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A KINGMAN CONVOLUTION IN $\mathbb{R}^{K+*}$

BY

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To my lovers

Abstract. In this paper we study a higher dimensional model of the Kingman convolution algebras.We also introduce a new class of k-dimensional Rayleigh distributions on $\mathbb{R}^{k+}$ which stands for an analogue of the class of k-dimensional Gaussian measures on $\mathbb{R}^k$. Moreover, we prove the Levy-Khinczyne theorem for infinitely divisible distributions in the k-dimensional Kingman convolution algebra.

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1. INTRODUCTION, NOTATION AND PRELIMILARIES

Let $\mathcal{P}$ denote the class of all probability measures (p.m.'s) on the positive half-line $\mathbb{R}^+$ endowed with the weak convergence and $*_{1,\delta}, \delta \geqslant 1$, denote the Kingman convolution which was introduced by Kingman [2] in connection with the addition of independent spherically symmetric random vectors (r.vec.) in an Euclidean space. Namely, for each continuous bounded function f on $\mathbb{R}^+$ we write :

$$(1.1)\quad \int_0^\infty f(x)\mu *_{1,\delta} \nu(dx) = \frac{\Gamma(s+1)}{\sqrt{\pi}\Gamma(s+\frac{1}{2})} \int_0^\infty \int_0^\infty \int_{-1}^1 f((x^2+2uxy+y^2)^{1/2})(1-u^2)^{s-1/2}\mu(dx)\nu(dy)du,$$

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where μ and $\nu \in P$ and $\delta = 2(s+1) \geqslant 1$ (cf. Kingman [2] and Urbanik [10]). The algebra $(\mathcal{P}, *_{1,\delta})$ is the most important example of Urbanik convolution algebras (cf Urbanik [10]). In language of the Urbanik convolution algebras, the *characteristic measure*, say σ_s, of the Kingman convolution has the Rayleigh density

$$d\sigma_s(y) = \frac{2(s+1)^{s+1}}{\Gamma(s+1)} y^{2s+1} \exp\left(-(s+1)y^2\right) dy \tag{1.2}$$

with the characteristic exponent $\varkappa = 2$ and the kernel Λ_s

$$\Lambda_s(x) = \Gamma(s+1) J_s(x)/(1/2x)^s, \tag{1.3}$$

where $J_s(x)$ denotes the Bessel function,

$$J_s(x) := \Sigma_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}. \tag{1.4}$$

It is known (cf. Kingman [2], Theorem 1), that the kernel Λ_s itself is an ordinary characteristic function (ch.f.) of a symmetric p.m., say F_s, defined on the interval [-1,1]. Thus, if θ_s denotes a random variable (r.v.) with distribution F_s then for each $t \in \mathbb{R}^+$,

$$\Lambda_s(t) = E\exp(it\theta_s) = \int_{-1}^{1} \exp(itx) dF_s(x). \tag{1.5}$$

Suppose that X is a nonnegative r.v. with distribution $\mu \in \mathcal{P}$ and X is independent of θ_s. The *radial characteristic function* (rad.ch.f.) of μ, denoted by $\hat{\mu}(t)$, is defined by

$$\hat{\mu}(t) = E\exp(itX\theta_s) = \int_0^{\infty} \Lambda_s(tx)\mu(dx), \tag{1.6}$$

for every $t \in \mathbb{R}^+$. In particular, the rad.ch.f. of σ_s is

$$\hat{\sigma}_s(t) = \exp\left(-\frac{t^2}{2}\right), \quad t \in \mathbb{R}^+. \tag{1.7}$$

It should be noted, since the rad.ch.f. is defined uniquely up to the mapping $x \to ax, a > 0, x \in \mathbb{R}^+$, that the representation (1.7) may be other than that in in Urbanik [10] and Kingman [2].

