J Glob Optim
DOI 10.1007/s10898-007-9243-0

On Nash–Cournot oligopolistic market equilibrium models with concave cost functions

Le D. Muu · V. H. Nguyen · N. V. Quy

Received: 9 November 2005 / Accepted: 16 September 2007

Abstract We consider Nash–Cournot oligopolistic market equilibrium models with concave cost functions. Concavity implies, in general, that a local equilibrium point is not necessarily a global one. We give conditions for existence of global equilibrium points. We then propose an algorithm for finding a global equilibrium point or for detecting that the problem is unsolvable. Numerical experiments on some randomly generated data show efficiency of the proposed algorithm.

Keywords Nonconvex Nash–Cournot model · Equilibrium · Concave cost · Variational inequality · Existence of solution · Algorithm

1 Introduction

Oligopolistic market equilibrium models have been introduced by Cournot and studied by some authors (see e.g. [7,9,15,20]). These models can be formulated as Nash equilibrium problems in the n-person noncooperative game theory. An oligopolistic market model concerns with n firms (producers) that produce a common homogeneous commodity. Each firm has a profit function which is the difference between the price and the cost. Each firm attempts to maximize its profit by choosing the corresponding production level on its strategy set. It has been shown [9,15] that an oligopolistic market model with concave profit functions

L. D. Muu (✉)
Institute of Mathematics, 18 Hoang Quoc Viet, 10137 Hanoi, Vietnam
e-mail: ldmuu@math.ac.vn

V. H. Nguyen
Département de Mathématiques, FUNDP unité d'Optimisation, Namur, Belgium
e-mail: Van.Hien.Nguyen@fundp.ac.be

N. V. Quy
Financial and Accounting Institute, Co Nhue, Tu Liem, 10137 Hanoi, Vietnam
e-mail: Quynv2002@yahoo.com

can be formulated equivalently as a monotone variational inequality. For finite dimensional variational inequalities the readers are refered to the interesting monographs [9,15].

Classical models have assumed linear or convex cost functions facilitating computation and proofs of existence. However, concave cost functions are an important feature of some practical problems, since the cost for a unit of the commodity decreases as its amount increases. In this paper we consider oligopolistic market equilibrium models with box constraints, where the price functions are linear, but not necessarily the same for every firm. Moreover the cost functions are assumed to be piecewise linear concave. Thus the profit functions are convex rather than concave as in classical models. Concavity of the cost function, in general, implies that a local equilibrium point is not necessarily a global one. We show that in this case the problem of finding a global equilibrium strategy can be formulated as a mixed variational inequality problem over a bounded rectangle, which, in contrast to the convex cost function case, may fail to exist an equilibrium point. We give conditions under which such a model has an equilibrium point. We then propose a decomposition algorithm for finding a global equilibrium point. The algorithm can also be used for detecting the existence of equilibria. We tested the algorithm by some randomly generated data. As it is expected, the numerical results show that the proposed algorithm is efficient when the number of the firms having concave cost functions is somewhat small. The number of the total variables may be much larger.

The paper is organized as follows. In the next section we describe the model and formulate it as a generalized variational inequality. Then we investigate the existence of global equilibria. The third section is devoted to description of the algorithm. We close the paper with some computational experiences and results.

2 The model with concave cost

In the oligopolistic market equilibrium model we are going to consider, it is assumed that there are n-firms producing a common homogenous commodity and that the price p_i of firm i depends on the total quantity $\sigma := \sum_{i=1}^{n} x_i$ of the commodity. Let $h_i(x_i)$ denote the cost of the firm i when its production level is x_i. Suppose that the profit of firm i is given by

$$f_i(x_1, \ldots, x_n) = x_i p_i \left(\sum_{i=1}^{n} x_i \right) - h_i(x_i)(i = 1, \ldots, n), \tag{2.1}$$

where h_i is the cost function of firm i that is assumed to be dependent only on its production level.

Let $U_i \subset I\!R$ $(i = 1, \ldots, n)$ denotes the strategy set of the firm i. Each firm seeks to maximize its own profit by choosing the corresponding production level under the presumption that the production of the other firms are parametric input. In this context, a Nash equilibrium is a production pattern in which no firm can increase its profit by changing its controlled variables. Thus under this equilibrium concept, each firm determines its best response given other firms' actions. Mathematically, a point $x^* = (x_1^*, \ldots, x_n^*) \in U := U_1 \times \cdots \times U_n$ is said to be a Nash-equilibrium if

$$f_i(x_1^*, \ldots, x_{i-1}^*, y_i, x_{i+1}^*, \ldots, x_n^*) \le f_i(x_1^*, \ldots, x_n^*), \quad \forall\, y_i \in U_i, \quad \forall\, i = 1, \ldots, n. \tag{2.2}$$

When h_i is affine, this market problem can be formulated as a special Nash equilibrium problem in the n-person noncooperative game theory, which in turn is a strongly monotone variational inequality (see e.g. [15]). Variational inequality formulation of exchange price equilibrium and network transportation models related to the oligopolistic market equilibrium problem are presented in some books and research papers (see e.g. [2,6,9,21,23] and the references thererein).

