Intersection theorems, coincidence theorems and maximal-element theorems in GFC-spaces[1]

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Abstract. We propose a definition of GFC-spaces to encompass G-convex spaces, FC-spaces and many recent existing spaces with generalied convexity structures. Intersection, coincidence and maximal-eliment theorems are then established under relaxed assumptions in GFC-spaces. These results contain as true particular cases a number of counterparts which were recently developed in the literature.

Key Words. GFC-spaces, generalized T-KKM mappings, better admissible mappings, compact openness, compact closedness.

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1 Introduction

Intersection theorems, coincidence theorems and maximal-element theorems are among the fundamental theorems of nonlinear analysis and play crucial roles in the existence study of wide-ranging problems of optimization and applied mathematics. They are closely related to other important theorems like fixed-point theorems, minimax theorems, invariant-point theorems, KKM-type theorems. Recently, these theorems have been generalized and extended along with generalizations of KKM mappings to general spaces with relaxed convexity structures. For G-convex spaces [16], intersection theorems was studied in [6]; coincidence theorems were established in [15, 17]; maximal-element theorems were investigated in [4, 5]. FC-spaces were introduced in [7] with maximal-element theorems established. In [8-10] intersection and coincidence theorems in FC-spaces were considered. In this paper we propose a definition of a generalized FC-space (GFC-space in short) and establish these three kinds of theorems. Our results contain several recent existing results in the literature as special cases. We leave applications of our results to optimization-related problems, especially to studies of the solution existence, to a forthcoming paper.

Our paper is splitted into four sections. In the remaining part of this section we propose a notion of GFC-spaces and generalize some definitions of known classes of multivalued mappings from FC-spaces to GFC-spaces. Section 2 is devoted to intersection theorems. In Section 3 some coincidence theorems and fixed-point consequences are proved. Maximal-element theorems are developed in the last Section 4.

For a set A, by $\langle A \rangle$ we denote the family of all finite subsets of A. If $A \subseteq X$, X being a topological space, then $\overline{A}$ and A^c signify the closure and the complement $X \backslash A$, respectively, of A. A is called compactly open (compactly closed, respectively) if for each nonempty compact subset K of X, $A \cap K$ is open (closed, respectively) in K. The compact interior and compact clusure of A are defined by, respectively,

$$\text{cint}A = \bigcup\{B \subseteq X : B \subseteq A \text{ and } B \text{ is compactly open in } X\},$$
$$\text{ccl}A = \bigcap\{B \subseteq X : B \supseteq A \text{ and } B \text{ is compactly closed in } X\}.$$

For topological spaces X, Y and a multivalued mapping $F : X \longrightarrow 2^Y$, F is said to be upper semicontinuous (usc, in short) at x if, for each open subset U of $F(x)$, there is a neighborhood V of x such that $U \supseteq F(V)$. F is called transfer-compactly-open-valued if, for each $x \in X$ and each compact subset K of Y, $y \in F(x) \cap K$ implies the existence of $x' \in X$ such that $y \in \text{int}_K(F(x') \cap K)$, where int_K stands for the interior in K, i.e. interior with respect to the topology of K induced by the topology of Y. $\mathbb{N}$ stands for the set of all natural numbers. Δ_n, $n \in \mathbb{N}$, denotes the n - simplex with the vertices being the unit vectors $e_1, e_2, ..., e_{n+1}$, which form a basis of R^{n+1}.

Definition 1.1

(i) Let X be a topological space, Y be a nonempty set and Φ be a family of continuous mappings $\varphi : \Delta_n \longrightarrow X, n \in \mathbb{N}$. Then a triple (X, Y, Φ) is said to be a generalized finitely continuous topological space (GFC-space in short) if for each finite subset $N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$, there is $\varphi_N : \Delta_n \longrightarrow X$ of the family Φ. Later we also use $(X, Y, \{\varphi_N\})$ to denote (X, Y, Φ).

(ii) Let $S : Y \longrightarrow 2^X$ be a multivalued mapping. A subset D of Y is called an S-subset of Y if, for each $N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$ and each $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N \cap D$, one has $\varphi_N : \Delta_n \longrightarrow X$ of Φ such that $\varphi_N(\Delta_k) \subseteq S(D)$, where Δ_k is the face of Δ_n corresponding to $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\}$.