2. CARTESIAN PRODUCT OF KINGMAN CONVOLUTIONS

Denote by $\mathbb{R}^{+k}, k = 1, 2, ...$ the k-dimensional nonnegative cone of $\mathbb{R}^k$ and $\mathcal{P}(\mathbb{R}^{+k})$ the class of all p.m.'s on $\mathbb{R}^{+k}$ equipped with the weak convergence. In the sequel, we will denote the multidimensional vectors and distributions and r.v.'s by bold letters. For each point z of any set Z let δ_z denote the Dirac measure (the unit mass) at the point z. In particular, if $\mathbf{x} = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^{k+}$, then

$$\delta_{\mathbf{x}} = \delta_{x_1} \times \delta_{x_2} \times \cdots \times \delta_{x_k}, \tag{2.1}$$

where the sign " $\times$ " denotes the Cartesian product of measures. We put, for $\mathbf{x} = (x_1, x_2, \cdots, x_k)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_k) \in \mathbb{R}^{+k}$,

$$\delta_{\mathbf{x}} \bigcirc_k \delta_{\mathbf{y}} = \{\delta_{x_1} \circ \delta_{y_1}\} \times \{\delta_{x_2} \circ \delta_{y_2}\} \times \cdots \times \{\delta_{x_k} \circ \delta_{y_k}\}, \tag{2.2}$$

here and somewhere bellow for the sake of simplicity we denote the Kingman convolution operation $*_{1,\delta}$ simply by $\circ$. Since convex combinations of p.m.'s of the form (2.1) are dense in $\mathcal{P}(\mathbb{R}^{+k})$ the relation (2.2) can be extended to arbitrary p.m.'s $\mathbf{F}$ and $\mathbf{G} \in \mathcal{P}(\mathbb{R}^{+k})$. Namely, we put

$$\mathbf{F} \bigcirc_k \mathbf{G} = \iint\limits_{\mathbb{R}^{+k}} \delta_{\mathbf{x}} \bigcirc_k \delta_{\mathbf{y}} \mathbf{F}(d\mathbf{x})\mathbf{G}(d\mathbf{y}). \tag{2.3}$$

In the sequel, the binary operation $\bigcirc_k$. will be called *the k-times Cartesian product of Kingman convolutions*. It is easy to show that the binary operation $\bigcirc_k$ is continuous in the weak topology which together with (1.1) and (2.3) implies the following theorem.

THEOREM 2.1. *The pair $(\mathcal{P}(\mathbb{R}^{+k}), \bigcirc_{\mathbf{k}})$ is a commutative topological semigroup with $\delta_{\mathbf{0}}$ as the unit element. Moreover, the operation $\bigcirc_k$ is distributive w. r. t. convex combinations of p.m.'s in $\mathcal{P}(\mathbb{R}^{+k})$.*

In the sequel, the pair $(\mathcal{P}(\mathbb{R}^{+k}), \bigcirc_k)$ will be called a *k-dimensional Kingman convolution algebra*[2]. For every $\mathbf{F} \in \mathcal{P}(\mathbb{R}^{+k})$ the k-dimensional rad.ch.f. $\hat{\mathbf{F}}(\mathbf{t}), \mathbf{t} = (t_1, t_2, \cdots t_k) \in \mathbb{R}^{k+}$, is defined by

$$\hat{\mathbf{F}}(\mathbf{t}) = \int\limits_{\mathbb{R}^{+k}} \prod_{j=1}^{k} \Lambda_s(t_j x_j)\mathbf{F}(\mathbf{dx}), \tag{2.4}$$

where $\mathbf{x} = (x_1, x_2, \cdots x_k) \in \mathbb{R}^{+k}$.

As noted by Kingman ([2], P.30) that the characteristic measure σ_s, being the Rayleigh distribution, plays the role of the normal distribution, one may expect that the multidimensional standard normal distribution has its counter part in the multidimensional Kingman convolution being the k-dimensional Rayleigh distribution, say $\mathbf{\Sigma}_s$, which is defined by

$$\mathbf{\Sigma}_s = \sigma_s \times \sigma_s \times \cdots \times \sigma_s \quad (k\ times). \tag{2.5}$$

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[2]Higher dimensional Urbanik convolution algebras can be introduced in the same way as here for the Kingman convolution case but this subject will be treated systematically else where.