In classical Cournot models [9,15], the price and the cost functions for each firm are assumed to be affine of the forms.

$$\begin{aligned} &p_i(\sigma) \equiv p(\sigma) = \alpha_0 - \beta\sigma, \quad \alpha_0 \geq 0, \quad \beta > 0, \quad \text{with } \sigma = \textstyle\sum_{i=1}^n x_i, \\ &h_i(x_i) = \mu_i x_i + \xi_i, \quad \mu_i \geq 0, \quad \xi_i \geq 0 \; (i = 1, \ldots, n). \end{aligned} \tag{2.3}$$

In this case, it has been shown in [9,15] that the problem can be formulated equivalently as the convex quadratic problem

$$\min_{x\in U} \left\{ \frac{1}{2} x^T Qx + (\mu - \alpha)^T x \right\} \tag{QP}$$

where

$$Q := \begin{pmatrix} 2\beta & \beta & \beta & \ldots & \beta \\ \beta & 2\beta & \beta & \ldots & \beta \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta & \beta & \beta & \ldots & 2\beta \end{pmatrix}.$$

Since $\beta > 0$, Q is a symmetric and positive definite matrix. Hence problem (QP) has a unique optimal solution which is also the unique equilibrium point of the classical oligopolistic market equilibrium model.

The assumption that the cost depends linearly on the quantity of the commodity, in general, is not practical, since usually the cost per a unit of the action does decrease when the quantity of the commodity exceeds a certain amount. Taking into account this fact, in the sequel we consider oligopolistic market equilibrium models with piecewise-linear concave cost functions. Actually, we suppose that the cost functions h_i $(i = 1, \ldots, n)$ are increasingly piecewise-linear concave and that the price function $p(\sum_{j=1}^n x_j)$ can change from firm by firm. Namely, the price has the following form:

$$p_i(\sigma) := p_i\left(\sum_{j=1}^n x_j\right) = \alpha_i - \beta_i \sum_{j=1}^n x_j, \quad \alpha_i \geq 0, \quad \beta_i \geq 0 \quad (i = 1, \ldots, n). \tag{2.4}$$

In this case, we take

$$B := \begin{pmatrix} \beta_1 & 0 & 0 & \ldots & 0 \\ = 0 & \beta_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \beta_n \end{pmatrix}, \; \tilde{B} := \begin{pmatrix} 0 & \beta_1 & \beta_1 & \ldots & \beta_1 \\ \beta_2 & 0 & \beta_2 & \ldots & \beta_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta_n & \beta_n & \beta_n & \ldots & 0 \end{pmatrix} \tag{2.5}$$

and $h(x) := \sum_{i=1}^n h_i(x_i)$ with $h_i (i = 1, \ldots, n)$ being concave functions.

Obviously, B is a symmetric positive semidefinite matrix. Let

$$\alpha^T = (\alpha_1, \alpha_2, \ldots, \alpha_n), \quad F(x) := \tilde{B}x - \alpha, \quad \varphi(x) := x^T Bx + h(x). \tag{2.6}$$

Then the problem of finding a Nash equilibrium point defined by (2.2) with f given by (2.1) becomes the following variational inequality:

$$(P) \begin{cases} \text{find a point } x^* \in U \text{ such that} \\ \Phi(x^*, y) := \langle F(x^*), y - x^* \rangle + \varphi(y) - \varphi(x^*) \geq 0 \quad \forall y \in U \end{cases}$$

where F is affine and φ is a d.c function (difference of two convex funxtions) defined by (2.6) (see [13]). As usual, in the sequel, we shall refer to F and to φ as the cost mapping and the cost function respectively. In literature problem (P) is often called a mixed variational inequality (see e.g. [15]) because of the apperance of function φ.

Note that since φ is not convex, (P) is not necessarily equivalent to the following local problem

$$(LP) \begin{cases} \text{find a point } x^* \in U \text{ such that} \\ \Phi(x^*, y) := \langle F(x^*), y - x^* \rangle + \varphi(y) - \varphi(x^*) \geq 0 \quad \forall y \in U_* \cap U \end{cases}$$

where U_* is a neighbourhout of x^*. In what follows an equilibrium point of the model or a solution of (P) is understood a global one. Moreover, the variational inequality (P), in general, has no solution even if the feasible set U is compact and F, φ are continuous (see example below).

3 Existence of solution

In this section, we investigate the question for existence of solution to the oligopolistic market equilibrium models where the price is given by (2.4) and the cost of each firm is a concave function of its production quantity. It has been shown in the preceeding section that such a model can be formulated as the variational inequality (P) with F and φ defined by (2.6). As we have mentioned, in this case, variational inequality (P) may have no solution. To see this let us consider the following simple examples.

Example 3.1 Let $U = [-1, 1] \subset \mathbb{R}$. Find $x \in U$ such that:

$$\langle x, y - x \rangle + x^2 - y^2 \geq 0 \quad \forall y \in U. \tag{3.1}$$

In this example, $F(x) = x$ is a strongly monotone mapping, and $\varphi(x) = -x^2$ is a concave function. It is easy to verify that (3.1) has no solution.

Now, we look for conditions under which the variational inequality (P) admits a solution. We recall [3] that a multivalued mapping H is said to be upper semicontinuous at a point x if for every open set G containing $H(x)$, there exists a neighbouhout I of x such that $H(I) \subset G$. The mapping H is said to be upper semicontinuous on a set D if it is upper semicontinuous at every point of D.

We shall use the well known Kakutani fixed point theorem.

Theorem 3.1 (Kakutani fixed point theorem) *Let U be a convex, compact subset in $\mathbb{R}^n$ and $H : U \to 2^U$ be an upper semicontinuous mapping on U. Suppose that $H(x)$ is nonempty compact, convex for every $x \in U$. Then H has a fixed point, i.e., $x \in H(x)$.*

Let us define, for each $x \in U$,

$$\theta(x) := \min_{y \in U} \{\Phi(x, y) := \langle F(x), y - x \rangle + \varphi(y) - \varphi(x)\}. \tag{3.2}$$

$$H(x) := \arg\min_{y \in U} \{\langle F(x), y \rangle + \varphi(y)\}. \tag{3.2a}$$

Since U is compact and φ is continuous. θ is finite, and $H(x) \neq \emptyset$ for every x.

