.

Lemma 6.1 (penalization of bilevel infinite programs). *Let $(\bar{x},\bar{y})$ be a partially calm feasible solution to the bilevel program* (6.1) *with $G\colon X \rightrightarrows Y$ given in* (6.2)*, and let the upper-level objective $f(\cdot)$ be continuous at this point. Then $(\bar{x},\bar{y})$ is a local optimal solution to the penalized problem*

$$\begin{cases} \text{minimize } \ \nu^{-1}f(x,y)+\varphi(x,y)-\mu(x) \\ \text{subject to } \ \varphi_t(x,y)\le 0,\ t\in T, \end{cases} \tag{6.6}$$

where $\nu>0$ is the constant from the partial calmness condition (6.5).

Proof. By the partial calmness of (6.1) we have $\nu>0$ and a neighborhood U of $(\bar{x},\bar{y},0)$ for which (6.5) is satisfied. It follows from the continuity of f at $(\bar{x},\bar{y})$ that there are $\gamma>0$ and $\eta>0$ such that $V:=[(\bar{x},\bar{y})+\eta I\!B]\times(-\gamma,\gamma)\subset U$ and that

$$|f(x,y)-f(\bar{x},\bar{y})|\le\nu\gamma \ \text{ whenever } \ (x,y)-(\bar{x},\bar{y})\in\eta I\!B.$$

This allows us to establish the relationship

$$f(x,y)-f(\bar{x},\bar{y})+\nu\big(\varphi(x,y)-\mu(x)\big)\ge 0 \ \text{ for all } \ (x,y)\in[(\bar{x},\bar{y})+\eta I\!B]\cap\operatorname{gph}G \tag{6.7}$$

with $G\colon X\rightrightarrows Y$ defined in (6.2). If $\big(x,y,\mu(x)-\varphi(x,y)\big)\in V$, then (6.7) follows directly from the partial calmness condition in (6.5). If otherwise $\big(x,y,\mu(x)-\varphi(x,y)\big)\notin V$, we get $\varphi(x,y)-\mu(x)\ge\gamma$ and hence $\nu\big(\varphi(x,y)-\mu(x)\big)\ge\nu\gamma$. This also implies (6.7) due to $f(x,y)-f(\bar{x},\bar{y})\ge-\nu\gamma$. To complete the proof of the lemma, it remains to observe that $\varphi(\bar{x},\bar{y})-\mu(\bar{x})=0$, since $(\bar{x},\bar{y})$ is a feasible solution to (6.1). △

The next theorem provides an efficient *upper estimate* for the convex *subdifferential* of the *value function* (6.3) at *partially calm* feasible solutions to the bilevel program. It is certainly of its own interest while playing a crucial rule, together with Theorem 5.11 of Section 5, in establishing the main result of this section (Theorem 6.3) on necessary optimality conditions for the bilevel problems under consideration.

Theorem 6.2 (subgradients of value functions at partially calm feasible solutions to bilevel programs). *Let $(\bar{x},\bar{y})$ be a partially calm feasible solution to the bilevel program* (6.1). *In addition to the standing assumption of this section, suppose that the qualification condition* (4.1) *is satisfied for the lower-level problem* (6.2) *and that the cost function $f(\cdot)$ of the upper-level problem is continuous at $(\bar{x},\bar{y})$. Then there is a number $\nu>0$ such that*

$$\partial\mu(\bar{x})\times\{0\}\subset\nu^{-1}\partial f(\bar{x},\bar{y})+\partial\varphi(\bar{x},\bar{y})+\bigcup_{\lambda\in A(\bar{x},\bar{y})}\Big[\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial\varphi_t(\bar{x},\bar{y})\Big] \tag{6.8}$$

for the convex value function (6.3), *where the set $A(\bar{x},\bar{y})$ of active constraint multipliers is defined in* (5.21). *In particular, we have the upper estimate*

$$\partial\mu(\bar{x})\subset\nu^{-1}\partial_x f(\bar{x},\bar{y})+\partial_x\varphi(\bar{x},\bar{y})+\bigcup_{\lambda\in A(\bar{x},\bar{y})}\Big[\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial_x\varphi_t(\bar{x},\bar{y})\Big]. \tag{6.9}$$