Furthermore, for any nonnegative numbers $\lambda_r, r = 1, 2, \cdots$ the distribution

$$\mathbf{F} = \{T_{\lambda_1}\sigma_s\} \times \{T_{\lambda_2}\sigma_s\} \times \cdots \{T_{\lambda_k}\sigma_s\}, \tag{2.6}$$

stands for a *k-dimensional Rayleighian distribution.*

By virtue of formulas (1.7, 2.4, 2.5 and 2.6) we have the following

THEOREM 2.2. *Suppose distributions $\mathbf{\Sigma}$ and $\mathbf{F}$ are of the form (2.5) and (2.6) then, for any $\mathbf{t} \in \mathbb{R}^{+k}$,*

$$\hat{\mathbf{\Sigma}}_s(\mathbf{t}) = \exp(-\frac{\sum\limits_{j=1}^{k} t_j^2}{2}) \tag{2.7}$$

and

$$\hat{\mathbf{F}}(\mathbf{t}) = \exp(-\frac{\sum\limits_{j=1}^{k} \lambda_j^2 t_j^2}{2}) \tag{2.8}$$

Let $\theta, \theta_1, \theta_2, \ldots, \theta_k$ be independent identically distributed (i.i.d.) r.v's with the common distribution F_s. We set

$$\mathbf{\Theta}_s = (\theta_1, \theta_2, \ldots, \theta_k) \tag{2.9}$$

Assume that $\mathbf{X} = (X_1, X_2, ..., X_k)$ is a k-dimensional r.vec. with distribution $\mathbf{F}$ and $\mathbf{X}$ is independent of $\mathbf{\Theta}$. We put

$$[\mathbf{\Theta}, \mathbf{X}] = (\theta_1 X_1, \theta_2 X_2, ..., \theta_k X_k). \tag{2.10}$$

Then, the following formula is the multidimensional generalization of (1.6) and is equivalent to (2.4)

$$\widehat{\mathbf{F}}(\mathbf{t}) = E e^{i<\mathbf{t},[\mathbf{\Theta},\mathbf{X}]>}, \tag{2.11}$$

where $\mathbf{X}$ and Θ are assumed to be independent and $\mathbf{t} = (t_1, t_2, ..., t_k) \in \mathbb{R}^{+k}$ and the symbol $<\ ,\ >$ denotes the inner product in $\mathbb{R}^k$. In fact, we have

$$\begin{aligned} E e^{i<(\theta_1 t_1, \theta_2 t_2, \ldots, \theta_k t_k), \mathbf{X}>} &= \int\limits_{\mathbb{R}^{+k}} E e^{i \sum\limits_{j=1}^{k} t_j x_j \theta_j} \mathbf{F}(d\mathbf{x}) \\ &= \int\limits_{\mathbb{R}^{+k}} \Pi_{j=1}^{k} \Lambda_s(t_j x_j) \mathbf{F}(d\mathbf{x}) \\ &= \widehat{\mathbf{F}}(\mathbf{t}). \end{aligned} \tag{2.12}$$

As a consequence of the representation (2.11) we have

COROLLARY 2.1. *For each $\mathbf{F} \in \mathcal{P}(\mathbb{R}^{k+})$ the rad.ch.f. $\hat{\mathbf{F}}(\mathbf{t})$ is also an ordinary k-dimensional ch.f. and hence, it is uniformly continuous.*

The following lemma will be used in the representation of k-dimensional ID p.m.'s

LEMMA 2.1. *(i) For every $t \geqslant 0$*

$$\lim_{x\to 0}\frac{1-\Lambda_s(tx)}{x^2} = \lim_{x\to 0}\frac{1-Ee^{it\theta}}{x^2} = \frac{t^2}{2}. \tag{2.13}$$

(ii) For any $\mathbf{x} = (x_0, x_1, \cdots, x_k)$ and $\mathbf{t} = (t_0, t_1, \cdots, t_k) \in \mathbb{R}^{k+1},\ k = 1, 2, ...$

$$lim_{\rho\to 0}\frac{1-\prod_{r=0}^{k}\Lambda_s(t_r x_r)}{\rho^2} = \Sigma_{r=0}^{k}\lambda_r(\mathbf{x})t_r^2, \tag{2.14}$$

where $\rho = \|\mathbf{x}\|$ and $\lambda_r(\mathbf{x}),\ r = 0, 1, ..., k$ are given by