Note that if $Y = X$, then (X, Y, Φ) is rewritten as (X, Φ) and becomes an FC-space [6]. If in addition, S is the identity map then an S-subset of Y coincides with a FC-subspace of Y [7]. Note also that FC-spaces and GFC-spaces have no convexity structure but they are generalizations of spaces with convexity structures, see e.g. [7]; in particular, of a nonempty convex subset of a vector space. If $Y \subseteq X$ and (X, Y, Γ) is a G-convex space (Γ is a generalized convex hull operator, see [16]), and Φ is the family of continuous mappings $\varphi_N : \Delta_n \longrightarrow \Gamma(N)$ as defined in [16], then (X, Y, Φ) is a GFC-space. Both G-convex space and FC-space are general and include many spaces mentioned in the literature, but they are incomparable. We have seen that both of them are special cases of GFC-spaces. The notion of GFC-space helps us also to encompass many generalized KKM-mappings as shown after Definition 1.3 below.

Definition 1.2 Let (X, Y, Φ) be a GFC-space and Z be a topological space. A multivalued mapping $T : X \longrightarrow 2^Z$ is called better admissible if T is usc and compact-valued such that for each $N \in \langle Y \rangle$ and each continuous mapping $\psi : T(\varphi_N(\Delta_n)) \longrightarrow \Delta_n$, the composition $\psi \circ T|_{\varphi_N(\Delta_n)} \circ \varphi_N : \Delta_n \longrightarrow 2^{\Delta_n}$ has a fixed point, where $\varphi_N \in \Phi$ is corresponding to N.

The class of all such better admissible mapping from X to Z is denoted by $\mathcal{B}(X, Y, Z)$. If $Y = X$, we simply write $\mathcal{B}(X, Z)$. This class was proposed for the particular case where X is a nonempty convex subset of a vector space in [13], extended later for G-convex spaces in [14] and for FC-spaces in [7].

Definition 1.3 Let (X, Y, Φ) be a GFC-space and Z be a topological space. Let $F : Y \longrightarrow 2^Z$ and $T : X \longrightarrow 2^Z$ be multivalued mappings. F is said to be a generalized KKM mapping with respect to (wrt) T (T-KKM mapping in short) if for each $N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$ and each $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$ one has $T(\varphi_N(\Delta_k)) \subseteq \bigcup_{j=0}^{k} F(y_{i_j})$, where $\varphi_N \in \Phi$ is corresponding to N and Δ_k is the k-simplex corresponding to $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\}$ in Definition 1.1.

T-KKM mappings were introduced for X being a convex subset of a topological vector space in [1] and extended for FC-spaces in [7]. Definition 1.3 includes these definitions as particular cases. It encompasses also many other kinds of generalized KKM mappings. We mention here some of them, while devoting a fothcoming paper to generlized KKM types theorems in GFC-spaces. Let $(X, \{\varphi_N\})$ be an FC-space, Y be a nonempty set and $S : Y \longrightarrow X$ be a mapping. We define a GFC-space $(X, Y, \{\varphi_N\})$ by setting $\varphi_N = \varphi_{s(N)}$ for each $N \in \langle Y \rangle$. Then a generalized s-KKM mapping wrt T introduced in [8] becomes a T-KKM mapping by Definition 1.3. A multivalued mapping $F : Y \longrightarrow 2^X$, being an R-KKM mapping as defined in [2], is a special case of T-KKM mappings on GFC-space when $X = Z$ and T is the identity map. The definition of generalized KKM mappings wrt to T in [12] is as well a particular case of Definition 1.3.