Lemma 3.1 *Suppose that U is compact and F and φ are continuous. Then*

(i) *θ is continuous and H is upper semicontinous on U.*
(ii) *$x^* \in U$ is a solution to (P) if and only if at least one of the following equivalent conditions holds*
(a) *$\theta(x^*) = 0$,*
(b) *$x^* \in H(x^*)$, i.e., x^* is a fixed point of the multivalued mapping H.*

Proof Statement (i) follows from the maximum theorem (Proposition 21 in [3], see also [4]).
(ii) It is easy to verify that if either $\theta(x^*) = 0$ or $x^* \in H(x^*)$, then x^* is a solution to (P). Conversely, suppose that x^* is a solution to (P). By definition we have

$$\langle F(x^*), y - x^* \rangle + \varphi(y) - \varphi(x^*) \geq 0 \quad \forall y \in U,$$

and $\theta(x^*) = \Phi(x^*, x^*) = 0$. On the other hand,

$$\langle F(x^*), x^* \rangle + \varphi(x^*) \leq \langle F(x^*), y \rangle + \varphi(y).$$

This means that x^* is a solution to the problem:

$$\min_{y \in U} \{ \langle F(x^*), y \rangle + \varphi(y) \}.$$

By the definition of H, it follows that $x^* \in H(x^*)$. □

Since $\theta(x) \leq 0$ and, by Lemma 3.1, x^* is a solution to (P) if and only if $\theta(x^*) = 0$, this function can be considered as a merit function for variational inequality (P). The merit function is a useful tool for study and for developing methods for variational inequalities [5,9,10,23,24,26]. In our case the function θ is very useful for checking whether a given point is a solution to (P) or not depending on its value at this point is zero or negative. This stopping criteria will be used in our algorithm to be described in the next section..

For simplicity of presentation we adopt the following notations.

Let $x^{-i} := (x_j)_{j \neq i}^n \in \mathbb{R}^{n-1}$, i.e., x^{-i} is the $(n-1)$-dimensional vector obtaining from $x \in I\!R^n$ by deleting ith component. Let $U^{-i} := \{x^{-i} \mid x \in U\}$. For each $y_i \in U_i$ and $x^{-i} \in U^{-i}$, we define

$$f_i(x^{-i}, y_i) := \alpha_i y_i - \left(\beta_i \sum_{j \neq i}^{n} x_j \right) y_i - \beta_i y_i^2 - h_i(y_i). \tag{3.3}$$

This is the profit of firm i when its production level is y_i and the production level of $(n-1)$-remaining firms is x^{-i}. Then we compute the optimal production level of firm i by solving the following n-optimization problems, each of them has one-variable:

$$\max_{y_i \in U_i} \{ f_i(x^{-i}, y_i) := \left(\alpha_i - \beta_i \sum_{j \neq i}^{n} x_j \right) y_i - \beta_i y_i^2 - h_i(y_i) \} (i = 1, \ldots, n) \tag{3.4}$$

Lemma 3.2 *Suppose $U_i \quad (i = 1, \ldots, n)$ are compact and $U = U_1 \times U_2 \times \cdots \times U_n$. If, for each fixed $x^{-i} \in U^{-i}$, the solution-set of every problem given in (3.4) is convex, then (P) has at least one solution.*

Proof Note that Problems (3.4) can be rewritten equivalently as

$$-\min_{y_i \in U_i} \left\{ -f_i(x^{-i}, y_i) := \left(\beta_i \sum_{j \neq i}^{n} x_j - \alpha_i \right) y_i + \beta_i y_i^2 + h_i(y_i) \right\} \; (i = 1, \ldots, n). \tag{3.4a}$$

Since

$$F(x) = \tilde{B}x - \alpha, \; \varphi(y) = y^T B y + \sum_{i=1}^{n} h_i(y_i),$$

using the definition of matrices $\tilde{B}$ and B we have

$$\langle F(x), y - x \rangle + \varphi(y) = -\sum_{i=1}^{n} f_i(x^{-i}, y_i).$$

Thus problem (3.2a), since $U = U_1 \times \cdots \times U_n$, is equivalent to n problems of one-variable

$$\max\{f_i(x^{-i}, y_i) \,|y_i \in U_i\} \; (i = 1, \ldots, n)$$

in the sense that a point $y^* := y^*(x) = (y_1^*(x), \ldots, y_n^*(x))$ is an optimal solution to (3.2a) if and only if $y_i^*(x)$ is an optimal solution to the problem $\max\{f_i(x^{-i}, y_i) \,|y_i \in U_i\}$ for every $i = 1, \ldots, n$.

Since all diagonal entries of the matrix $\tilde{B}$ are zero, for each fixed point $x \in U$, we have

$$\langle F(x), y \rangle + \varphi(y) = \langle \tilde{B}x - \alpha, y \rangle + y^T B y + h(y) = -\sum_{i=1}^{n} f_i(x^{-i}, y_i)$$

where $f_i(x^{-i}, y_i)$ is given by (3.3). Thus, $H(x)$ is convex if and only if the solution-set of (3.4) is convex for every i. From Lemma 3.1, H is upper semicontinuous, and since U is compact, by the Kakutani fixed point theorem, H has a fixed point which is a solution of (P). □