Proof. Fix $(\bar{x},\bar{y})$ satisfying the assumptions of the theorem. Lemma 6.1 ensures that $(\bar{x},\bar{y})$ is a local minimizer to the penalized problem (6.6), which is a *DC infinite program* of type (3.1) described in the space $X\times Y$ by the l.s.c. convex functions

$$\vartheta(x,y):=\nu^{-1}f(x,y)+\varphi(x,y),\quad \theta(x,y):=\mu(x),\ \text{ and }\ \vartheta_t(x,y):=\varphi_t(x,y) \tag{6.10}$$

with $\Theta=X\times Y$ in (3.1). Let us show that the assumed qualification condition (4.1) implies the fulfillment of the CQC condition from Definition 3.1 in the space $X^*\times Y^*\times I\!R$ for the functions ϑ and ϑ_t defined in (6.10). Using the structure of the feasible set

$$\Xi:=\big\{(x,y)\in X\times Y\big|\ \varphi_t(x,y)\le 0\ \text{ for all }\ t\in T\big\}$$

to the DC infinite program (6.6), the conjugate epigraphical rule (2.4), and the qualification condition (4.1), we get the chain of equalities:

$$\begin{aligned}\operatorname{epi}\big(\varphi+\delta(\cdot;\Xi)\big)^* &= \operatorname{cl}^*\big(\operatorname{epi}\varphi^*+\operatorname{epi}\delta^*(\cdot;\Xi)\big)=\operatorname{cl}^*\Big\{\operatorname{epi}\varphi^*+\operatorname{cl}^*\Big(\operatorname{cone}\Big[\bigcup_{t\in T}\operatorname{epi}\varphi_t^*\Big]\Big)\Big\}\\ &= \operatorname{cl}^*\Big\{\operatorname{epi}\varphi^*+\operatorname{cone}\Big[\bigcup_{t\in T}\operatorname{epi}\varphi_t^*\Big]\Big\}=\operatorname{epi}\varphi^*+\operatorname{cone}\Big[\bigcup_{t\in T}\operatorname{epi}\varphi_t^*\Big].\end{aligned}$$

Further, the refined conjugate epigraphical rule from Lemma 2.1(ii) applied to the sum of functions in (6.10) by the assumed continuity of $f(\cdot)$ at $(\bar{x},\bar{y})$ gives the equalities

$$\begin{aligned}\operatorname{epi}\vartheta^*+\operatorname{cone}\Big[\bigcup_{t\in T}\operatorname{epi}\vartheta_t^*\Big] &= \operatorname{epi}\big(\nu^{-1}f\big)^*+\operatorname{epi}\varphi^*+\operatorname{cone}\Big[\bigcup_{t\in T}\operatorname{epi}\varphi_t^*\Big]\\ &= \operatorname{epi}\big(\nu^{-1}f\big)^*+\operatorname{epi}\big(\varphi+\delta(\cdot;\Xi)\big)^*=\operatorname{epi}\big(\vartheta+\delta(\cdot;\Xi)\big)^*.\end{aligned}$$

This allows us to conclude that the set

$$\operatorname{epi}\vartheta^*+\operatorname{cone}\Big[\bigcup_{t\in T}\operatorname{epi}\vartheta_t^*\Big]\ \text{ is weak}^*\text{ closed in }\ X^*\times Y^*\times I\!R,$$

which is exactly the CQC requirement for the application of Theorem 3.2 to the DC problem (6.6). Employing the latter result and the subdifferential sum rule

$$\partial\vartheta(\bar{x},\bar{y})=\partial\big(\nu^{-1}f+\varphi\big)(\bar{x},\bar{y})=\nu^{-1}\partial f(\bar{x},\bar{y})+\partial\varphi(\bar{x},\bar{y})$$

held by the continuity of $f(\cdot)$, we arrive at the general inclusion (6.8) for subgradients of the value function claimed the theorem. The upper estimate in (6.9) immediately follows from (6.8) due to the relationships (4.11) between the full and partial subdifferentials of convex functions. This completes the proof of the theorem. $\triangle$

Now we are ready to establish the main result of this section providing subdifferential necessary optimality conditions for the fully convex bilevel programs with infinitely many (in particular, finitely many) inequality constraints.