Remark 1.1. Since a multivalued mapping $S : Y \longrightarrow 2^X$ is equivalent to a relation $\beta(y, x)$ linking $y \in Y$ and $x \in X$ by setting: $x \in S(y)$ if and only if $\beta(y, x)$ holds, the above-mentioned notions related to multivalued mappings can be expressed in terms of relations. For instance the definition of an S-subset of Y can be restated as follows: for a relation β linking $y \in Y$ and $x \in X$, a subset D of Y is called a β-subset of Y if for each $N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$ and each $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N \cap D$, one has $\varphi_N(\Delta_k) \subseteq \bigcup_{y \in D}\{x \in X : \beta(y, x) \text{ holds}\}$. The class of better admissible relations $\mathcal{R}(X, Y, Z)$ and, for a relation $\alpha(x, z)$ linking $x \in X$ and $z \in Z$, an α-KKM relation $\mathcal{F}(y, z)$ linking $y \in Y$ and $z \in Z$ are defined similarly, corresponding to Definitions 1.2 and 1.3, respectively. Formulations of results involving multivalued mappings in terms of relations may be very convenient when many variables and compositions of mappings are involved, see e.g. [11], where variational inclusion problems are stated in terms of variational relations.

2 Intersection theorems

Lemma 2.1 *Let $(X, Y, \{\varphi_N\})$ be a GFC-space and Z be a topological space. Let $F : Y \longrightarrow 2^Z$ and $T : X \longrightarrow 2^Z$ be multivalued mappings. Assume that*

(i) *for each $y \in Y$, $F(y)$ is compactly closed*;

(ii) $T \in \mathcal{B}(X, Y, Z)$ *and* F *is* T*-KKM.*

Then, for each $N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$,

$$T(\varphi_N(\Delta_n)) \cap \bigcap_{y_i \in N} F(y_i) \neq \emptyset.$$

Proof. Suppose to the contrary that $N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$ exists such that

$$T(\varphi_N(\Delta_n)) = \bigcup_{i=0}^{n} [(Z \backslash F(y_i)) \cap T(\varphi_N(\Delta_n))],$$

i.e. $\{(Z \backslash F(y_i)) \cap T(\varphi_N(\Delta_n))\}_{i=0}^n$ is an open covering of the compact set $T(\varphi_N(\Delta_n))$. Let $\{\psi_i\}_{i=0}^n$ be a continuous partition of unity associated with this covering and $\psi : T(\varphi_N(\Delta_n)) \longrightarrow \Delta_n$ be defined by $\psi(z) = \sum_{i=0}^n \psi_i(z) e_i$. Then ψ is continuous. Since T is better admissible, there is a fixed point of $\psi \circ T|_{\varphi_N(\Delta_n)} \circ \varphi_N$, i.e. there is $z_0 \in T(\varphi_N(\Delta_n))$ such that $z_0 \in T(\varphi_N(\psi(z_0)))$. We have

$$\psi(z_0) = \sum_{j \in J(z_0)} \psi_j(z_0) e_j \in \Delta_{J(z_0)},$$

where $J(z_0) = \{j \in \{0, 1, ..., n\} : \psi_j(z_0) \neq 0\}$. As F is T-KKM, we also have

$$\begin{aligned} z_0 &\in T(\varphi_N(\psi(z_0))) \\ &\subseteq T(\varphi_N(\Delta_{J(z_0)})) \\ &\subseteq \textstyle\bigcup_{j \in J(z_0)} F(y_j). \end{aligned}$$

Hence, there exists $j \in J(z_0)$, $z_0 \in F(y_j)$.

On the other hand, by the definitions of $J(z_0)$ and of the partition $\{\psi_i\}_{i=0}^n$,

$$\begin{aligned} z_0 &\in \{z \in T(\varphi_N(\Delta_n)) : \psi_j(z) \neq 0\} \\ &\subseteq (Z \backslash F(y_j)) \cap T(\varphi_N(\Delta_N)) \\ &\subseteq (Z \backslash F(y_j)), \end{aligned}$$

a contradiction.