Now, we study conditions under which the solution-set of each problem given by (3.4) is singleton, hence convex. To this end, we suppose that the strategy set $U_i := [\eta_0^i, \eta_{n_i}^i]$ of firm i is divided into n_i intervals $0 \leq \eta_0^i < \eta_1^i \cdots < \eta_{n_i}^i < +\infty$ $(n_i \geq 1)$ and that on each subinterval the cost function h_i is increasingly affine. Thus the cost function h_i is piecewise-linear increasingly concave on U_i. More precisely, let

$$h_i(x_i) := \begin{cases} a_0^i x_i + b_0^i & \text{if} \quad \eta_0^i \leq x_i \leq \eta_1^i, \\ a_1^i x_i + b_1^i & \text{if} \quad \eta_1^i \leq x_i \leq \eta_2^i, \\ \ldots & \ldots \quad \ldots \\ a_{n_i-1}^i x_i + b_{n_i-1}^i & \text{if} \quad \eta_{n_i-1}^i \leq x_i \leq \eta_{n_i}^i \end{cases} \tag{3.5}$$

where we suppose that

$$+\infty > a_0^i > a_1^i > \cdots > a_{n_i}^i > 0, \quad \alpha_i > a_0^i \quad \forall i, \tag{3.5a}$$

$$0 \leq b_0^i < b_1^i < \cdots < b_{n_i}^i \quad (i = 1, \ldots, n). \tag{3.5b}$$

Hence

$$h_i(x_i) = \min_{0 \leq j \leq n_i - 1} \{a_j^i x_i + b_j^i\}.$$

The assumptions (3.5a) and (3.5b) are quite natural, since they mean that the variable coefficient cost decreases as the production quantity gets larger. Clearly, $n_i = 1$ means that h_i is affine on the whole interval U_i.

Proposition 3.1 *Suppose that, for each $i = 1, \dots, n$, the utility function f_i is given by (3.3) with the cost function h_i given by (3.5) satisfying (3.5a) and (3.5b). Assume that the condition*

$$\alpha_i - a_0^i \geq \beta_i \Big(\sum_{j \neq i}^{n} \eta_{n_j}^j + 2\eta_{n_i - 1}^i \Big) \quad \forall\, i \in \bar{I} \tag{3.6}$$

holds true, where $\bar{I} = \{i : 1 \leq i \leq n, n_i > 1, \beta_i > 0\}$. Then problem (P) has at least one solution.

Proof We will prove that, for every $i = 1, \dots, n$, the solution-set of the problem

$$\max_{y_i \in U_i} \left\{ f(x^{-i}, y_i) := \alpha_i y_i - \left(\beta_i \sum_{j \neq i}^{n} x_j \right) y_i - \beta_i y_i^2 - h_i(y_i) \right\} \tag{3.7}$$

is convex.

Clearly, if $n_i = 1$, then h_i is affine on U_i. Hence the objective function of (3.7) is concave, and therefore its solution-set is convex.

Now let $I_0 := \{i \| \beta_i = 0\}$ and suppose $i \in \bar{I} \cup I_0$. Since $\alpha_i > a_0^i$, condition (3.6) is satisfied for every $i \in \bar{I} \cup I_0$. We will show that, for $i \in \bar{I} \cup I_0$, the objective function of (3.7) is monotone on U_i. Fix $y_i \in (\eta_{k-1}^i, \eta_k^i)$ for some $k \in \{1, 2, \dots, n_i - 1\}$. Then, by (3.5), $h_i(y_i) = a_{k-1}^i y_i + b_{k-1}^i$. The function $f_i(x^{-i}, .)$ is differentiable at y_i and

$$f'(x^{-i}, y_i) = \alpha_i - \beta_i \sum_{j \neq i}^{n} x_j - 2\beta_i y_i - a_{k-1}^i.$$

Thus,

$$f'(x^{-i}, y_i) > 0 \iff \alpha_i - a_{k-1}^i > \beta_i \left[\sum_{j \neq i}^{n} x_j + 2y_i \right]. \tag{3.8}$$

Since $x_j \in U_j = [\eta_0^j, \eta_{n_j}^j]$ and $y_i \in [\eta_0^i, \eta_{n_i-1}^i]$ with $\eta_0^i \geq 0$, we have

$$\beta_i \Big[\sum_{j \neq i}^{n} x_j + 2y_i \Big] \leq \beta_i \left[\sum_{j \neq i}^{n} \eta_{n_j}^j + 2\eta_{n_i-1}^i \right]. \tag{3.9}$$

Note that, by the assumption (3.5a), $a_0^i > a_k^i$ for every $k > 0$. Thus we have

$$\alpha_i - a_{k-1}^i > \alpha_i - a_0^i \quad \forall k. \tag{3.10}$$

From (3.9) to (3.10) we can deduce that if

$$\alpha_i - a_0^i \geq \beta_i \left(\sum_{j \neq i}^{n} \eta_{n_j}^j + 2\eta_{n_i-1}^i \right),$$

then, by (3.8), $f'(x^{-i}, y_i) > 0$ which implies strict monotonicity of $f(x^{-i}, .)$ on the interval (η_{k-1}^i, η_k^i). This is true for every $k = 1, \dots, n_i - 1$. Since $f(x^{-i}, .)$ is continuous on U_i, it must be increasing on the interval $[\eta_0^i, \eta_{n_i-1}^i]$. Observe that on the interval $[\eta_{n_i-1}^i, \eta_{n_i}^i]$ the objective function $f(x^{-i}, .)$ is strictly concave. Thus $f(x^{-i}, .)$ must attain its maximum on U_i at a unique point in $[\eta_{n_i-1}^i, \eta_{n_i}^i]$. Consequently, every problem given by (3.4) has a unique solution. Then by virtue of Theorem 3.1, problem (P) admits an solution. □

Remark 3.1 The condition

$$\alpha_i - a_0^i \geq \beta_i \Big(\sum_{j \neq i}^{n} \eta_{n_j}^j + 2\eta_{n_i-1}^i \Big) \quad \forall\, i \in \bar{I}$$

means that the difference between the price and the variable cost coefficient $\alpha_i - a_i^0$ must be greater or equal to $\beta_i \Big(\sum_{j \neq i}^{n} \eta_{n_j}^j + 2\eta_{n^i-1}^i \Big)$. The latter is just the lost of the profit caused by the decrease of the price. In some practical models the coefficient β_i is small while α_i is high, since α_i is the price of the goods when there is no product at firm i. In these models, condition (3.6) is satisfied. In other cases, for example, when the upper bound $\eta_{n_i}^i$ is large (unbounded models), condition (3.6), in general, is not satisfied.