Theorem 6.3 (necessary optimality condition for bilevel infinite programs). *Let $(\bar{x},\bar{y})$ be a partially calm optimal solution to the bilevel program* (6.1) *satisfying the standing assumptions of this section. Suppose in addition that the qualification condition* (4.1) *is fulfilled for the lower-level problem* (6.2)*, that the upper objective $f(\cdot)$ is continuous at $(\bar{x},\bar{y})$, and that $\partial\mu(\bar{x})\neq\emptyset$ for the convex value function* (6.3)*. Then for each $\widetilde{y}\in M(\bar{x})$ from the argminimum set in* (6.1) *there exist a number $\nu>0$ and multipliers $\lambda=(\lambda_t)\in\widetilde{I\!R}^T_+$ and $\beta=(\beta_t)\in\widetilde{I\!R}^T_+$ from the positive cone in* (3.3) *such that we have the relationships*

$$\begin{aligned} 0\ &\in\partial_x f(\bar{x},\bar{y})+\nu\big[\partial_x\varphi(\bar{x},\bar{y})-\partial_x\varphi(\bar{x},\widetilde{y})\big]+\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial_x\varphi_t(\bar{x},\bar{y})\\ &-\nu\sum_{t\in\operatorname{supp}\beta}\beta_t\partial_x\varphi_t(\bar{x},\widetilde{y}), \end{aligned}\tag{6.11}$$

$$0\in\partial_y f(\bar{x},\bar{y})+\nu\partial_y\varphi(\bar{x},\bar{y})+\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial_y\varphi_t(\bar{x},\bar{y}),\tag{6.12}$$

$$0\in\partial_y\varphi(\bar{x},\widetilde{y})+\sum_{t\in\operatorname{supp}\beta}\beta_t\partial_y\varphi_t(\bar{x},\widetilde{y}),\tag{6.13}$$

$$\lambda_t\varphi_t(\bar{x},\bar{y})=\beta_t\varphi_t(\bar{x},\widetilde{y})=0\ \text{ for all }\ t\in T.\tag{6.14}$$

Proof. Since $\partial\mu(\bar{x})\neq\emptyset$, we take $x^*\in\partial\mu(\bar{x})$ and by Theorem 6.2 find $\nu>0$ and $\lambda\in\widetilde{I\!R}^T_+$ satisfying the inclusion

$$\nu(x^*,0)\in\partial f(\bar{x},\bar{y})+\nu\partial\varphi(\bar{x},\bar{y})+\sum_{t\in\operatorname{supp}\lambda}\lambda_t\partial\varphi_t(\bar{x},\bar{y})\tag{6.15}$$

with $\lambda_t\varphi_t(\bar{x},\bar{y})=0$ for all $t\in\operatorname{supp}\lambda$. On the other hand, picking any $\widetilde{y}\in M(\bar{x})$ and applying to $x^*\in\partial\mu(\bar{x})$ the result of Theorem 5.11 and taking into account the partial subdifferential relationships (4.11), we find $\beta\in\widetilde{I\!R}^T_+$ such that

$$x^*\in\partial_x\varphi(\bar{x},\widetilde{y})+\sum_{t\in\operatorname{supp}\,\beta}\partial_x\varphi_t(\bar{x},\widetilde{y}),\quad 0\in\partial_y\varphi(\bar{x},\widetilde{y})+\sum_{t\in\operatorname{supp}\,\beta}\partial_y\varphi_t(\bar{x},\widetilde{y}),\tag{6.16}$$

and $\beta_t\varphi_t(\bar{x},\widetilde{y})=0$ for all $t\in\operatorname{supp}\beta$. Combining (6.15) and (6.16) and remembering the definition of "supp" in Section 3, we arrive at the optimality conditions (6.11)–(6.14) and thus complete the proof of the theorem. $\triangle$

As an immediate consequence of Theorem 6.3, we get the following necessary optimality conditions for the bilevel program (6.1) involving only the reference optimal solution $(\bar{x},\bar{y})$.