Theorem 2.2 *Let* $(X, Y, \{\varphi_N\})$ *be a GFC-space,* Z *be a topological space. Let* $T \in \mathcal{B}(X, Y, Z)$ *and* $F : Y \longrightarrow 2^Z$ *satisfy the following conditions*

(i) *for each* $y \in Y$, $F(y)$ *is compactly closed*;

(ii) *F is* T-KKM;

(iii) *one of the following three conditions holds*

(iii$_1$) *there are* $N_0 \in \langle Y \rangle$ *and a nonempty compact subset* K *of* Z *such that* $\bigcap_{y_i \in N_0} F(y_i) \subseteq K$;

(iii$_2$) *there is* $S : Y \longrightarrow 2^X$ *such that for each* $N \in \langle Y \rangle$, *there exists an* S*-subset* L_N *of* Y, *containing* N, *so that* $S(L_N)$ *is a compact subset and, for some nonempty and compact subset* K *of* Z,

$$(T \circ S)(L_N) \cap (\bigcap_{y \in L_N} F(y)) \subseteq K;$$

(iii$_3$) *there are* $S : Y \longrightarrow 2^X$ *and a nonempty subset* $Y_0 \subseteq Y$ *such that* $K := \bigcap_{y \in Y_0} F(y)$ *is nonempty and compact and that, for each* $N \in \langle Y \rangle$, *there exists an* S*-subset* L_N *of* Y *containing* $Y_0 \cup N$ *so that* $S(L_N)$ *is compact.*

Then

$$K \cap \overline{T(X)} \cap (\bigcap_{y \in Y} F(y)) \neq \emptyset.$$

Furthermore, for the case of (iii$_3$), *if* $K = \emptyset$ *then*

$$\overline{T(X)} \cap (\bigcap_{y \in Y} F(y)) \neq \emptyset.$$

Proof. Case of (iii$_1$). For $y \in Y$, set

$$U(y) = \overline{T(X)} \cap (\bigcap_{y_i \in N_0} F(y_i)) \cap F(y).$$

By (i) and (iii$_1$), $\{U(y)\}_{y \in Y}$ is a family of sets which are closed in K. For each $N \in \langle Y \rangle$, setting $M = N \cup N_0$ and $m = |N| + |N_0|$, by Lemma 2.1 we have

$$\emptyset \neq T(\varphi_M(\Delta_m)) \cap (\bigcap_{y \in M} F(y))$$

$$\subseteq \overline{T(X)} \cap (\textstyle\bigcap_{y\in M} F(y))$$

$$= \bigcap_{y\in N} U(y).$$

Since K is compact, this implies that

$$\emptyset \neq \bigcap_{y\in Y} U(y) \subseteq K \cap \overline{T(X)} \cap (\bigcap_{y\in Y} F(y)).$$

Case of (iii$_2$). By (i), $\{K \cap \overline{T(X)} \cap F(y)\}_{y\in Y}$ is a family of sets which are closed in K. Suppose that

$$\emptyset = K \cap \overline{T(X)} \cap (\bigcap_{y\in Y} F(y))$$

$$= \bigcap_{y\in Y} (K \cap \overline{T(X)} \cap F(y)).$$

Then there exists $N \in \langle Y \rangle$ such that

$$\emptyset = \bigcap_{y\in N} (K \cap \overline{T(X)} \cap F(y))$$

$$= K \cap \overline{T(X)} \cap (\bigcap_{y\in N} F(y)),$$

i.e.

$$\overline{T(X)} \cap (\bigcap_{y\in N} F(y)) \subseteq Z \backslash K.$$

In view of the assumption (iii$_2$), there is an S-subset L_N of Y containing N such that

$$(T \circ S)(L_N) \cap (\bigcap_{y\in L_N} F(y)) \subseteq K.$$

On the other hand,

$$(T \circ S)(L_N) \cap (\bigcap_{y\in L_N} F(y)) \subseteq \overline{T(X)} \cap (\bigcap_{y\in N} F(y)) \subseteq Z \backslash K.$$

Thus,

$$(T \circ S)(L_N) \cap (\bigcap_{y\in L_N} F(y)) = \emptyset.$$

As L_N is an S-subset of Y, by virtue of Lemma 2.1 we have, for each $M \in \langle L_N \rangle$,

$$\emptyset \neq T(\varphi_M(\Delta_m)) \cap (\bigcap_{y \in M} F(y))$$
$$\subseteq (T \circ S)(L_N) \cap (\bigcap_{y \in M} F(y)).$$

By the compactness of $(T \circ S)(L_N)$, this implies that

$$(T \circ S)(L_N) \cap (\bigcap_{y \in L_N} F(y)) \neq \emptyset,$$

a contradiction.