Remark 3.2 From the proof of Proposition 3.1 one can see that when the condition (3.6) is satisfied, the strategy set U_i of firm i can be replaced by the interval $[\eta_{n_i-1}^i, \eta_{n_i}^i]$.

4 A solution method for piecewise-linear concave cost-models

In this section we propose a solution method for finding a global equilibrium point of a Cournot–Nash oligopolistic market equilibrium model with a piecewise-linear concave cost function which, as we have seen, can be formulated as the variational inequality problem (P). Some solution approaches have been developed for solving variational inequalities (see; e.g., [1,2,8,9,11,12,14–18,21,22,25] and the references thererein). To our knowledges, most of the available methods can be used only for the case when the cost function is convex. Since in our case, the cost function φ is not convex, the available methods for variational inequalities having convex cost functions can not be applied.

The method we are going to describe proceeds by dividing the strategy feasible rectangular set into subrectangles on each of them the cost function h is affine. Then we solve (P) on each subrectangle. Due to its special structure, each subvariational inequality can be solved by minimizing a certain strongly convex quadratic function on a subrectangle. By evaluating the merit function θ at each iterate point, we can detect the optimality of the current iterate. By this way we may avoid searching all of generated subrectangles.

Let $U^{-i} := \prod_{j \neq i} U_j$ and define a $(n-1)$-dimensional problem from the n-dimensional problem (P) by deleting item ith. Namely, we define the problem

$$(P^{-i_0}) \begin{cases} \text{find } x^{-i_0} \in U^{-i_0} \text{ such that} \\ \langle F^{-i_0}(x^{-i_0}), y^{-i_0} - x^{-i_0} \rangle + \varphi^{-i_0}(y^{-i_0}) - \varphi^{-i_0}(x^{-i_0}) \geq 0, \quad \forall\, y^{-i_0} \in U^{-i_0} \end{cases}$$

where

$$\begin{cases} F^{-i_0}(x^{-i_0}) := \tilde{B}^{-i_0} x^{-i_0} - \alpha^{-i_0}, \\ \varphi^{-i_0}(x^{-i_0}) := \sum_{i \neq i_0}^{n} \varphi_i(x_i) \end{cases} \tag{4.1}$$

with $\tilde{B}^{-i_0}$ being the $(n-1) \times (n-1)$ matrix obtained from matrix $\tilde{B}$ by deleting row i_0 and column i_0.

We note that when $\beta_i = 0$ for some i, the price of firm i is just equal to the constant α_i. In this case an equilibrium strategy for firm i can be computed by maximizing its profit function as stated by the following lemma.

Lemma 4.1 *Suppose that $\beta_{i_0} = 0$. Let $\hat{x}_{i_0}$ be an optimal solution of the one-dimensional problem*

$$\min\{h_{i_0}(y_{i_0}) - \alpha_{i_0} y_{i_0} \mid y_{i_0} \in U_{i_0}\} \tag{4.2}$$

and let $\hat{x}^{-i_0}$ be a solution to the variational inequality problem (P^{-i_0}). Then $\hat{x} := (\hat{x}^{-i_0}, \hat{x}_{i_0})$ is a solution to the variational problem (P). Conversely, every solution $\hat{x}$ of (P) has the form $\hat{x} = (\hat{x}^{-i_0}, \hat{x}_{i_0})$ with $\hat{x}_{i_0}$ and $\hat{x}^{-i_0}$ being solutions of (4.2) and (P^{-i_0}) respectively.

Proof The variational inequality problem (P) can be equivalently rewritten as the problem

$$\begin{cases} \text{find } x \in U \text{ such that} \\ \sum_{i=1}^n (\beta_i \sum_{j \neq i}^n x_j - \alpha_i)(y_i - x_i) \\ + \sum_{i=1}^n \beta_i (y_i^2 - x_i^2) + \sum_{i=1}^n (h_i(y_i) - h_i(x_i)) \geq 0, \quad \forall\, y \in U. \end{cases} \tag{4.3}$$

Suppose that $x = (x_1, \ldots, x_n)^T \in U$ is a solution of this problem. Fix i and take $y_j = x_j$ for every $j \neq i$. Then, since $U = U_1 \times \cdots \times U_n$, we have from (4.3) that

$$\left(\beta_i \sum_{j \neq i}^n x_j - \alpha_i\right)(y_i - x_i) + \beta_i (y_i^2 - x_i^2) + h_i(y_i) - h_i(x_i) \geq 0, \quad \forall y_i \in U_i \tag{4.4}$$

This is true for every $i = 1, \ldots, n$. Conversely, if (4.4) holds for every $i = 1, \ldots, n$, then, clearly, (4.3) holds.