Corollary 6.4 (specification of necessary optimality conditions for bilevel programs). *Let* $(\bar{x},\bar{y})$ *be an optimal solution to the bilevel program* (6.1) *under all the assumptions of Theorem* 6.3. *Then there are* $\nu>0$ *and* $\lambda,\beta\in\widetilde{I\!R}^T_+$ *such that*

$$0\in\partial_x f(\bar{x},\bar{y})+\nu\big[\partial_x\varphi(\bar{x},\bar{y})-\partial_x\varphi(\bar{x},\bar{y})\big]+\sum_{t\in T}\Big[\big(\lambda_t-\nu\beta_t\big)\partial_x\varphi_t(\bar{x},\bar{y})\Big],$$

$$0\in\partial_y f(\bar{x},\bar{y})+\nu\partial_y\varphi(\bar{x},\bar{y})+\sum_{t\in T}\lambda_t\partial_y\varphi_t(\bar{x},\bar{y}),$$

$$0\in\partial_y\varphi(\bar{x},\bar{y})+\sum_{t\in T}\beta_t\partial_y\varphi_t(\bar{x},\bar{y}),$$

$$\lambda_t\varphi_t(\bar{x},\bar{y})=\beta_t\varphi_t(\bar{x},\bar{y})=0\ \ \textit{for all}\ \ t\in T.$$

Proof. Follows from Theorem 6.3 by taking $\widetilde{y}=\bar{y}\in M(\bar{x})$ in (6.11)–(6.14). △

Let us finally discuss the assumption $\partial\mu(\bar{x})\neq\emptyset$ in Theorem 6.3 and compare the results obtained above with those known in the literature.

Remark 6.5 (subdifferentiability of value functions in the lower-level problems). We have a number of verifiable conditions, which ensure that $\partial\mu(\bar{x})\neq\emptyset$ in the assumptions of Theorem 6.3 and Corollary 6.4, i.e., that the convex value function of the lower-level problem is *subdifferentiable* at $\bar{x}$. It has been recently shown in [5] that $\partial\mu(\bar{x})\neq\emptyset$ for a large class of convex infinite programs in arbitrary Banach spaces under some *closedness qualification condition* of the CQC type. If on the other hand the space X is *Asplund*, then the required subdifferentiability of the value function $\mu(\cdot)$ at $\bar{x}$ is implied by its *local Lipschitz continuity*, which in turn is ensured by the dual qualification condition (5.18) of the *Mangasarian-Fromovitz type* introduced and justified for infinite programs in Theorem 5.10.

Remark 6.6 (comparison with known results on optimality conditions for fully convex bilevel programs). To the best of our knowledge, Theorem 6.3 is the *first result in the literature* on necessary optimality conditions for *bilevel infinite* as well as *semi-infinite programs*. It turns out furthermore that the specifications of Theorem 6.3 and its Corollary 6.4 for finite index sets T provide *significant improvements* over previously known necessary optimality conditions for *fully convex* bilevel programs with *finitely many constraints*. The most advanced results for problems of the latter type have been recently obtained in [4, Section 4.1] in the finite-dimensional setting; see also the references and commentaries in [4]. In comparison with our Theorem 6.3, Theorem 4.1 from [4] establishes necessary optimality conditions of type (6.10)–(6.14) for such bilevel problems (6.1) with *some* (vs. *any*) element $\widetilde{y}\in M(\bar{x})$ therein assuming in addition that $M(\cdot)$ is uniformly bounded around $\bar{x}$ and imposing a more restrictive constraint qualification/regularity condition in the lower-level problem, which automatically implies the local Lipschitzian continuity of the value function $\mu(\cdot)$ around $\bar{x}$ and hence its subdifferentiability at this point. The possibility of choosing $\widetilde{y} = \bar{y}$ in (6.10)–(6.14) is justified in [4, Theorem 4.1] under the additional *inner semicontinuity* of $M(\cdot)$ at $(\bar{x}, \bar{y})$, which is *not required* in our Theorem 6.3 and Corollary 6.4. The latter condition is also not required in [4, Theorem 4.4] for bilevel problems of this type under the additional *smoothness* assumption imposed on all the data in (6.1) that is essentially employed in the proof. Nothing like that is needed in Theorem 6.3 and Corollary 6.4, which are proved by using variational techniques significantly different from those in [4].

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