Case of (iii$_3$). If K is nonempty and compact, then $\{K \cap \overline{T(X)} \cap F(y)\}_{y \in Y}$ is a family of closed subsets of K. Suppose that

$$K \cap \overline{T(X)} \cap (\bigcap_{y \in Y} F(y)) = \emptyset.$$

Then there is $N \in \langle Y \rangle$ such that

$$\emptyset = \bigcap_{y \in N} (K \cap \overline{T(X)} \cap F(y))$$
$$= \overline{T(X)} \cap (\bigcap_{y \in Y_0 \cup N} F(y)).$$

Therefore,

$$(T \circ S)(L_N) \cap (\bigcap_{y \in L_N} F(y)) \subseteq \overline{T(X)} \cap (\bigcap_{y \in Y_0 \cup N} F(y)) = \emptyset.$$

If K is empty, then, for each $N \in \langle Y \rangle$,

$$(T \circ S)(L_N) \cap (\bigcap_{y \in L_N} F(y)) \subseteq K = \emptyset.$$

Since in both subcases the set on the left hand side is empty, we can argue similarly as for the case (iii$_2$) to get a contradiction.

Note that Theorem 2.2 contains Theorem 3 of [17] and Theorems 1-3 of [3] as special cases for the case of G-convex spaces.

3 Coincidence theorems

Theorem 3.1 *Let $(X, Y, \{\varphi_N\})$ be a GFC-space and Z be a topological space. Let $S : Y \longrightarrow 2^X$, $T : X \longrightarrow 2^Z$ and $F : Z \longrightarrow 2^Y$ be multivalued mappings with $T \in \mathcal{B}(X, Y, Z)$. Assume that*

(i) *for each $x \in X$ and each $z \in T(x)$, $F(z)$ is an S-subset of Y;*

(ii) *for each $y \in Y, F^{-1}(y)$ contains a compactly open O_y (O_y may be empty) of Z such that $K := \bigcup_{y\in Y} O_y$ is nonempty and compact;*

(iii) *one of the following three conditions holds:*

(iii_1) *there is $N_0 \in \langle Y\rangle$ such that $\bigcap_{y\in N_0} O_y^c \subseteq K$;*

(iii_2) *for each $N \in \langle Y\rangle$, there is an S-subset L_N of Y, containing N such that $S(L_N)$ is compact and*

$$(T \circ S)(L_N) \cap (\bigcap_{y\in L_N} O_y^c) \subseteq K;$$

(iii_3) *$K = Z$; there is a nonempty subset Y_0 of Y such that $\bigcap_{y\in Y_0} O_y^c$ is compact or empty; and for each $N \in \langle Y\rangle$ there is an S-subset L_N of Y containing $Y_0 \cup N$ so that $S(L_N)$ is compact.*

Then a point $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Z$ exists such that $\bar{x} \in S(\bar{y})$, $\bar{y} \in F(\bar{z})$ and $\bar{z} \in T(\bar{x})$.

Proof. Define a new multivalued mapping $G : Y \longrightarrow 2^Z$ by setting, $\forall y \in Y$, $G(y) = O_y^c$, which is compactly closed by (ii), i.e. assumption (i) of Theorem 2.2 for G in the place of F is fulfilled. It is clear that (iii_1), (iii_2), and (iii$_3$) imply the corresponding assumptions of Theorem 2.2 for G. By (ii) of Theorem 3.1,

$$K \cap \overline{T(X)} \cap (\bigcap_{y \in Y} G(y)) \subseteq K \cap (\bigcap_{y \in Y} O_y^c) = \emptyset,$$

which means that the conclusion of Theorem 2.2 for G in the place of F does not hold. Therefore assumption (ii) of this theorem must be violated, i.e. G is not T-KKM. This means that there are $N = \{y_0, y_1, ..., y_n\} \in \langle Y \rangle$ and $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$ such that

$$T(\varphi_N(\Delta_k)) \not\subseteq \bigcup_{j=0}^{k} G(y_{i_j}) = \bigcup_{j=0}^{k} O_{y_{i_j}}^c.$$

This in turn is equivalent to the existence of $\bar{x} \in \varphi_N(\Delta_k)$ and $\bar{z} \in T(\bar{x})$ such that $\bar{z} \in O_{y_{i_j}}$, for all $j = 0, 1, ..., k$. Since $O_{y_{i_j}} \subseteq F^{-1}(y_{i_j})$ by (ii), $y_{i_j} \in F(\bar{z})$, which is an S-subset of Y. Hence

$$\bar{x} \in \varphi_N(\Delta_k) \subseteq S(F(\bar{z})),$$

which means that there is $\bar{y} \in F(\bar{z})$ such that $\bar{x} \in S(\bar{y})$.