Since $\beta_{i_0} = 0$, it follows from (4.4) that

$$-\alpha_{i_0}(y_{i_0} - x_{i_0}) + h_{i_0}(y_{i_0}) - h_{i_0}(x_{i_0}) \geq 0, \quad \forall y_{i_0} \in U_{i_0}.$$

This implies that x_{i_0} is an optimal solution to problem (4.2). With $i = i_0$ and $x_{i_0} = \hat{x}_{i_0}$, the inequality (4.4) is reduced to

$$\left(\beta_i \sum_{j \neq i, i_0}^n x_j - \alpha_i + \beta_i \hat{x}_{i_0}\right)(y_i - x_i) + \beta_i (y_i^2 - x_i^2) + h_i(y_i) - h_i(x_i) \geq 0. \tag{4.5}$$

This is true for all $y_i \in U_i$; $i = 1, \ldots, n; i \neq i_0$. By a similar way, we can see that, $\hat{x}^{-i_0}$ satisfies (4.5) if and only if $\hat{x}^{-i_0}$ is a solution to problem (P^{-i_0}). Hence, $\hat{x} = (\hat{x}^{-i_0}, \hat{x}_{i_0})$ is a solution to problem (P). □

In virtue of Lemma 4.1, in what follows, without lost of generality, we may assume that $\beta_i > 0$ for all i.

Since $U_i = [\eta_0^i, \eta_{n_i}^i]$ for $i = 1, \ldots, n$, we have

$$U = [\eta_0^1, \eta_{n_1}^1] \times [\eta_0^2, \eta_{n_2}^2] \times \cdots \times [\eta_0^n, \eta_{n_n}^n]. \tag{4.6}$$

Let Γ_i be the family of all consecutive subintervals of $U_i (i = 1, \ldots, n)$, i.e.,

$$\Gamma_i = \{[\eta_{j-1}^i, \eta_j^i] \mid j = 1, \ldots, n_i\}.$$

Let

$$\Sigma := \{I \mid I = I_1 \times \cdots \times I_n : \ I_i \in \Gamma_i, \ i = 1, \ldots, n\}.$$

Clearly, for all $i = 1, \dots, n$,

$$U_i = \bigcup_{I_i \in \Gamma_i} I_i \quad \text{and} \quad U = \bigcup_{I \in \Sigma} I.$$

For each subbox $I \in \Sigma$, we solve the following subvariational inequality

$$(P^I) \begin{cases} \text{find a point } x^I \in I \quad \text{such that} \\ \Phi(x^I, y) := \langle F(x^I), y - x^I \rangle + \varphi(y) - \varphi(x^I) \geq 0 \quad \forall y \in I. \end{cases}$$

Note that the subproblem (P^I) can be rewritten in the form

$$(VIP^I) \begin{cases} \text{find} x^I \in I \quad \text{such that} \\ \langle \tilde{Q} x^I + q, y - x^I \rangle \geq 0 \quad \forall y \in I \end{cases}$$

where

$$\tilde{Q} = \begin{pmatrix} 2\beta_1 & \beta_1 & \beta_1 & \dots & \beta_1 \\ \beta_2 & 2\beta_2 & \beta_2 & \dots & \beta_2 \\ \dots & \dots & \dots & \dots & \dots \\ \beta_n & \beta_n & \beta_n & \dots & 2\beta_n \end{pmatrix} \tag{4.7}$$

and $q^T = (q_1, \dots, q_n)$ with

$$q_i = a^i_{j_i - 1} - \alpha_i \ (i = 1, \dots, n). \tag{4.8}$$

Thanks to the special structure of $\tilde{Q}$ the affine variational inequality (VIP^I) is equivalent to a strongly convex quadratic program as stated by the following lemma.

Lemma 4.2 (VIP^I) *is equivalent to the strongly convex quadratic problem*

$$\min_{x \in I} \left\{ \frac{1}{2} x^T C x + c^T x \right\} \tag{QP^I}$$

where

$$C = \begin{pmatrix} 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 2 \end{pmatrix} \tag{4.9}$$

is symmetric positive definite and $c^T = (c_1, \dots, c_n)$ *with* $c_i = q_i/\beta_i$ $(i = 1, \dots, n)$ *and* q_i *defined by* (4.8).

Proof For simplicity of notation, we suppose that

$$I := \left\{ x \in \mathbb{R}^n \,|\, a_i \leq x_i \leq b_i \quad \forall\, i = 1, \dots, n \right\}. \tag{4.10}$$

Note that x is a solution of (VIP^I) if and only if x is a solution of the linear programming problem

$$\min_y \left\{ y^T \tilde{Q} x + q^T y : a_i \leq y_i \leq b_i \quad \forall\, i = 1, \dots, n \right\}. \tag{4.11}$$

By the Kuhn–Tucker theorem, x is a solution to (4.11) if and only if there exist $\lambda_1, \lambda_2, \ldots \lambda_{2n} \geq 0$ such that

$$\begin{cases} \beta_i(x_i + \sum_{j=1}^n x_j) + q_i + \lambda_{2i-1} - \lambda_{2i} = 0, \\ \lambda_{2i-1}(x_i - a_i) = 0, \\ \lambda_{2i}(-x_i + b_i) = 0, \\ a_i \leq x_i \leq b_i, \\ \lambda_{2i-1} \geq 0, \lambda_{2i} \geq 0 \quad (i = 1, \ldots, n). \end{cases}$$

Since $\beta_i > 0 \quad \forall\, i = 1, \ldots, n$, this system can be rewritten equivalently as the following one

$$\begin{cases} \left(x_i + \sum_{j=1}^n x_j\right) + \frac{1}{\beta_i} q_i + \frac{1}{\beta_i}\lambda_{2i-1} - \frac{1}{\beta_i}\lambda_{2i} = 0, \\ \frac{1}{\beta_i}\lambda_{2i-1}(x_i - a_i) = 0, \\ \frac{1}{\beta_i}\lambda_{2i}(-x_i + b_i) = 0, \\ a_i \leq x_i \leq b_i, \\ \frac{1}{\beta_i}\lambda_{2i-1} \geq 0, \quad \frac{1}{\beta_i}\lambda_{2i} \geq 0 \quad (i = 1, \ldots, n). \end{cases} \tag{4.12}$$