$$\lambda_r(\mathbf{x}) = \begin{cases} \frac{1}{2}\cos^2\phi & r = 0,\\ \frac{1}{2}\big(\sin\phi\sin\phi_1\cdots\sin\phi_{r-1}\cos\phi_r\big)^2 & 1 \leqslant r \leqslant k-2),\\ \frac{1}{2}\big(\sin\phi\sin\phi_1...\sin\phi_{k-2}\cos\psi\big)^2 & r = k-1,\\ \frac{1}{2}\big(\sin\phi\sin\phi_1...\sin\phi_{k-2}\sin\psi\big)^2 & r = k, \end{cases} \tag{2.15}$$

where $0 \leqslant \psi,\ \phi,\ \phi_r\ \leqslant \frac{\pi}{2},\ r = 1, 2, ..., k-2$ are angles of $\mathbf{x}$ appearing in its polar form.

P r o o f. (i) The equation (1.5) in conjunction with the l'Hôpital rule implies that

$$\lim_{x\to 0}\frac{1-\Lambda_s(tx)}{x^2} = \lim_{x\to 0}\frac{1-Ee^{it\theta}}{x^2} = \frac{t^2}{2},$$

which proves (2.13).
(ii) In order to prove (2.14) let the points $\mathbf{x} = (x_0, x_1, ..., x_k) \in \mathbb{R}^{k+1}$ be of the polar form

$$x_r = \begin{cases} \rho\cos\phi, & r = 0,\\ \rho\sin\phi\sin\phi_1\cdots\sin\phi_{r-1}\cos\phi_r, & 1 \leqslant r \leqslant k-2\\ \rho\sin\phi\sin\phi_1...\sin\phi_{k-2}\cos\psi, & r = k-1,\\ \rho\sin\phi\sin\phi_1...\sin\phi_{k-2}\sin\psi, & r = k. \end{cases} \tag{2.16}$$

where $0 \leqslant \psi, \phi, \phi_r \leqslant \pi/2, r = 1, 2, ..., k-2$. Putting

$$(2.17)\qquad A(\mathbf{\Theta}, \mathbf{t}, \mathbf{\Phi}) = \begin{cases} t_0\theta_0 \cos\phi \\ + \sum\limits_{r=1}^{k-2} t_r\theta_r \sin\phi \sin\phi_1 \cdots sin\phi_{r-1} \cos\phi_r \\ +t_{k-1}\theta_{k-1} \sin\phi \sin\phi_1 \cdots \sin\phi_{k-2} \cos\psi \\ +t_k\theta_k \sin\phi \sin\phi_1 \cdots \sin\phi_{k-2} \sin\psi \end{cases}$$

and

$$(2.18)\qquad V(\mathbf{\Theta}, \mathbf{t}, \mathbf{\Phi}) = \sum_{r=0}^{k} t_r x_r \theta_r,$$

where the $\theta_r, r = 0, 1, 2, ...$ are symmetric i.i.d. r.v.'s with distribution σ_s, $\mathbf{\Phi} = (\psi, \phi, \phi_1, \cdots, \phi_k)$ and $\mathbf{\Theta} := (\theta_0, \theta_1, ..., \theta_k)$. By virtue of (2.12) and (2.16) and applying l'Hôpital rule, we have

(2.19)

$$lim_{\rho\to 0}\frac{1 - \prod_{r=0}^{k} \Lambda_s(t_r x_r)}{\rho^2} = lim_{\rho\to 0}\frac{1 - E(e^{i\sum\limits_{r=0}^{k} t_r x_r \theta_r})}{\rho^2}$$

$$= \frac{\frac{d^2}{d\rho^2}\left(1 - Ee^{i\rho A(\mathbf{\Theta},\mathbf{t},\mathbf{\Phi})}\right)}{\frac{d^2}{d\rho^2}\rho^2}|_{\rho=0} = \frac{1}{2}EV^2(\mathbf{\Theta}, \mathbf{t}, \mathbf{\Phi})e^{i\rho V(\mathbf{\Theta},\mathbf{t},\mathbf{\Phi})}|_{\rho=0}.$$

Since σ_s has expectation zero and variance 1 it follows that

$$(2.20)\qquad EV^2(\theta, \mathbf{t}, \phi) = \sum_{j=1}^{k} t_j^2 x_j^2$$

which together with (2.19) implies (2.14). ■

Proceeding successively, we have the following theorem:

THEOREM 2.3. *Every p.m.* $\mathbf{F} \in \mathcal{P}(\mathbb{R}^{+k})$ *is uniquely determined by its k-dimensional rad.ch.f.* $\hat{\mathbf{F}}$ *and the following formula holds:*

$$(2.21)\qquad \widehat{\mathbf{F}_1 \bigcirc_k \mathbf{F}_2}(\mathbf{t}) = \widehat{\mathbf{F}_1}(\mathbf{t})\widehat{\mathbf{F}_2}(\mathbf{t}),$$

where $\mathbf{F}_1, \mathbf{F}_2 \in \mathcal{P}(\mathbb{R}^{+k})$ *and* $\mathbf{t} \in \mathbb{R}^{+k}$.

P r o o f. The formula (2.21) follows from (1.1) and (2.3). Next, using the formulas (2.3) and (2.4) and integrating the function $\hat{\mathbf{F}}(t_1u_1, ..., t_ku_k)$, k-times w. r. t. σ_s , we get

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^{+k}} \hat{\mathbf{F}}(t_1u_1, ..., t_ku_k)\sigma_s(du_1)...\sigma_s(du_k) \\
(2.22) \qquad &= \iint\limits_{\mathbb{R}^{+}} ... \int\limits_{\mathbb{R}^{+}} \prod_{j=1}^{k} \Lambda_s(t_jx_ju_j)\mathbf{F}(\mathbf{dx})\sigma_s(du_1)...\sigma_s(du_k) \\
&= \int\limits_{\mathbb{R}^{+k}} \prod_{j=1}^{k} \exp\{-t_j^2x_j^2\}\mathbf{F}(\mathbf{d}x),
\end{aligned}
$$

which, by change of variables $y_j = x_j^2, j = 1, ..., k$ and by the uniqueness of the k-dimensional Laplace transform, implies that $\mathbf{F}$ is uniquely determined by the left-hand side of (2.22). ∎

As a consequence of the formula (2.22) we have the following corollary which is an analogue of the continuity theorem for multidimensional Laplace transforms.

THEOREM 2.4. *Suppose that $\{\mathbf{F}_n\}$ is a sequence of distributions on $\mathbb{R}^{k+}$ and $\{\phi_n\}$ is a sequence of the corresponding rad.ch.f.'s. Then, $\mathbf{F}_n$ converges weakly to a distribution $\mathbf{F}$ if, and only if, $\{\phi_n\}$ converges uniformly on every compact subsets of $\mathbb{R}^{k+}$ to a rad.ch.f. ϕ.*

For any $\mathbf{x} \in \mathbb{R}^{+k}$ the generalized translation operators (g.t.o.'s) $\mathbf{T^x}$ acting on the Banach space $\mathbb{C}_b(\mathbb{R}^{+k})$ of real bounded continuous functions f on $\mathbb{R}^{+k}$ are defined, for each $\mathbf{y} \in \mathbb{R}^{+k}$, by

$$
\mathbf{T^x}f(\mathbf{y}) = \int\limits_{\mathbb{R}^{+k}} f(\mathbf{u})\{\delta_{\mathbf{x}} \bigcirc_{\mathbf{k}} \delta_{\mathbf{y}}\}(d\mathbf{u}). \tag{2.23}
$$

In terms of these g.t.o.'s the k-dimensional rad.ch.f. of p.m.'s on $\mathbb{R}^{+k}$ can be characterized as the following:

THEOREM 2.5. *A real bounded continuous function f on $\mathbb{R}^{+k}$ is a (k-dimensional) rad.ch.f. of a p.m., if and only if $f(\mathbf{0}) = 1$ and f is $\{\mathbf{T^x}\}$-nonnegative definite in the sense that for any $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k \in \mathbb{R}^k$ and $\lambda_1, \lambda_2, ..., \lambda_k \in \mathbb{C}$*

$$
\sum_{i,j=1}^{k} \lambda_i\bar{\lambda}_j\mathbf{T}^{\mathbf{x}_i}f(\mathbf{x}_j) \geqslant 0. \tag{2.24}
$$

(See Vólkovich [12] for the proof).