Remark 3.1.

(i) By setting $H(y) = F^{-1}(y)$ for $y \in Y$, Theorem 3.1 can be restated with the conclusion that there is $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Z$ such that $\bar{x} \in S(\bar{y})$ and $\bar{z} \in H(\bar{y}) \cap T(\bar{x})$. So Theorem 3.1 includes properly Theorem 1 of [15].

(ii) For the special case, where $X = Y = Z$ and $T = S$ is the identity mapping, Theorem 3.1 becomes a fixed-point theorem for FC-spaces.

4 Maximal-element theorems

Theorem 4.1 *Let $(X, Y, \{\varphi_N\})$ be a GFC-space, Z be a topological space* *and* $K \subseteq Z$ *be nonempty and compact. Let* $F : Z \longrightarrow 2^Y$ *and* $T : X \longrightarrow 2^Z$ *be multivalued mapping such that* $T \in \mathcal{B}(X, Y, Z)$ *and the following assumptions are satisfied*

(i) *for each* $y \in Y$, $F^{-1}(y)$ *includes a compactly open subset* O_y (O_y *may be empty*)*of* Z *such that*

$$\bigcup_{y\in Y}(O_y \cap K) = \bigcup_{y\in Y}(F^{-1}(y) \cap K);$$

(ii) *for each* $N = \{y_0, y_1, ..., y_n\} \in \langle Y\rangle$ *and each* $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$,

$$T(\varphi_N(\Delta_k)) \cap (\bigcap_{j=0}^{k} O_{y_{i_j}}) = \emptyset;$$

(iii) *one of the following conditions hold :*

(iii$_1$) *there is* $N_0 \in \langle Y\rangle$ *such that* $Z\backslash K \subseteq \bigcup_{y_i\in N_0} O_{y_i}$;

(iii$_2$) *there is a multivalued map* $S : Y \longrightarrow 2^X$ *such that, for each* $N \in \langle Y\rangle$, *there is an* S*-subset* L_N *of* Y *containing* N *so that* $S(L_N)$ *is compact and*

$$(T \circ S)(L_N)\backslash K \subseteq \bigcup_{y\in L_N} O_y.$$

Then a point $\bar{z} \in K$ *exists such that* $F(\bar{z}) = \emptyset$.

Proof. We check the assumptions of Theorem 2.2 in order to apply it for, instead of F, a new multivalued mapping $G : Y \longrightarrow 2^Z$ defined by $G(y) = O_y^c$. Assumption (i) is clearly fulfilled. For (ii), with arbitrary $N = \{y_0, y_1, ..., y_n\} \in \langle Y\rangle$ and $\{y_{i_0}, y_{i_1}, ..., y_{i_k}\} \subseteq N$, we have, by (ii) of Theorem 4.1,

$$T(\varphi_N(\Delta_k)) \subseteq \bigcup_{j=0}^{k} O^c_{y_{i_j}} = \bigcup_{j=0}^{k} G(y_{i_j}),$$

i.e. G is T-KKM as required. By (iii$_1$) of this theorem, we obtain (iii$_1$) since

$$\bigcap_{y_i \in N_0} G(y_i) \;=\; \bigcap_{y_i \in N_0} O^c_{y_i} \;\subseteq\; Z\backslash(Z\backslash K) \;=\; K.$$