By setting

$$c_i = \frac{1}{\beta_i} q_i, \ \nu_{2i-1} = \frac{1}{\beta_i}\lambda_{2i-1}, \ \nu_{2i} = \frac{1}{\beta_i}\lambda_{2i} \quad \forall\, i = 1, \ldots, n.$$

we can write (4.12) as

$$\begin{cases} \left(x_i + \sum_{j=1}^n x_j\right) + c_i + \nu_{2i-1} - \nu_{2i} = 0, \\ \nu_{2i-1}(x_i - a_i) = 0, \\ \nu_{2i}(-x_i + b_i) = 0, \\ a_i \leq x_i \leq b_i, \\ \nu_{2i-1} \geq 0, \nu_{2i} \geq 0 \quad (i = 1, \ldots, n). \end{cases} \tag{4.13}$$

Using again the Kuhn–Tucker theorem, we see that system (4.13) is a necessary condition for x to be an optimal solution to the quadratic optimization problem (QP^I). This condition is also sufficient for optimality of x to (QP^I), because C is symmetric positive definite. □

Proposition 4.1 *(i) For each $I \in \Sigma$, the subproblem (P^I) has a unique solution, (ii) The variational inequality (P) has a solution if and only if there exists $I_* \in \Sigma$ such that $\theta(x^{I_*}) = 0$, where θ is defined by (3.2).*

Proof (i) is immediate from Lemma 4.2, since (P^I) is equivalent to the strongly convex quadratic program (QP^I).

(ii) Suppose that, there exists $x^{I_*} \in U$ such that $\theta(x^{I_*}) = 0$. By Lemma 2.1, x^{I_*} is a solution to (P).

Conversely, suppose that $x^* \in U$ is a solution to (P). Since

$$U = \bigcup_{I \in \Sigma} I,$$

there exists a subbox $I_* \in \Sigma$ such that $x^* \in I_*$. By part (i), x^* must be equal to x^{I_*}. Again by Lemma 2.1, we have $\theta(x^*) = 0$. □

In the algorithm we are going to describe, for ech $I \in \Sigma$ we have to solve the strongly convex quadratic problem (QP^I). Suppose $I = I_1 \times \cdots \times I_n$ with

$$I_i := [\eta^i_{j_i-1}, \eta^i_{j_i}] \quad (i = 1, \ldots, n).$$

To construct vector $c^T = (c_1, \ldots, c_n)$ in the object function of problem (QP^I), first we need to compute

$$a^I = (a^1_{j_1-1}, \ldots, a^n_{j_n-1}).$$

Then we take

$$c_i = \frac{1}{\beta_i}(a^i_{j_i-1} - \alpha_i) \quad (i = 1, \ldots, n).$$

Let x^I be the unique optimal solution to problem (QP^I). Using the obtained x^I, we compute $\theta(x^I)$ as the optimal value of the problem

$$\min_{y \in U} \left\{ \langle \tilde{B}x^I - \alpha, y - x^I \rangle + \varphi(y) - \varphi(x^I) \right\} \qquad (OP^I)$$

where, since h is separable, by (2.6), $\varphi(y) := \sum_{i=1}^n \varphi_i(y_i)$. Since the objective function of (OP^I) is separable, this problem is reduced to n-optimization problems of one-variable, each of them has the following form

$$f_i^* := \min_{y_i \in U_i} \{f_i(y_i)\} \quad (i = 1, \ldots, n). \qquad (4.14)$$

By definition, we have $\theta(x^I) = \sum_{i=1}^n f_i^*$.

Below are the steps of proposed algorithm.

Algorithm
Choose a tolerance $\epsilon \geq 0$.
Step 1: Select a subbox $I \in \Sigma$.
Step 2: Solve the strongly convex quadratic problem (QP^I) to obtain its unique solution x^I.
Step 3: Solve n one-dimensional optimization problems (4.14) to obtain f_i^* $(i = 1, \ldots, n)$. Take

$$\theta(x^I) = \sum_{i=1}^n f_i^*.$$

(a) If $\theta(x^I) \geq -\epsilon$, then terminate: we call x^I an ϵ-equilibrium point.
(b) If $\theta(x^I) < -\epsilon$ and $\Sigma = \emptyset$, then terminate: the model has no equilibrium point.

Otherwise, replace Σ by $(\Sigma \setminus \{I\})$ and return to Step 1.

The validity of the algorithm is ensured by Proposition 4.1. Since Σ is a finite set, the algorithm must terminate at Case (a) or Case (b). In the worst case the algorithm searches for all subboxes in Σ.

5 Computational results and experiences

We have tested the proposed algorithm for more than 20 problems with random generated data on a PC computer by using Matlab for solving strongly convex quadratic subprograms.