The k-dimensional ID elements w.r.t. $\bigcirc_k$ can be defined as the following:

DEFINITION 2.1. A p.m. $\mu \in \mathcal{P}(\mathbb{R}^{+k})$ is called ID, if for every natural m there exists a p.m. μ_m such that $\mu = \mu_m \bigcirc_k \ \ldots \bigcirc_k \mu_m \quad (m\, times)$.

The simplest but most important example of k-dimensional ID distributions are the k-dimensional Rayleigh distributions. More generally, if $\mathbf{F}$ *is a k-dimensional Rayleighian distribution, then it is also ID. Let us denote by* $ID(\bigcirc_k)$ *the class of all i.d.p.m.'s in* $(\mathcal{P}(\mathbb{R}^{+k}), \bigcirc_k)$*. The following theorem, being a generalization of Theorem 7 in Kingman [2], stands for an analogue of the Lévy-Khintchine representation for rad. ch. f.'s of i.d.p.m.'s in the k-dimensional Kingman convolution.*

THEOREM 2.6. *A p.m.* $\mu \in ID(\bigcirc_k)$ *if and only if there exist a* σ*-finite measure M (a Lévy's measure) on* $\mathbb{R}^{+k}$ *with the property that* $M(\{\mathbf{0}\}) = 0$*, M is finite outside every neighborhood of* $\mathbf{0}$ *and*

$$\int\limits_{\mathbb{R}^{+k}} \frac{\|\mathbf{x}\|^2}{1+\|\mathbf{x}\|^2} M(d\mathbf{x}) < \infty \tag{2.25}$$

and for each $\mathbf{t} = (t_1, ..., t_k) \in \mathbb{R}^{k+}$

$$-\log \hat{\mu}(\mathbf{t}) = \int\limits_{\mathbb{R}^{+k}} \{1 - \prod_{j=1}^{k} \Lambda_s(t_j x_j)\} \frac{1+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2} M(d\mathbf{x}), \tag{2.26}$$

where, at the origin $\mathbf{0}$*, the integrand on the right-hand side of (2.26) is assumed to be*

$$\Sigma_{j=1}^{k} \lambda_j(\mathbf{x}) t_j^2 = lim_{\|\mathbf{x}\| \to 0} \{1 - \prod_{j=1}^{k} \Lambda_s(t_j x_j)\} \frac{1+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2} \tag{2.27}$$

for nonnegative $\lambda_j(\mathbf{x}), j = 1, 2, ..., k$ *and* $\mathbf{x} \in \mathbb{R}^{k+}$*, given by equations (2.15) in Lemma 2.1. In particular, if* $M = 0$*, then* μ *becomes a Rayleighian distribution with the rad. ch. f.*

$$-\log \hat{\mu}(\mathbf{t}) = \frac{1}{2} \sum_{j=1}^{k} \lambda_j t_j^2, \quad \mathbf{t} \in \mathbb{R}^{k+}, \tag{2.28}$$

for some nonnegative $\lambda_j, j = 1, ..., k$ *such that* $\Sigma_{j=1}^{k} \lambda_j = 1$*.*
Moreover, the representation (2.26) is unique.

P r o o f. The proof is carried out in several steps:

(i) *If* ϕ *is a k-dimensional ID rad.ch.f., then it does not vanish on* $\mathbb{R}^{k+}$.
Indeed, denote by Φ_k the totality of k-dimensional ID rad.ch.f.'s (of the fixed index s). Then, we have

$$\Phi_k = \cap_{n=1}^{\infty} \{\phi : \phi^{1/n} \in \Phi_n\} \tag{2.29}$$

which in conjunction with (2.12) and (2.21) implies that every k-dimensional ID rad.ch.f. is a symmetric ordinary ID ch.f. and, consequently, it does not vanish on $\mathbb{R}^{k+}$.