From (iii$_2$) of this theorem it follows that

$$\begin{aligned}(T\circ S)(L_N)\cap(\bigcap_{y\in L_N} G(y)) \;&=\; (T\circ S)(L_N)\cap(Z\backslash\bigcup_{y\in L_N} O_y)\\ &\subseteq (T\circ S)(L_N)\cap\{Z\backslash((T\circ S)(L_N)\backslash K)\}\\ &\subseteq K,\end{aligned}$$

i.e. (iii$_2$) is satisfied. Now that all the assumptions of Theorem 2.2 have been checked, we obtain from this theorem

$$\begin{aligned}\emptyset \;\neq\; K\cap\overline{T(X)}\cap(\bigcap_{y\in Y} G(y)) \;&=\; K\cap\overline{T(X)}\cap(Z\backslash\bigcup_{y\in Y} O_y)\\ &\subseteq\; K\cap\overline{T(X)}\cap(Z\backslash\bigcup_{y\in Y}(O_y\cap K))\\ &=\; K\cap\overline{T(X)}\cap(Z\backslash\bigcup_{y\in Y}(F^{-1}(y)\cap K)).\end{aligned}$$

Therefore an element $\bar z \in K$ exists such that $\bar z \notin F^{-1}(y)\cap K$ for every $y \in Y$, i.e. $F(\bar z) = \emptyset$.

Remark 4.1 Assumption (i) of Theorem 4.1 is satisfied with $O_y = \mathrm{cintF^{-1}(y)}$ if F^{-1} is transfer-compactly-open-valued (by Lemma 1.2 of [7]). This case of Theorem 4.1 extends Theorem 2.2 of [7] to the case of GFC-spaces.

Theorem 4.2 *Let* $(X, Y, \{\varphi_N\})$, Z, F *and* T *be defined as in Theorem* 4.1 *such that* (ii) *is satisfied and* (i) *and* (iii) *are replaced respectively by*

- (i') *for each* $y \in Y$, $F^{-1}(y)$ *contains a compactly open subset* O_y, *which may be empty, of* Z *such that, for each nonempty compact subset* Z_0 *of* Z,

$$\bigcup_{y\in Y}(O_y\cap Z_0) = \bigcup_{y\in Y}(F^{-1}(y)\cap Z_0);$$

(iii') *there are a multivalued map* $S : Y \longrightarrow 2^X$ *and a nonempty subset* Y_0 *of* Y *such that* $K := \bigcap_{y\in Y_0} O_y^c$ *is an* S*-subset* L_N *of* Y *containing* $Y_0 \cup N$ *so that* $S(L_N)$ *is compact.*

Then an element $\bar{z} \in Z$ *exists with* $F(\bar{z}) = \emptyset$.

Proof. It is not hard to see that all the assumptions (i), (ii) and (iii_3) of Theorem 2.2 are fulfilled with $G : Y \longrightarrow 2^Z$ defined by $G(y) = O_y^c$ in the place of F. By this theorem we have, if K is nonempty,

$$\begin{aligned}\emptyset \neq K \cap \overline{T(X)} \cap (\bigcap_{y\in Y} G(y)) &\subseteq K \cap \overline{T(X)} \cap (Z \backslash \bigcup_{y\in Y}(O_y \cap K)) \\ &= K \cap \overline{T(X)} \cap (Z \backslash \bigcup_{y\in Y}(F^{-1}(y) \cap K)).\end{aligned}$$

Consequently, there is $\bar{z} \in K$ such that $\bar{z} \notin F^{-1}(y)$ for each $y \in Y$, which means that $F(\bar{z}) = \emptyset$.

If K is empty, Theorem 2.2 gives that

$$\emptyset \neq \overline{T(X)} \cap (\bigcap_{y\in Y} G(y)) \subseteq \overline{T(X)} \cap (Z \backslash \bigcup_{y\in Y}(F^{-1}(y) \cap K)).$$

Hence, $\bar{z} \in Z$ also exists such that $\bar{z} \notin F^{-1}(y)$ for each $y \in Y$, i.e. $F(\bar{z}) = \emptyset$.

Similarly as mentioned in Remark 4.1, if F^{-1} is transfer-compactly-open-valued, by taking $O_y = \mathrm{cintF^{-1}(y)}$, all the assumptions of Theorem 4.2 are satisfied. This case of Theorem 4.2 includes Theorem 2.1 of [7] as a special case for the FC-space setting.

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