In Table 1 the number of firms having concave cost function is $nc = 15$. The number of all subboxes is $K_{max} = 5184$. In Table 2 $nc = 4$ and $K_{max} = 72$. All problems in Table 1

Table 1 with $n = 100, nc = 15$ and $\epsilon = 10^{-5}$

Problem	Ite.	CPU-times/s	β
1	3893	2953.7	10^{-4}
2	1128	660.2	10^{-4}
3	1728	1427.8	10^{-5}
4	5184	3424.2	10^{-5}
5	1728	2287.4	10^{-5}
6	1117	846.5	10^{-5}
7	1123	769.8	10^{-5}
8	605	390.7	10^{-5}
9	1124	737.3	10^{-6}
10	1	1.4	10^{-6}
11	1	0.61	10^{-3}
12	37	21.56	10^{-3}
13	1	0.5	10^{-7}
14	1	0.6	10^{-8}
15	1	1.392	10^{-9}

Table 2 with $n = 10, nc = 4$ and $\epsilon = 10^{-5}$

Problem	Ite.	CPU-times/s	β	Solution
1	72	28.18	10^{-2}	No
2	72	27.96	10^{-3}	No
3	14	6.42	10^{-4}	Yes
4	15	6.97	10^{-5}	Yes
5	15	6.98	10^{-6}	Yes
6	72	22.74	10^{-9}	Yes
7	72	20.98	10^{-10}	Yes

have a solution whereas in Table 2 the "yes" means that the problem has equilibrium point and the "no" means that the problem has no equilibrium point.

From the obtained results one can see that the proposed algorithm can be used for solving the model with moderate number of producers having concave cost function. For high dimensional problems we suggest portioning the family of subboxes Σ into two or more disjunctive subsets and run simultaneously the algorithm on two or more computers. Each of them solves the problem on a subset of boxes. The computation will terminate when one computer finds a solution x^I of problem (OP^I) with a box I such that $\theta(x^I) \geq -\epsilon$. It is expected that, by using this independent and parallel computation, the proposed algorithm can solve large-scale problems.

Acknowledgements The work presented here was completed while the first and third authors were on leave at the University of Namur, Belgium. They wish to thank the University of Namur for financial support in the form of FUND Research Scholarship.

References

1. Anh, P.N., Muu, L.D., Nguyen, V.H., Strodiot, J.J.: On the contraction and nonexpensiveness properties of the marginal mapping in generalized variational inequalities involving cocoercive operators. In: Eberhard, A., Hadjisavvas, N., Luc, D.T.(eds). Generalized Convexity, and Generalized Monotonicity and Applications, pp. 89–111, Chapter 5, Springer (2005)

2. Anh, P.N., Muu, L.D., Nguyen, V.H., Strodiot, J.J.: Using the Banach contraction principle to implement the proximal point method for solving multivalued monotone variational inequalities. J. Optim. Theory Appl. **124**, 285–306 (2005)
3. Aubin, J.P., Ekeland, I.: Applied Nonlinear Analysis. Wiley, New York (1984)
4. Berge, C.: Topological Spaces. MacMillan, New York (1968)
5. Blum, E., Oettli, W.: From optimization and variational inequality to equilibrium problems. Mathe. Stud. **63**, 127–149 (1994)
6. Dafermos, S.: Exchange price equilibria and variational inequalities. Math. Program. **46**, 391–402 (1990)
7. Dafermosm, S., Nagurney, A.: Oligopolistic and competitive behavior of spatially separated markets. Reg. Sci. Urban Econ. **17**, 225–254 (1997)
8. Cohen, G.: Auxiliary problem principle extended to variational inequalities. J. Optim. Theory Appl. **59**, 325–333 (1998)
9. Facchinei, F., Pang, J.S.: Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer, Berlin (2002)
10. Fukushima, M.: Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems. Math. Program. **53**, 99–110 (1992)
11. Harker, P.T., Pang, J.S.: Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms, and applications. Mathe. Program. **48**, 161–220 (1990)
12. Hue, T.T., Strodiot, J.J., Nguyen, V.H.: Convergence of the Approximate Auxiliary Problem Method for Solving Generalized Variational Inequalities. J. Optim. Theory Appl. **121**, 119–145 (2004)
13. Horst, R., Tuy, H.: Global Optimization (Deterministic Approach). Springer, Berlin (1990)
14. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Academic Press (1980)
15. Konnov, I.: Combined Relaxation Methods for Variational Inequalities. Springer, Berlin (2001)
16. Konnov, I., Kum, S.: Descent methods for mixed variational inequalities in a Hilbert space. Nonlinear Anal. Theory Meth. Appl. **47**, 561–572 (2001)
17. Marcotte, P.: A new algorithm for solving variational inequalities. Mathematical Programming **33**, 339–351 (1995)
18. Muu, L.D.: An augmented penalty function method for solving a class of variational inequalities. Soviet. Comput. Math. Phys. **12**, 1788–1796 (1986)
19. Muu, W., Oettli, L.D.: Convergence of an adaptive scheme for finding constraint equilibria. Nonlinear Anal Theory Meth Appl. **18**, 1159–1166 (1992)
20. Muu, L.D., Quy, N.V.: A global optimization method for solving convex quadratic bilevel programming problems. J. Global Optim. **26**, 199–219 (2003)
21. Nagurney, A.: Network Economics: A Variational Inequality Approach. Kluwer, Academic Publishers (1993)
22. Noor, M.A.: Iterative schemes for quasimonotone mixed variational inequalities. Optimization **50**, 29–44 (2001)
23. Patriksson, M.: Nonlinear Programming and Variational Inequality Problems: A Unified Approach. Kluwer, Dordrecht (1999)
24. Patriksson, M.: Merit function and descent algorithms for a class of variational inequality problems.. Optimization **41**, 37–55 (1997)
25. Salmon, G., Nguyen, V.H., Strodiot, J.J.: Coupling the auxiliary problem principle and epiconvergence theory to solve general variational inequalities. J. Optim Theory Appl. **104**, 629–657 (2000)
26. Taji, T., Fukushima, M.: A new merit function and a successive quadratic programming algorithm for variational inequality problem. SIAM J. Optim. **6**, 704–713 (1996)