(ii) *Any $\nu \in ID(\bigcirc_k)$ with rad.ch.f. $\hat{\nu} = \psi \in \Phi_k$ can be expressed in the form (2.26).*

Accordingly, we have, for every n, $\psi = \psi_n^n$. By virtue of (i), $\psi(\mathbf{t}) > 0$ for each $\mathbf{t}$. Therefore,

$$\log\psi(\mathbf{t}) = lim_{n\to\infty} n\{\psi_n(\mathbf{t}) - 1\}. \tag{2.30}$$

Let H_n be a p.m. such that

$$\psi_n(\mathbf{t}) = \int_{\mathbb{R}^{k+}} \Pi_{j=1}^k \Lambda_s(t_j x_j)\mathbf{H}_n(d\mathbf{x}), \quad \mathbf{t} \in \mathbb{R}^{k+}. \tag{2.31}$$

Putting

$$\mathbf{G}_n(A) = n \int_A \frac{\|\mathbf{x}\|^2}{1+\|\mathbf{x}\|^2}\mathbf{H}_n(d\mathbf{x}) \tag{2.32}$$

and taking into account the equations (2.30) and (2.31) we get

$$-log\psi(\mathbf{t}) = lim_{n\to\infty} \int_{\mathbb{R}^{k+}} \{1 - \prod_{j=1}^k \Lambda_s(t_j x_j)\}\frac{1+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2}\mathbf{G}_n(d\mathbf{x}). \tag{2.33}$$

which can be rewritten as

$$-log\psi(\mathbf{t}) = lim_{n\to\infty} \int_{\mathbb{R}^{k+}} \{1 - \prod_{j=1}^k \Lambda_s(t_j x_j)\}\mathbf{K}_n(d\mathbf{x}), \tag{2.34}$$

where K_n are finite measures vanishing at **0** defined by

$$\mathbf{K}_n(d\mathbf{x}) := \frac{1+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2}\mathbf{G}_n(d\mathbf{x}), \qquad (n = 1, 2, ...).$$

Replacing $\mathbf{t}$ in (2.35) by $[\mathbf{t}, \mathbf{u}], \mathbf{t}, \mathbf{u} \in \mathbb{R}^{k+}$ and integrating w. r. t. $\sigma_s \times \cdots \times \sigma_s(d\mathbf{u})$ it follows that

$$\begin{aligned}
&- \int_{\mathbb{R}^{k+}} log\psi([\mathbf{t}, \mathbf{u}])\sigma_s \times \cdots \times \sigma_s(d\mathbf{u}) \\
&= \int_{\mathbb{R}^{k+}} lim_{n\to\infty} \int_{\mathbb{R}^{k+}} \{1 - \prod_{j=1}^k \Lambda_s(t_j u_j x_j)\}\mathbf{K}_n(d\mathbf{x})\sigma_s \times \cdots \times \sigma_s(d\mathbf{u}) \\
&= lim_{n\to\infty} \int_{\mathbb{R}^{k+}} \{1 - \prod_{j=1}^k e^{-t_j^2 x_j^2}\}\mathbf{K}_n(d\mathbf{x}),
\end{aligned}$$

which, by changing variables $x_j^2 \to u_j, j = 1, 2, ..., k$ and applying the Continuity Theorem for the classical infinitely divisible Laplace transforms on $\mathbb{R}^{k+}$, implies that there exists a finite measure $\mathbf{K}$ vanishing at **0** and a subsequence $\{\mathbf{K}_{m_r}\}$ which converges to K in the sense that for any bounded continuous function f from $\mathbb{R}^{k+}$ to $\mathbb{R}$ vanishing on a neighborhood of **0** and

$$lim_{r\to\infty} \int\limits_{\mathbb{R}^{k+}} f(\mathbf{x})\mathbf{K}_{m_r}(d\mathbf{x}) = \int\limits_{\mathbb{R}^{k+}} f(\mathbf{x})\mathbf{K}(d\mathbf{x}).$$

which together with (2.33) and (2.14) implies that every ψ is of the form (2.26) for a Lévy's measure $\mathbf{M}$.

(iii) Now, if $\mathbf{M}$ tends to the zero measure it follows that, at the origin **0**, the integrand on the right-hand side of (2.26) is determined by (2.1) which is a consequence of Lemma 2.1.

(iv) Conversely, the uniqueness of the formula (2.26) can be proved in the same way as in the classical case (cf. Sato [4], Theorems 8.1 and 8.7).

■

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